CO-SPECTRA OF JOINT SPECTRA OF STRUCTURES

A. A. SOSKOVA AND I. N. SOSKOV

Abstract. We introduce and study the notion of joint spectrum of finitely many abstract structures. A characterization of the lower bounds of the elements of the joint spectrum is obtained.

1. Introduction

Let $\mathfrak{A} = (\mathbb{N}; R_1, \dots, R_k)$ be a structure with domain the set of all natural numbers $\mathbb{N}$, where each R_i is a subset of $\mathbb{N}^{r_i}$ and "$=$" and "$\neq$" are among $R_1, \dots, R_k$.

An enumeration f *of* $\mathfrak{A}$ is a total mapping from $\mathbb{N}$ onto $\mathbb{N}$.

For every $A \subseteq \mathbb{N}^a$ define

$$f^{-1}(A) = \{\langle x_1 \dots x_a\rangle : (f(x_1), \dots, f(x_a)) \in A\}$$

Let

$$f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k).$$

For any sets of natural numbers A and B the set A is enumeration reducible to B ($A \leq_e B$) if there is an enumeration operator Γ_z such that $A = \Gamma_z(B)$. By $d_e(A)$ we denote the enumeration degree of the set A. The set A is total if $A \equiv_e A^+$, where $A^+ = A \oplus (\mathbb{N} \setminus A)$. An enumeration degree is called total if it contains a total set.

1.1. Definition. *The Degree Spectrum of* $\mathfrak{A}$ is the set

$$DS(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) : f \text{ is an enumeration of } \mathfrak{A}\}.$$

The notion is introduced by [6] for bijective enumerations. In [2, 5, 4, 7] several results about degree spectra of structures are obtained. In [7] it is shown that if $\mathbf{a} \in DS(\mathfrak{A})$ and $\mathbf{b}$ is a total e-degree, $\mathbf{a} \leq \mathbf{b}$, then $\mathbf{b} \in DS(\mathfrak{A})$. In other words, the Degree Spectrum of $\mathfrak{A}$ is closed upwards.

The co-spectrum of the structure $\mathfrak{A}$ is the set of all lower bounds of the degree spectra of $\mathfrak{A}$. Co-spectra are introduced and studied in [7].

The aim of the present paper is to study a generalization of the notions of degree spectra and co-spectra for finitely many structures and to give a normal form of the sets which generates the elements of the generalized co-spectra in terms of recursive Σ^+ formulae.

In what follows we shall use the following Jump Inversion Theorem proved in [8]. Notice that the jump operation "$'$" denotes here the enumeration jump introduced by Cooper [3].

Given $n+1$ sets $B_0, \dots, B_n$, for every $i \leq n$ define the set $\mathcal{P}(B_0, \dots, B_i)$ by means of the following inductive definition:

1991 *Mathematics Subject Classification.* 03D30.

Key words and phrases. enumeration reducibility, enumeration jump, enumeration degrees, forcing.

(i) $\mathcal{P}(B_0) = B_0$;
(ii) If $i < n$, then $\mathcal{P}(B_0, \dots, B_{i+1}) = (\mathcal{P}(B_0, \dots, B_i))' \oplus B_{i+1}$.

1.2. Theorem. *Let $n > k \geq 0$, $B_0, \dots, B_n$ be arbitrary sets of natural numbers. Let $A \subseteq \mathbb{N}$ and let Q be a total subset of $\mathbb{N}$ such that $\mathcal{P}(B_0, \dots, B_n) \leq_e Q$ and $A^+ \leq_e Q$. Suppose also that $A \not\leq_e \mathcal{P}(B_0, \dots, B_k)$. Then there exists a total set F having the following properties:*
(i) *For all $i \leq n$, $B_i \leq_e F^{(i)}$;*
(ii) *For all $i, 1 \leq i \leq n$, $F^{(i)} \equiv_e F \oplus \mathcal{P}(B_0, \dots, B_{i-1})'$;*
(iii) *$F^{(n)} \equiv_e Q$.*
(iv) *$A \not\leq_e F^{(k)}$.*

2. Joint spectra of structures

Let us fix the structures $\mathfrak{A}_0, \dots, \mathfrak{A}_n$.

2.1. Definition. *The Joint Spectrum of* $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ is the set

$$DS(\mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \{\mathbf{a} : \mathbf{a} \in DS(\mathfrak{A}_0), \mathbf{a}' \in DS(\mathfrak{A}_1), \dots, \mathbf{a}^{(\mathbf{n})} \in DS(\mathfrak{A}_n)\}.$$

2.2. Definition. Let $k \leq n$. *The k - th Jump Spectrum of* $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ is the set

$$DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n) = \{\mathbf{a}^{(\mathbf{k})} : \mathbf{a} \in DS(\mathfrak{A}_0, \dots, \mathfrak{A}_n)\}.$$

2.3. Proposition. *$DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$ is closed upwards, i.e. if $\mathbf{a}^{(\mathbf{k})} \in DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$, $\mathbf{b}$ is a total e-degree and $\mathbf{a}^{(\mathbf{k})} \leq \mathbf{b}$, then $\mathbf{b} \in DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$.*

Proof. Suppose that $\mathbf{a}^{(\mathbf{k})} \in DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$, $\mathbf{b}$ is a total degree and $\mathbf{b} \geq \mathbf{a}^{(\mathbf{k})}$. By the Jump Inversion Theorem 1.2 there is a total e-degree $\mathbf{f}$ such that:
(1) $\mathbf{a}^{(\mathbf{i})} \leq \mathbf{f}^{(\mathbf{i})}$, for all $i \leq k$;
(2) $\mathbf{f}^{(\mathbf{k})} = \mathbf{b}$.
Clearly $\mathbf{a}^{(\mathbf{i})} \leq \mathbf{f}^{(\mathbf{i})}$ for $i \leq n$. Since $\mathbf{a}^{(\mathbf{i})} \in DS(\mathfrak{A}_i)$ and $\mathbf{f}^{(\mathbf{i})}$ is total, $\mathbf{f}^{(\mathbf{i})} \in DS(\mathfrak{A}_i)$, $i \leq n$. Therefore $\mathbf{f} \in DS(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$ and hence $\mathbf{b} = \mathbf{f}^{(\mathbf{k})} \in DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$. □

2.4. Definition. Let $k \leq n$. *The k - th Co-spectrum of* $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ is the set of all lower bounds of $DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$, i.e.

$$CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n) = \{\mathbf{b} : \mathbf{b} \in \mathcal{D}_e \& (\forall \mathbf{a} \in DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n))(\mathbf{b} \leq \mathbf{a})\}.$$

2.5. Proposition. *Let $k \leq n$. Then*

$$CS_k(\mathfrak{A}_0, \dots \mathfrak{A}_k \dots, \mathfrak{A}_n) = CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k).$$

Proof. Clearly $DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k, \dots, \mathfrak{A}_n) \subseteq DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k)$ and hence

$$CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k) \subseteq CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k, \dots, \mathfrak{A}_n).$$

To show the reverse inclusion let $\mathbf{c} \in CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$, i.e. $\mathbf{c} \leq \mathbf{a}^{(\mathbf{k})}$ for all $\mathbf{a} \in DS(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$. Suppose that $\mathbf{c} \notin CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k)$. Then there exist sets C and A such that $d_e(C) = \mathbf{c}$ and $d_e(A) \in DS(\mathfrak{A}_0, \dots, \mathfrak{A}_k)$ and $C \not\leq_e A^{(k)}$. Notice that $\mathcal{P}(A, A', \dots, A^{(k)}) \equiv_e A^{(k)}$ and therefore $C \not\leq_e \mathcal{P}(A, A', \dots, A^{(k)})$. Fix some sets $B_1, \dots, B_{n-k}$ such that $d_e(B_1) \in DS(\mathfrak{A}_{k+1}), \dots, d_e(B_{n-k}) \in DS(\mathfrak{A}_n)$. Applying the Jump Inversion Theorem 1.2 we obtain a total set F such that:
(i) For all $i \leq k$, $A^{(i)} \leq_e F^{(i)}$;
(ii) For all $j, 1 \leq j \leq n-k$, $B_j \leq_e F^{(k+j)}$;
(iii) $C \not\leq_e F^{(k)}$.

Since the degree spectra are closed upwards, $d_e(F^{(i)}) \in DS(\mathfrak{A}_i), i = 0, \dots, n$ and hence $d_e(F) \in DS(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$. On the other hand $C \not\leq_e F^{(k)}$ and hence $c \notin CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$. A contradiction. $\square$

2.6. Theorem. *Let $A \subseteq \mathbb{N}$. Then the following are equivalent:*

(1) $d_e(A) \in CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k)$.
(2) *For every $k+1$ enumerations $f_0, \dots, f_k$,*

$$A \leq_e \mathcal{P}(f_0^{-1}(\mathfrak{A}_0), \dots, f_k^{-1}(\mathfrak{A}_k))).$$

Proof. Suppose that A satisfies (2) and consider a $\mathbf{b} \in DS(\mathfrak{A}_0, \dots, \mathfrak{A}_k)$. We shall show that $d_e(A) \leq \mathbf{b^{(k)}}$.

Let $i \leq k$. Then $\mathbf{b^{(i)}} \in DS(\mathfrak{A}_i)$ and hence there exists an enumeration f_i such that $\mathbf{b^{(i)}} = d_e(f_i^{-1}(\mathfrak{A}_i))$. Clearly $d_e(A) \leq d_e(\mathcal{P}(f_0^{-1}(\mathfrak{A}_0), \dots, f_k^{-1}(\mathfrak{A}_k))) = \mathbf{b^{(k)}}$.

Suppose now that $d_e(A) \in CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k)$ and $f_0, \dots, f_k$ are enumerations. Set $B_i = f_i^{-1}(\mathfrak{A}_i)$, $i = 0, \dots, k$. Towards a contradiction assume that $A \not\leq_e \mathcal{P}(f_0^{-1}(\mathfrak{A}_0), \dots, f_k^{-1}(\mathfrak{A}_k))$. By the Jump Inversion Theorem 1.2 there is a total set F such that: $B_i \leq_e F^{(i)}$, $i \leq k$, and $A \not\leq_e F^{(k)}$. Clearly $d_e(F) \in DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k)$ and $d_e(A) \not\leq F^{(k)}$. So, $d_e(A) \notin CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k)$. A contradiction. $\square$

3. Generic enumerations and forcing

3.1. **The satisfaction relation.** Given $k+1$ enumerations $f_0, \dots, f_k$, denote by $\bar{f}$ the sequence $f_0, \dots, f_k$ and set for $i \leq k$, $\mathcal{P}_i^{\bar{f}} = \mathcal{P}(f_0^{-1}(\mathfrak{A}_0), \dots, f_i^{-1}(\mathfrak{A}_i))$.

Let $W_0, \dots, W_z, \dots$ be a Goedel enumeration of the r.e. sets and D_v be the finite set having canonical code v.

For every $i \leq k$, e and x in $\mathbb{N}$ define the relations $\bar{f} \models_i F_e(x)$ and $\bar{f} \models_i \neg F_e(x)$ by induction on i:

(i) $\bar{f} \models_0 F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&\ D_v \subseteq f_0^{-1}(\mathfrak{A}_0))$;

(ii)
$$\begin{aligned} \bar{f} \models_{i+1} F_e(x) \iff & (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(\\ & u = \langle 0, e_u, x_u\rangle \ \&\ \bar{f} \models_i F_{e_u}(x_u) \vee \\ & u = \langle 1, e_u, x_u\rangle \ \&\ \bar{f} \models_i \neg F_{e_u}(x_u) \vee \\ & u = \langle 2, x_u\rangle \ \&\ x_u \in f_{i+1}^{-1}(\mathfrak{A}_{i+1}))); \end{aligned}$$

(iii) $\bar{f} \models_i \neg F_e(x) \iff \bar{f} \not\models_i F_e(x)$.

From the above definition it follows easily the truth of the following:

3.1. Proposition. *Let $A \subseteq \mathbb{N}$ and $i \leq k$. Then*

$$A \leq_e \mathcal{P}_i^{\bar{f}} \iff (\exists e)(A = \{x : \bar{f} \models_i F_e(x)\}).$$

3.2. **Finite parts and forcing.** The forcing conditions which we shall call *finite parts* are $k-$ tuples $\bar{\tau} = (\tau_0, \dots, \tau_k)$ of finite mappings $\tau_0, \dots, \tau_k$ of $\mathbb{N}$ in $\mathbb{N}$. We shall use the letters $\bar{\delta}, \bar{\tau}, \bar{\rho}, \bar{\mu}$ to denote finite parts.

For every $i \leq k$, e and x in $\mathbb{N}$ and every finite part $\bar{\tau}$ we define the forcing relations $\bar{\tau} \Vdash_i F_e(x)$ and $\bar{\tau} \Vdash_i \neg F_e(x)$ following the definition of relations "$\models_i$".

3.2. Definition.

(i) $\bar{\tau} \Vdash_0 F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&\ D_v \subseteq \tau_0^{-1}(\mathfrak{A}_0))$;

(ii)
$$\begin{aligned}\bar\tau \Vdash_{i+1} F_e(x) \iff \exists v(\langle v,x\rangle \in W_e \ \&\\ (\forall u \in D_v)(u = \langle 0, e_u, x_u\rangle \ \&\ \bar\tau \Vdash_i F_{e_u}(x_u) \vee\\ u = \langle 1, e_u, x_u\rangle \ \&\ \bar\tau \Vdash_i \neg F_{e_u}(x_u) \vee\\ u = \langle 2, x_u\rangle \ \&\ x_u \in \tau_{i+1}^{-1}(\mathfrak{A}_{i+1})));\end{aligned}$$

(iii) $\bar\tau \Vdash_i \neg F_e(x) \iff (\forall \bar\rho \supseteq \bar\tau)(\bar\rho \nVdash_i F_e(x))$.

Given finite parts $\bar\delta = (\delta_0, \dots, \delta_k)$ and $\bar\tau = (\tau_0, \dots, \tau_k)$, let

$$\bar\delta \subseteq \bar\tau \iff \delta_0 \subseteq \tau_0, \dots, \delta_k \subseteq \tau_k.$$

3.3. Proposition. *Let $i \le k, e, x \in \mathbb{N}$ and $\bar\delta = (\delta_0, \dots, \delta_k)$, $\bar\tau = (\tau_0, \dots, \tau_k)$ be finite parts.*

(1) *$\bar\delta \subseteq \bar\tau$, then $\bar\delta \Vdash_i (\neg)F_e(x) \Longrightarrow \bar\tau \Vdash_i (\neg)F_e(x)$;*
(2) *If $\delta_0 = \tau_0, \dots, \delta_i = \tau_i$ then $\bar\delta \Vdash_i (\neg)F_e(x) \iff \bar\tau \Vdash_i (\neg)F_e(x)$.*

Proof. The monotonicity condition (1) is obvious.

The proof of (2) is by induction on i. Skipping the obvious case $i = 0$ suppose that $i < k$ and

$$\bar\delta \Vdash_i (\neg)F_e(x) \iff \bar\tau \Vdash_i (\neg)F_e(x).$$

Let $\tau_j = \delta_j, j \le i+1$. From the definition of the relation $\Vdash_{i+1}$ it follows immediately that

$$\bar\delta \Vdash_{i+1} F_e(x) \iff \bar\tau \Vdash_{i+1} F_e(x).$$

Assume that $\bar\delta \Vdash_{i+1} \neg F_e(x)$ but $\bar\tau \nVdash_{i+1} \neg F_e(x)$. Then there exists a finite part $\bar\rho \supseteq \bar\tau$ such that $\bar\rho \Vdash_{i+1} F_e(x)$. Consider the finite part $\bar\mu$ such that $\mu_j = \rho_j, j \le i+1$ and $\mu_j = \delta_j$ for $i+1 < j \le k$. Clearly $\bar\mu \supseteq \bar\delta$ and $\bar\mu \Vdash_{i+1} F_e(x)$. A contradiction. □

3.4. Definition. If $\bar\delta = (\delta_0, \dots, \delta_k)$, $\bar\tau = (\tau_0, \dots, \tau_k)$ and $i \le k$ define

$$\bar\delta \subseteq_i \bar\tau \iff \delta_0 \subseteq \tau_0, \dots, \delta_i \subseteq \tau_i, \delta_{i+1} = \tau_{i+1}, \dots, \delta_k = \rho_k.$$

Let $\bar\tau \Vdash_i^* (\neg)F_e(x)$ be the same as $\bar\tau \Vdash_i (\neg)F_e(x)$ with the exception of

(iii) $\bar\tau \Vdash_i \neg F_e(x) \iff (\forall \bar\rho \supseteq_i \bar\tau)(\bar\rho \nVdash_i^* F_e(x))$.

As an immediate corrolary of the previous Proposition we get the following:

3.5. Lemma. *For each $i \le k, e, x \in \mathbb{N}$ and $\bar\tau$,*

$$\bar\tau \Vdash_i (\neg)F_e(x) \iff \bar\tau \Vdash_i^* (\neg)F_e(x).$$

3.3. **Generic enumerations.** For any $i \le k, e, x \in \mathbb{N}$ denote by $X^i_{\langle e,x\rangle} = \{\bar\rho : \bar\rho \Vdash_i F_e(x)\}$.

If $\bar f = (f_0, \dots, f_k)$ is an enumeration of $\mathfrak{A}_0, \dots, \mathfrak{A}_k$, then

$$\bar\tau \subseteq \bar f \iff \tau_0 \subseteq f_0, \dots, \tau_k \subseteq f_k.$$

3.6. Definition. An enumeration $\bar f$ of $\mathfrak{A}_0, \dots, \mathfrak{A}_k$ is *i-generic* if for every $j < i$, $e, x \in \mathbb{N}$

$$(\forall \bar\tau \subseteq \bar f)(\exists \bar\rho \in X^j_{\langle e,x\rangle})(\bar\tau \subseteq \bar\rho) \Longrightarrow (\exists \bar\tau \subseteq \bar f)(\bar\tau \in X^j_{\langle e,x\rangle}).$$

3.7. Lemma.

(1) *Let $\bar f$ be an i-generic enumeration. Then*

$$\bar f \models_i F_e(x) \iff (\exists \bar\tau \subseteq \bar f)(\bar\tau \Vdash_i F_e(x)).$$

(2) *Let f be an $i+1$-generic enumeration. Then*

$$\bar{f} \models_i \neg F_e(x) \iff (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \Vdash_i \neg F_e(x)).$$

Proof. Induction on i. Clearly for every $\bar{f}$ we have that

$$\bar{f} \models_0 F_e(x) \iff (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \Vdash_0 F_e(x)).$$

From the definition of the relations $\models_i$ and $\Vdash_i$ it follows immediately that if for some enumeration $\bar{f}$ we have the equivalences

$$\bar{f} \models_i F_e(x) \iff (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \Vdash_i F_e(x))$$

and

$$\bar{f} \models_i \neg F_e(x) \iff (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \Vdash_i \neg F_e(x)),$$

then we have also and

$$\bar{f} \models_{i+1} F_e(x) \iff (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \Vdash_{i+1} F_e(x)).$$

So to finish the proof we have to show that if for some $i < k$ the enumeration $\bar{f}$ is $i+1$-generic and (1) holds, then (2) holds as well. Indeed suppose that $\bar{f} \models_i \neg F_e(x)$. Assume that there is no $\bar{\tau} \subseteq \bar{f}$ such that $\bar{\tau} \Vdash_i \neg F_e(x)$. Then for every $\bar{\tau} \subseteq \bar{f}$ there exists a finite part $\bar{\rho} \supseteq \bar{\tau}$ such that $\bar{\rho} \Vdash_i F_e(x)$. From the $i+1$ - genericity of $\bar{f}$ it follows that there exists a finite part $\bar{\tau} \subseteq \bar{f}$ such that $\bar{\tau} \Vdash_i F_e(x)$. Hence $\bar{f} \models_i F_e(x)$. A contradiction.

Assume now that $\bar{\tau} \subseteq \bar{f}$ and $\bar{\tau} \Vdash_i \neg F_e(x)$. Assume that $\bar{f} \models_i F_e(x)$. Then we can find a finite part $\bar{\mu} \subseteq \bar{f}$ such that $\bar{\mu} \Vdash_i F_e(x)$ and $\bar{\mu} \supseteq \bar{\tau}$. A contradiction. □

3.4. **Forcing k - definable sets.**

3.8. Definition. The set $A \subseteq \mathbb{N}$ is *forcing k-definable* on $\mathfrak{A}_0, \ldots, \mathfrak{A}_k$ if there exist a finite part $\bar{\delta}$ and $e \in \mathbb{N}$ such that

$$x \in A \iff (\exists \bar{\tau} \supseteq \bar{\delta})(\bar{\tau} \Vdash_k F_e(x)).$$

3.9. Theorem. *Let $A \subseteq \mathbb{N}$.*

If $A \leq_e \mathcal{P}(f_0^{-1}(\mathfrak{A}_0), \ldots, f_k^{-1}(\mathfrak{A}_k))$ for all $f_0, \ldots, f_k$ enumerations of $\mathfrak{A}_0, \ldots, \mathfrak{A}_k$, respectively, then A is forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_k$.

Proof. Suppose that A is not forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_k$.

We shall construct a $k+1$ - generic enumeration $\bar{f}$ such that $A \not\leq \mathcal{P}_k^{\bar{f}}$.

The construction of the enumeration $\bar{f}$ will be carried out by steps. On each step j we shall define a finite part $\bar{\delta}^j = (\delta_0^j, \ldots, \delta_k^j)$, so that $\bar{\delta}^j \subseteq \bar{\delta}^{j+1}$, and take $f_i = \cup_j \delta_i^j$, for each $i \leq k$.

On the steps $j = 3q$ we shall ensure that each f_i is a total surjective mapping from $\mathbb{N}$ onto $\mathbb{N}$. On the steps $j = 3q+1$ we shall ensure that $\bar{f}$ is $k+1$- generic. On the steps $j = 3q+2$ we shall ensure that $A \not\leq \mathcal{P}_k^{\bar{f}}$.

Let $\bar{\delta}^0 = (\emptyset, \ldots, \emptyset)$.

Suppose that $\bar{\delta}^j$ is defined.

Case $j = 3q$. For every i, $0 \leq i \leq k$, let x_i be the least natural number which does not belong to the domain of δ_i^j and y_i be the least natural number which does not belong to the range of δ_i^j. Let $\delta_i^{j+1}(x_i) = y_i$ and $\delta_i^{j+1}(x) \simeq \delta_i^j(x)$ for $x \neq x_i$.

Case $j = 3\langle e, i, x\rangle + 1$, $i \leq k$. Check if there exists a finite part $\bar{\rho} \supseteq \bar{\delta}^j$ such that $\bar{\rho} \Vdash_i F_e(x)$. If so then let $\bar{\delta}^{j+1}$ be the least such ρ. Otherwise let $\bar{\delta}^{j+1} = \bar{\delta}^j$.

CASE $j = 3q + 2$. Consider the set

$$C = \{x : (\exists \bar{\tau} \supseteq \bar{\delta}^j)(\bar{\tau} \Vdash_k F_q(x))\}.$$

Clearly C is forcing k-definable on $\mathfrak{A}_0, \dots, \mathfrak{A}_k$ and hence $C \neq A$. Then there exists a x such that either $x \in A$ and $x \notin C$ or $x \in C$ and $x \notin A$. Take $\bar{\delta}^{j+1} = \bar{\delta}^j$ in the first case.

If the second case holds then there must exist a $\rho \supseteq \bar{\delta}^j$ such that $\rho \Vdash F_q(x)$. Let $\bar{\delta}^{j+1}$ be the least such ρ.

Let $\bar{\delta}^{j+1} = \bar{\delta}^j$ in the other cases.

To prove that the so received enumeration $\bar{f} = \cup_j \bar{\delta}^j$ is $k+1$-generic let us fix numbers $i \leq k$, $e, x \in \mathbb{N}$ and suppose that for every finite part $\bar{\tau} \subseteq \bar{f}$ there is an extention $\bar{\rho} \Vdash_i F_e(x)$. Then consider the step $j = 3\langle e, i, x\rangle + 1$. From the construction we have that $\bar{\delta}^{j+1} \Vdash_i F_e(x)$.

Suppose that there is an $q \in \mathbb{N}$, so that $A = \{x : \bar{f} \models_k F_q(x)\}$. Consider the step $j = 3q + 2$. From the construction there is a x such that one of the following two cases holds.

(a) $x \in A$ and $(\forall \bar{\rho} \supseteq \bar{\delta}^j)(\bar{\rho} \nVdash_k F_q(x))$. So, $\bar{\delta}^j \Vdash_k \neg F_q(x)$.

Since $\bar{f}$ is $k+1$-generic $x \in A$ & $\bar{f} \not\models_k F_q(x)$. A contradiction.

(b) $x \notin A$ & $\bar{\delta}^{j+1} \Vdash_k F_q(x)$. Since $\bar{f}$ is $k+1$-generic $\bar{f} \models_k F_q(x)$. A contradiction.

□

4. THE NORMAL FORM THEOREM

In this section we shall give an explicit form of the forcing k-definable on $\mathfrak{A}_0$, $\dots$, $\mathfrak{A}_k$ sets by means of *positive* recursive Σ^+_k formulae. These formulae can be considered as a modification of Ash's formulae introduced in [1].

4.1. **Recursive Σ^+_k formulae.** Let, for each $i \leq k$, $\mathcal{L}_i = \{T^i_1, \dots, T^i_{n_i}\}$ be the language of $\mathfrak{A}_i$, where every T^i_j is an r^i_j-ary predicate symbol, and $\mathcal{L} = \mathcal{L}_0 \cup \dots \cup \mathcal{L}_k$. We suppose that the languages $\mathcal{L}_0, \dots, \mathcal{L}_k$ are disjoint.

For each $i \leq k$ - fix a sequence $\mathbb{X}^i_0, \dots, \mathbb{X}^i_n, \dots$ of variables. The upper index i in the variable $\mathbb{X}^i_j$ shows that the possible values of $\mathbb{X}^i_j$ will be in $|\mathfrak{A}_i|$. By $\bar{X}^i$ we shall denote finite sequences of variables of the form $X^i_0, \dots, X^i_l$.

For each $i \leq k$, define the elementary Σ^+_i formulae and the Σ^+_i formulae by induction on i, as follows.

4.1. Definition.

(1) *An elementary Σ^+_0 formula* with free variables among $\bar{X}^0$ is an existential formula of the form:

$$\exists Y^0_1 \dots \exists Y^0_m \Phi(\bar{X}^0, Y^0_1, \dots, Y^0_m),$$

where Φ is a finite conjunction of atomic formulae in $\mathcal{L}_0$ with variables among $Y^0_1 \dots Y^0_m, \bar{X}^0$;

(2) *An elementary Σ^+_{i+1} formula* with free variables among $\bar{X}^0 \dots \bar{X}^{i+1}$ is in the form

$$\exists \bar{Y}^0 \dots \exists \bar{Y}^{i+1} \Phi(\bar{X}^0 \dots \bar{X}^{i+1}, \bar{Y}^0, \dots, \bar{Y}^{i+1})$$

where Φ is a finite conjunction of Σ^+_i formulae and negations of Σ^+_i formulae with free variables among $\bar{Y}^0 \dots \bar{Y}^i, \bar{X}^0 \dots \bar{X}^i$ and atoms of $\mathcal{L}_{i+1}$ with variables among $\bar{X}^{i+1}, \bar{Y}^{i+1}$;

(3) *A* Σ^+_i *formula* with free variables among $\bar{X}^0, \dots, \bar{X}^i$ is an r.e. infinitary disjunction of elementary Σ^+_i formulae with free variables among $\bar{X}^0, \dots, \bar{X}^i$.

Let Φ be a Σ^+_i formula $i \leq k$ with free variables among $\bar{X}^0, \dots, \bar{X}^i$ and let $\bar{t}^0, \dots, \bar{t}^i$ be elements of $\mathbb{N}$. Then by $(\mathfrak{A}_0, \dots, \mathfrak{A}_i) \models \Phi(\bar{X}^0/\bar{t}^0, \dots, \bar{X}^i/\bar{t}^i)$ we shall denote that Φ is true on $(\mathfrak{A}_0, \dots, \mathfrak{A}_i)$ under the variable assignment v such that $v(\bar{X}^0) = \bar{t}^0, \dots, v(\bar{X}^i) = \bar{t}^i$. More precisely we have the following:

4.2. Definition.

(1) If $\Phi = \exists Y^0_1 \dots \exists Y^0_m \Psi(\bar{X}^0, Y^0_1, \dots, Y^0_m)$ is a Σ^+_0 formula, then

$$(\mathfrak{A}_0) \models \Phi(\bar{X}^0/\bar{t}^0) \iff \exists s_1 \dots \exists s_m (\mathfrak{A}_0 \models \Psi(\bar{X}^0/\bar{t}^0, Y^0_1/s_1, \dots, Y^0_m/s_m)).$$

(2) If $\Phi = \exists \bar{Y}^0 \dots \exists \bar{Y}^{i+1} \Psi(\bar{X}^0, \dots, \bar{X}^{i+1}, \bar{Y}^0, \dots, \bar{Y}^{i+1})$, and $\Psi = (\varphi \ \&\ \alpha)$, where $\varphi(\bar{X}^0, \dots, \bar{X}^i, \bar{Y}^0, \dots, \bar{Y}^i)$ is a conjunction of Σ^+_i formulae and negations of Σ^+_i formulae and $\alpha(\bar{Y}^{i+1}, \bar{X}^{i+1})$ is a conjunction of atoms of $\mathcal{L}_{i+1}$, then

$$\begin{aligned}(\mathfrak{A}_0, \dots, \mathfrak{A}_{i+1}) \models \Phi(\bar{X}^0/\bar{t}^0, \dots, \bar{X}^{i+1}/\bar{t}^{i+1}) \iff \\ \exists \bar{s}^0 \dots \exists \bar{s}^{i+1} ((\mathfrak{A}_0, \dots, \mathfrak{A}_i) \models \varphi(\bar{X}^0/\bar{t}^0, \dots, \bar{X}^i/\bar{t}_i, \bar{Y}^0/\bar{s}^0, \dots, \bar{Y}^i/\bar{s}^i) \ \& \\ (\mathfrak{A}_{i+1}) \models \alpha(\bar{X}^{i+1}/\bar{t}^{i+1}, \bar{Y}^{i+1}/\bar{s}^{i+1})).\end{aligned}$$

4.2. **The formally k-definable sets.**

4.3. Definition. The set $A \subseteq \mathbb{N}$ is *formally k-definable* on $\mathfrak{A}_0, \dots, \mathfrak{A}_k$ if there exists a recursive sequence $\{\Phi\}^{\gamma(x)}$ of Σ^+_k formulae with free variables among $\bar{W}^0, \dots, \bar{W}^k$ and elements $\bar{t}^0, \dots, \bar{t}^k$ of $\mathbb{N}$ such that the following equivalence holds:

$$x \in A \iff (\mathfrak{A}_0 \dots \mathfrak{A}_k) \models \Phi^{\gamma(x)}(\bar{W}^0/\bar{t}^0 \dots \bar{W}^k/\bar{t}^k).$$

We shall show that every forcing k-definable set is formally k-definable.

Let for every i, $0 \leq i \leq k$, var_i be an effective bijective mapping of the natural numbers onto the variables with upper index i. Given a natural number x, by X^i we shall denote the variable $var_i(x)$.

Let $y_1 < y_2 < \dots < y_k$ be the elements of a finite set D, let Q be one of the quantifiers $\exists$ or $\forall$ an let Φ be an arbitrary formula. Then by $Q^i(y : y \in D)\Phi$ we shall denote the formula $QY^i_1 \dots QY^i_k \Phi$.

4.4. Proposition. *Let $\bar{E} = (E_0 \dots E_k)$ be a sequence of finite sets of natural numbers, where $E_j = \{w^j_0, \dots, w^j_{s_j}\}$. Let $i \leq k, x, e$ be elements of $\mathbb{N}$. There exists an uniform effective way to construct a Σ^+_i formula $\Phi^i_{\bar{E},e,x}$ with free variables among $\bar{W}^0, \dots, \bar{W}^k$, where $W^i_j = var(w^i_j)$, such that for every finite part $\bar{\delta} = (\delta_0 \dots \delta_k)$, $\mathrm{dom}(\delta_0) = E_0 \dots \mathrm{dom}(\delta_k) = E_k$*

$$(\mathfrak{A}_0, \dots, \mathfrak{A}_k) \models \Phi^i_{\bar{E},e,x}(\bar{W}^0/\delta_0(\bar{w}^0), \dots, \bar{W}^k/\delta_k(\bar{w}^k)) \iff \bar{\delta} \Vdash^*_i F_e(x).$$

Proof. We shall construct the formula $\Phi^i_{\bar{E},e,x}$ by induction on i following the definition of the forcing.

(1) Let $i = 0$. Let $V = \{v : \langle v, x \rangle \in W_e\}$. Consider an element v of V. For every $u \in D_v$ define the atom Π_u as follows

(a) If $u = \langle j, x^0_1, \dots, x^0_{r_j} \rangle$, where $1 \leq j \leq n_0$ and all $x^0_1, \dots, x^0_{r_j}$ are elements of E_0, then let $\Pi_u = T^0_j(X^0_1, \dots, X^0_{r_j})$.

(b) Let $\Pi_u = X^0_0 \neq X^0_0$ in the other cases.

Set $\Pi_v = \bigwedge_{u\in D_v} \Pi_u$ and $\Phi^0_{\bar{E},e,x} = \bigvee_{v\in V} \Pi_v$.

(2) Case $i+1$. Let $V = \{v : \langle v, x\rangle \in W_e\}$ and $v \in V$.

For every $u \in D_v$ define the formula Π_u as follows:

(a) If $u = \langle 0, e_u, x_u\rangle$, then let $\Pi_u = \Phi^i_{\bar{E},e_u,x_u}$.

(b) If $u = \langle 1, e_u, x_u\rangle$, then let

$$\Pi_u = \neg[\bigvee_{E_0^*\supseteq E_0\dots E_i^*\supseteq E_i} (\exists^0 y \in E_0^* \setminus E_0)\dots(\exists^i y \in E_i^* \setminus E_i)\Phi^i_{\bar{E}^*,e_u,x_u}],$$

where $\bar{E}^* = (E_0^*, \dots, E_i^*, E_{i+1}, \dots, E_k)$.

(c) If $u = \langle 2, x_u\rangle$, $x_u = \langle j, x_1^{i+1}, \dots, x_{r_j}^{i+1}\rangle$, $j \le n_{i+1}$ and $x_1^{i+1}, \dots, x_{r_j}^{i+1} \in E_{i+1}$ then let $\Pi_u = T_j^{i+1}(X_1^{i+1}, \dots, X_{r_j}^{i+1})$.

(d) Let $\Pi_u = \Phi^i_{\{\emptyset\},0,0} \wedge \neg\Phi^i_{\{\emptyset\},0,0}$ in the other cases.

Now let $\Pi_v = \bigwedge_{u\in D_v} \Pi_u$ and set $\Phi^{i+1}_{\bar{E},e,x} = \bigvee_{v\in V} \Pi_v$. An induction on i shows that for every i the Σ_i^+ formula $\Phi^i_{\bar{E},e,x}$ satisfies the requirements of the Proposition. □

4.5. Theorem. *Let $A \subseteq \mathbb{N}$ be forcing k-definable on $\mathfrak{A}_0, \dots, \mathfrak{A}_k$. Then A is formally k-definable on $\mathfrak{A}_0, \dots, \mathfrak{A}_k$.*

Proof. If A is forcing k-definable on $\mathfrak{A}_0, \dots, \mathfrak{A}_k$ then there exist a finite part $\bar{\delta} = (\delta_0, \dots, \delta_k)$ and $e \in \mathbb{N}$, such that

$$x \in A \iff (\exists \bar{\tau} \supseteq \bar{\delta})(\bar{\tau} \Vdash_k F_e(x)) \iff (\exists \bar{\tau} \supseteq \bar{\delta})(\bar{\tau} \Vdash_k^* F_e(x)).$$

Let for $i = 1, \dots, k$,$E_i = \mathrm{dom}(\delta_i) = \{w_1^i, \dots, w_r^i\}$ and let $\delta(w_j^i) = t_j^i$, $j = 1, \dots, r$. Set $\bar{E} = (E_0, \dots, E_k)$. From the previous Proposition we know that:

$$(\mathfrak{A}_0, \dots, \mathfrak{A}_k) \models \bigvee_{\bar{E}^*\supseteq\bar{E}} \exists(y \in \bar{E}^* \setminus \bar{E})\Phi^k_{E^*,e,x}(\bar{W}^0/\bar{t}_0, \dots, \bar{W}^k/\bar{t}_k) \iff$$
$$(\exists \bar{\tau} \supseteq \bar{\delta})(\mathrm{dom}(\bar{\tau}) = \bar{E}^*)(\bar{\tau} \Vdash_k^* F_e(x)).$$

Then for all $x \in \mathbb{N}$ the following equivalence is true:

$$x \in A \iff (\mathfrak{A}_0, \dots, \mathfrak{A}_k) \models \bigvee_{\bar{E}^*\supseteq\bar{E}} \exists(y \in \bar{E}^* \setminus \bar{E})\Phi^k_{E^*,e,x}(\bar{W}^0/\bar{t}_0, \dots, \bar{W}^k/\bar{t}_k).$$

From here we can conclude that A is formally k-definable on $\mathfrak{A}_0, \dots, \mathfrak{A}_k$. □

4.6. Theorem. *Let $A \subseteq \mathbb{N}$. Then the following are equivalent:*

(1) $d_e(A) \in CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$, $k \le n$.

(2) *For every enumeration $\bar{f}$ of $\mathfrak{A}_0, \dots, \mathfrak{A}_k$, $A \le_e \mathcal{P}_k(f_0^{-1}(\mathfrak{A}_0), \dots, f_k^{-1}(\mathfrak{A}_k))$.*

(3) *A is forcing k-definable on $\mathfrak{A}_0, \dots, \mathfrak{A}_k$.*

(4) *A is formally k-definable on $\mathfrak{A}_0, \dots, \mathfrak{A}_k$.*

Proof. The equivalence (1) $\iff$ (2) follows from the Theorem 2.6.

The implication (2) $\Rightarrow$ (3) follows from the Theorem 3.9.

The implication (3) $\Rightarrow$ (4) follows from the previous theorem.

The last implication (4) $\Rightarrow$ (2) follows by induction on i. □

References

[1] C. J. Ash, *Generalizations of enumeration reducibility using recursive infinitary propositional senetences*, Ann. Pure Appl. Logic **58** (1992), 173–184.

[2] C. J. Ash, C. Jockush, and J. F. Knight, *Jumps of orderings*, Trans. Amer. Math. Soc. **319** (1990), 573–599.

[3] S. B. Cooper, *Partial degrees and the density problem. Part 2: The enumeration degrees of the Σ_2 sets are dense*, J. Symbolic Logic **49** (1984), 503–513.

[4] R. G. Downey and J. F. Knight, *Orderings with α-th jump degree $\mathbf{0}^{(\alpha)}$*, Proc. Amer. Math. Soc. **114** (1992), 545–552.

[5] J. F. Knight, *Degrees coded in jumps of orderings*, J. Symbolic Logic **51** (1986), 1034–1042.

[6] L. J. Richter, *Degrees of structures*, J. Symbolic Logic **46** (1981), 723–731.

[7] I. N. Soskov, *Degree spectra and co-spectra of structures*, Submitted.

[8] ———, *A jump inversion theorem for the enumeration jump*, Arch. Math. Logic **39** (2000), 417–437.

Faculty of Mathematics and Computer Science, Sofia University, Blvd. "James Bourchier" 5, 1164 Sofia, Bulgaria

E-mail address: asoskova@fmi.uni-sofia.bg, soskov@fmi.uni-sofia.bg