ON COTOTALITY AND THE SKIP OPERATOR IN THE ENUMERATION DEGREES

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ABSTRACT. A set $A \subseteq \omega$ is *cototal* if it is enumeration reducible to its complement, $\overline{A}$. The *skip* of A is the uniform upper bound of the complements of all sets enumeration reducible to A. These are closely connected: A has cototal degree if and only if it is enumeration reducible to its skip. We study cototality and related properties, using the skip operator as a tool in our investigation. We give many examples of classes of enumeration degrees that either guarantee or prohibit cototality. We also study the skip for its own sake, noting that it has many of the nice properties of the Turing jump, even though the skip of A is not always above A (i.e., not all degrees are cototal). In fact, there is a set that is its own double skip.

1. Introduction

Enumeration reducibility was defined by Friedberg and Rogers in the late 1950's to capture a notion of reducibility between sets in which only positive information about membership in the set is either used or computed. This notion turns out to be as natural as Turing reducibility in a number of settings, e.g., in group theory and computable model theory.

For an arbitrary set $A \subseteq \omega$, the enumeration degree of A and the enumeration degree of $\overline{A}$, the complement of A, need not be comparable. By requiring that they are comparable, we can isolate two interesting subclasses of the enumeration degrees. The first was introduced at the same time as the enumeration degrees themselves. Call a set $A \subseteq \omega$ *total* if $\overline{A} \leq_e A$, and call an enumeration degree *total* if it contains a total set. Note that A is total if and only if $A \equiv_e A \oplus \overline{A}$, where $\oplus$ denotes the effective disjoint union of sets. Since every set of the form $A \oplus \overline{A}$ is total, the total degrees are exactly the degrees of sets $A \oplus \overline{A}$ for some $A \subseteq \omega$. In fact, the map $A \mapsto A \oplus \overline{A}$ induces an order-preserving isomorphism between the Turing degrees and the total enumeration degrees. The name "total" is due to the fact that an enumeration degree is total if and only if it contains the graph of a total function. In

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particular, if A is a total set, then $\deg_{\mathrm{e}}(A)$ contains the graph of the characteristic function of A.

It is important to note that total degrees[1] always contain nontotal sets as well. For example, all c.e. sets have total degree because they are all enumeration equivalent to the empty set, but only computable c.e. sets are total.

1.1. **Cototality.** What happens if we reverse the relationship between A and $\overline{A}$? Call a set $A \subseteq \omega$ *cototal* if $A \leq_{\mathrm{e}} \overline{A}$, and call an enumeration degree *cototal* if it contains a cototal set. While we are the first to isolate this property under this name, both the property and the name have appeared in the literature. The name was essentially first used, as far as we are aware, in an abstract of A.V. Pankratov from 2000 [18]. Pankratov used "кототальное" (Russian for "cototal") to refer to what *we* call the graph-cototal degrees, which turns out to be a proper subclass of the cototal degrees: For any total function $f\colon \omega \to \omega$, let $G_f = \{\langle n, m\rangle \colon f(n) = m\}$ be the graph of f. It is easy to see that $\overline{G_f} \leq_{\mathrm{e}} G_f$, so $\overline{G_f}$ is a cototal set. If an enumeration degree contains a set of the form $\overline{G_f}$, then we call it *graph-cototal*.

The graph-cototal sets and degrees were further studied by Solon, Pankratov's advisor. In [24], he used "co-total" to refer to what we call "graph-cototal". However, in the Russian version [23] of the same paper, Solon used "ко-тотальное" for a different property: Call a degree $\mathbf{a}$ *weakly cototal* if it contains a set A such that $\overline{A}$ has total enumeration degree. It is clear that every cototal degree is weakly cototal, since if $A \leq_{\mathrm{e}} \overline{A}$, then $\overline{A}$ is a total set. So we have

$$\text{graph-cototal} \implies \text{cototal} \implies \text{weakly cototal}.$$

We show that these three properties are distinct. The harder separation is given in Section 5, where we use an infinite-injury argument relative to $\mathbf{0}'$ to construct a cototal degree that is not graph-cototal. In Section 4, we give examples of weakly cototal degrees that are not cototal, as well as enumeration degrees that are not weakly cototal. Of these properties, we believe that there is a strong case that *cototal* is the most fundamental.

Our study of cototality was motivated by two examples of cototal sets that were pointed out to us by Jeandel [10]. He showed that the set of non-identity words in a finitely generated simple group is cototal (see also Thomas and Williams [26]). Jeandel also gave an example from symbolic dynamics: The set of words that appear in a minimal subshift is cototal. This is particularly interesting because the Turing degrees of elements of a minimal subshift are exactly the degrees that enumerate the set of words that appear in the subshift, so understanding the enumeration degree of this set is closely related to understanding the Turing degree spectrum of the subshift.

In Section 2, we explain Jeandel's examples in more detail, and we give several other examples of cototal sets and degrees. We show that every Σ^0_2-set is cototal, in fact, graph-cototal. We show that the complement of a maximal independent subset of a computable graph is cototal, and that every cototal degree contains the complement of a maximal independent subset of $\omega^{<\omega}$. Ethan McCarthy proved that the same is true of complements of maximal antichains in $\omega^{<\omega}$. We show that joins of nontrivial K-pairs are cototal, and that the natural embedding of the continuous degrees into the enumeration degrees maps into the cototal degrees. Finally, we note that Harris [9] proved that sets with a good approximation have cototal degree.

[1]We will sometimes use the term *degree* to refer to an enumeration degree.

The earliest reference to a cototality notion seems to be in Case's dissertation [4, p. 14] from 1969; he wrote "The author does not know if there are sets A such that A lies in a total partial degree and $\overline{A}$ lies in a non-total partial degree, but he conjectures that there are no such sets." In our language, Case is conjecturing that if $\overline{A}$ has weakly cototal degree, then it has total degree. The same question also appears in the journal version [5, p. 426]. Gutteridge [8, Chapter II] disproved this conjecture by constructing a quasiminimal graph-cototal degree. Recall that an enumeration degree $\mathbf{a}$ is *quasiminimal* if it is nonzero and the only total degree below $\mathbf{a}$ is $\mathbf{0}_\mathrm{e} = \deg_\mathrm{e}(\emptyset)$; in particular, quasiminimal degrees are nontotal. At least two other independent constructions of nontotal cototal degrees appear in the literature: Sanchis [20], apparently unaware of Case's conjecture, gave an explicit construction of a cototal set that is not total. Aware of Case's conjecture but not Gutteridge's example, Sorbi [25] constructed a quasiminimal cototal degree. Neither of these constructions explicitly produce a graph-cototal degree.

In the abstract mentioned above, Pankratov [18] claimed that there is a graph-cototal Σ^0_2-enumeration degree that forms a minimal pair with every incomplete Π^0_1-enumeration degree.[2] The graph-cototal degrees were studied more extensively by Solon [23, 24].[3] He proved that every total enumeration degree above $\overline{K}$ contains the graph G_f of a total function $f\colon \omega \to \omega$ such that $\overline{G_f}$ is quasiminimal. He also showed that for every total enumeration degree $\mathbf{b}$, there is a graph-cototal enumeration degree $\mathbf{a}$ that is quasiminimal over $\mathbf{b}$. Finally, Solon proved that for every total enumeration degree $\mathbf{b}$ above $\overline{K}$, there is a graph-cototal quasiminimal enumeration degree $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{b}$ (see below for more about the enumeration jump). This strengthens a result of McEvoy [15], who proved that the quasiminimal enumeration degrees have all possible enumeration jumps. Note that all three of Solon's results can also be seen as generalizations of Gutteridge's construction of a quasiminimal graph-cototal degree.

1.2. **The skip.** Cototality is closely related to the other main subject of this paper: the skip operator. Let $\{\Gamma_e\}_{e\in\omega}$ be an effective list of all enumeration operators and let $K_A = \bigoplus_{e\in\omega} \Gamma_e(A) = \{\langle e, x\rangle : x \in \Gamma_e(A)\}$. Note that $K_A \equiv_\mathrm{e} A$. We define the *skip* of A to be $A^\Diamond = \overline{K_A}$. It is easy to see that the skip is degree invariant, so it induces an operator on enumeration degrees. We use $\mathbf{a}^\Diamond$ to denote the skip of $\mathbf{a}$. Note that the complements of elements of $\deg_\mathrm{e}(A)$ are enumeration reducible to $A^\Diamond$; indeed, they are columns of $A^\Diamond$. In other words, $\deg_\mathrm{e}(A^\Diamond)$ is the maximum possible degree of the complement of an element of $\deg_\mathrm{e}(A)$. One consequence of this characterization is the connection between the skip and cototality:

Proposition 1.1. *A set $A \subseteq \omega$ has cototal degree if and only if $A \leq_\mathrm{e} A^\Diamond$.*

Proof. If A has cototal degree, then there is $B \equiv_\mathrm{e} A$ such that $B \leq_\mathrm{e} \overline{B}$. So $A \equiv_\mathrm{e} B \leq_\mathrm{e} \overline{B} \leq_\mathrm{e} A^\Diamond$. For the other direction, assume that $A \leq_\mathrm{e} A^\Diamond$. So $K_A \equiv_\mathrm{e} A \leq_\mathrm{e} A^\Diamond = \overline{K_A}$, hence A has cototal degree. □

This connection is quite useful; the separations we prove in Section 4 rely on our study of the skip operator in Section 3.

[2]This result does not appear to be published and we do not know the proof that Pankratov had in mind, but note that graph-cototality is free because every Σ^0_2-enumeration degree is graph-cototal.

[3]We note here a slight confusion in Solon's papers between cototal sets and cototal degrees, which does not, however, affect his main results.

In some ways, the skip is analogous to the jump operator in the Turing degrees. For example, a standard diagonalization argument shows that $A^\Diamond \not\leq_{\mathrm{e}} A$. In Proposition 3.1, we restate the well-known fact that $A \leq_{\mathrm{e}} B$ if and only if $A^\Diamond \leq_1 B^\Diamond$, mirroring the jump in the Turing degrees. Finally, in Theorem 3.3, we prove a skip inversion theorem analogous to Friedberg jump inversion: If $S \geq_{\mathrm{e}} \overline{K}$, then there is a set A such that $A^\Diamond \equiv_{\mathrm{e}} S$.

The biggest difference between the skip and the Turing jump is that it is not always the case that $A \leq_{\mathrm{e}} A^\Diamond$ (because not all enumeration degrees are cototal). In fact, as we will see in Section 3.2.3, there is a *skip* 2*-cycle*, i.e., a set $A \subseteq \omega$ such that $A = A^{\Diamond\Diamond}$. If we modify the skip to ensure that it is increasing in the enumeration degrees, then we recover the definition of the enumeration jump as introduced by Cooper[4] [6].

The *enumeration jump* of a set $A \subseteq \omega$ is $J_{\mathrm{e}}(A) = K_A \oplus \overline{K_A} \equiv_{\mathrm{e}} A \oplus A^\Diamond$. (We will also use A' to denote $J_{\mathrm{e}}(A)$). So A has cototal degree if and only if $J_{\mathrm{e}}(A) \equiv_{\mathrm{e}} A^\Diamond$. Of course, the enumeration jump is degree invariant and induces an operator on the enumeration degrees; we use $\mathbf{a}'$ for the jump of $\mathbf{a}$. The definition of the enumeration jump ensures that $A <_{\mathrm{e}} J_{\mathrm{e}}(A)$, as we expect from a jump. On the other hand, we lose two of the properties that the skip shares with the Turing jump. The enumeration jump is always total, so it cannot possibly map onto all enumeration degrees above $\mathbf{0}_{\mathrm{e}}'$. However, by Friedberg jump inversion, it does map onto the total degrees above $\mathbf{0}_{\mathrm{e}}'$. We will also see, in Proposition 3.23, that $J_{\mathrm{e}}(A) \leq_1 J_{\mathrm{e}}(B)$ does not necessarily imply that $A \leq_{\mathrm{e}} B$. So neither the skip nor the enumeration jump is the perfect analogue of the Turing jump; we believe that both have a role in the study of the enumeration degrees.

We assume that the reader is conversant with the main notions in computability theory and refer the reader to [17] for further information on the Turing degrees and the enumeration degrees.

2. Examples of cototal sets and degrees

2.1. **Total degrees.** For any set $A \subseteq \omega$, the set $A \oplus \overline{A}$ is clearly cototal. Therefore, every total degree is cototal.

2.2. **The complement of the graph of a total function.** As we have noted, if $f\colon \omega \to \omega$ is total, then $\overline{G}_f$, the complement of the graph of f, is a cototal set. This is because $\langle n, m\rangle \in \overline{G}_f$ if and only if there is $m' \neq m$ such that $\langle n, m'\rangle \in G_f$. The class of graph-cototal enumeration degrees turns out to lie strictly between the total degrees and the cototal degrees. The hard part will be showing that there is a cototal degree that is not graph-cototal. We will do that in Section 5. To see that every total degree is graph-cototal, recall that each total degree contains the graph of the characteristic function χ_A of some total set A; it also contains the complement of the graph of A. We already saw that $\overline{G}_{\chi_A} \leq_{\mathrm{e}} G_{\chi_A}$. But now since $\langle n, m\rangle \in G_{\chi_A}$ if and only if $m \in \{0, 1\}$ and $\langle n, 1-m\rangle \in \overline{G}_{\chi_A}$, we have that $\overline{G}_{\chi_A} \equiv_{\mathrm{e}} G_{\chi_A}$. The next result implies that there are nontotal graph-cototal degrees.

Proposition 2.1. *Every enumeration degree* $\mathbf{a} \leq \mathbf{0}_{\mathrm{e}}'$ *is graph-cototal.*

[4]Cooper [6] thanks his student McEvoy for helping provide the correct definition of the enumeration jump operator. Sorbi recalled (in private communication) that Cooper's original "incorrect" definition was actually our definition of the skip operator.

Proof. The enumeration degrees below $\mathbf{0}'_{\mathrm{e}}$ consist entirely of Σ^0_2-sets. So, fix an enumeration degree $\mathbf{a} \leq \mathbf{0}'_{\mathrm{e}}$ and a Σ^0_2-set $A \in \mathbf{a}$. We must show that there is a set $\overline{G} \equiv_{\mathrm{e}} A$ that is the complement of the graph G of a total function. We can think of $\overline{G}$ as an infinite table such that each column contains all but one element.

Fix a Σ^0_2-approximation $\{A_s\}_{s<\omega}$ to the set A. This is a uniformly computable sequence of finite sets such that $a \in A$ if and only if $a \in A_s$ for all but finitely many s. So, to every $a \in A$ we can associate the first stage s_a such that $a \in A_s$ for every $s \geq s_a$. We may assume that $A_0 = \emptyset$. Consider the set

$$U = \{\langle a, s\rangle : s \neq s_a\} = \{\langle a, s\rangle : a \in A_{s-1} \vee (\exists t \geq s)[a \notin A_t]\}.$$

Note that U is a c.e. table such that the a-th column of U contains all natural numbers if $a \notin A$ and all but one natural number if $a \in A$. We combine U and A to define the set $\overline{G}$:

$$\langle a, m\rangle \in \overline{G} \text{ if and only if } m = 0 \,\&\, a \in A \vee m > 0 \,\&\, \langle a, m-1\rangle \in U.$$

The set $\overline{G}$ is clearly in the degree $\mathbf{a}$ and is the complement of the graph of the total function $g : \omega \to \omega$ such that $g(a) = s_a + 1$ if $a \in A$ and $g(a) = 0$ if $a \notin A$. □

It is worth pointing out that the argument above cannot be extended to further levels of the arithmetical hierarchy. In Section 4, we will show that there are Π^0_2-sets that do not even have cototal enumeration degree. Another way to see this is to use a theorem of Badillo and Harris [2] proving the existence of a Π^0_2-enumeration degree that contains only properly Π^0_2-sets. Such a degree must have skip equal to $\mathbf{0}'_{\mathrm{e}}$ and hence cannot be cototal. On the other hand, it is easy to see that every Π^0_2-set has weakly cototal degree. This is because every set A is enumeration equivalent to $A \oplus K$, where K is the halting set. So, if A is Π^0_2 then $\overline{A \oplus K} = \overline{A} \oplus \overline{K} \equiv_{\mathrm{e}} \overline{K} \in \mathbf{0}'_{\mathrm{e}}$. As for higher levels of the arithmetical hierarchy, we will see in Section 4 that there are Δ^0_3-sets that are not even weakly cototal.

Let $\overline{G}$ be the complement of the graph G of a total function. Notice that the reduction Γ witnessing that $\overline{G} \leq_{\mathrm{e}} G$ described above has the following interesting feature: If $x \in \overline{G}$, then there is a unique axiom in Γ that enumerates x into $\Gamma(G)$. We say that $\overline{G}$ reduces to G via *a unique axiom reduction.* We will next see that this property characterizes the graph-cototal enumeration degrees among all cototal enumeration degrees.

Proposition 2.2 (Unique Axiom Characterization)**.** *An enumeration degree* $\mathbf{a}$ *is graph-cototal if and only if it contains a cototal set A that reduces to $\overline{A}$ via a unique axiom reduction.*

Proof. We have already seen that graph-cototal degrees have this property. For the reverse direction, let $\mathbf{a}$ be an enumeration degree and let $A \in \mathbf{a}$ be a cototal set that reduces to $\overline{A}$ via a unique axiom reduction Γ. We will, in this case as well, construct an infinite table $\overline{G}$, the first row of which will contain only elements in columns corresponding to members of A. For the remaining rows, we will use the c.e. set Γ. Note that if $\langle a, D\rangle \in \Gamma$ and $a \notin A$, then D must contain an element of A, and if $a \in A$, then there is a unique axiom $\langle a, D\rangle$ such that $D \cap A = \emptyset$. Intuitively, the idea is to assign the axioms of Γ to the remaining undecided elements in each column and enumerate into $\overline{G}$ an element in the a-th column unless it corresponds to the unique correct axiom for a. We formalize this idea below.

Fix a computable function s that lists Γ without repetitions. Without loss of generality, we may assume that Γ is infinite, as a finite unique axiom reduction can enumerate only a finite set and we already know that $\mathbf{0}_{\mathrm{e}}$ is graph-cototal. We define the set $\overline{G}$ as follows:

$$\begin{aligned}\langle a, m\rangle \in \overline{G} \text{ if and only if } & [m = 0 \ \&\ a \in A] \\ \text{or } & \Big[m > 0 \ \&\ \big[s(m-1) \text{ is not an axiom for } a \\ & \qquad \text{or } (s(m-1) = \langle a, D\rangle \ \&\ D \cap A \neq \emptyset)\big]\Big].\end{aligned}$$

The set $\overline{G}$ is clearly in the degree $\mathbf{a}$ and is the complement of the graph of the total function $g\colon \omega \to \omega$ such that $g(a) = d + 1$, where d codes the unique correct axiom for a if $a \in A$, and $g(a) = 0$ if $a \notin A$. $\square$

We can make this characterization even tighter by noting that the reduction Γ used to witness that $\overline{G} \leq_{\mathrm{e}} G$ is furthermore a *singleton operator*: every axiom in Γ is of the form $\langle a, \{b\}\rangle$ where $a \neq b$.

We will therefore be interested in finding examples of cototal enumeration degrees that do not satisfy the Unique Axiom Characterization, as we would like to separate the cototal degrees from the graph-cototal degrees. The next example, which comes from graph theory, is motivated by this desire.

2.3. **Complements of maximal independent sets.** Recall that an (undirected) graph is a pair $G = (V, E)$, where V is a set of vertices and E is a set of unordered pairs of vertices, called the edge relation.

Definition 2.3. An *independent set* for a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that no pair of distinct vertices in S is connected by an edge. An independent set is *maximal* if it has no proper independent superset.

In other words, an independent set S is maximal if and only if every vertex $v \in V$ is either in S or is connected by an edge to an element of S. The maximal independent sets for the graph of the cube are illustrated in the figure below, courtesy of David Eppstein and Wikipedia.

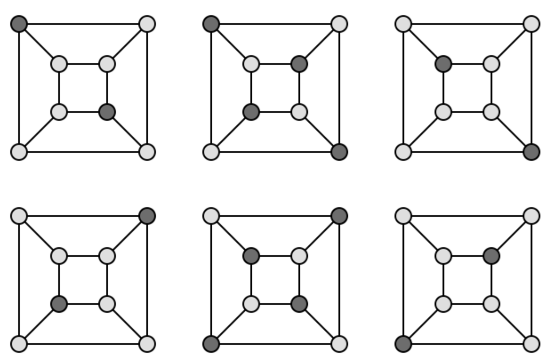

FIGURE 1. Maximal independent sets for the cube

Consider an infinite graph $G = (\omega, E)$ with a computable edge relation. For example, we can think of the tree $\omega^{<\omega}$ as a computable graph on the natural numbers by fixing an effective coding of the finite sequences of natural numbers and putting an edge between any non-root node and its immediate predecessor. If S is a maximal independent set for G, then S can enumerate its complement:

$$\overline{S} = \{v\colon (\exists u \in S)[\{u,v\} \in E]\}.$$

It follows that complements of maximal independent sets in computable graphs on ω are cototal. Our main reason for considering this example is that, in general, this reduction does not have the unique axiom property. This is well illustrated by Figure 1: the maximal independent set in the middle of the first row, for example, would enumerate each element of its complement with three distinct correct axioms. Hence we might hope that complements of maximal independent sets allow us to move beyond the graph-cototal degrees. They do, and in fact, they are universal for the cototal enumeration degrees.

Theorem 2.4. *Every cototal degree contains the complement of a maximal independent set for $\omega^{<\omega}$.*

Proof. Fix a cototal set A and let $A = \Gamma(\overline{A})$. We will build a set $G \subseteq \omega^{<\omega}$ which will be the complement of a maximal independent set for $\omega^{<\omega}$. In this case, we will again assume that A is not c.e. and so Γ is an infinite c.e. set, as there are easy examples of computable maximal independent sets, e.g., the set of all odd-length strings in $\omega^{<\omega}$. So let g be a computable listing of Γ without repetitions. We will further assume that no axiom in Γ is of the form $\langle a, \emptyset\rangle$. We can easily replace Γ with an operator that fits this description by replacing every such axiom by $\langle a, \{b_0\}\rangle$, where b_0 is some fixed member of $\overline{A}$. We will also fix a number $a_0 \in A$.

To every node $\sigma \in \omega^{<\omega}$ other than the root $\langle\rangle$, we will computably assign a finite set D_σ. The set G will then be defined as

$$G = \{\sigma\colon D_\sigma \cap A \neq \emptyset\} \cup \{\langle\rangle\}.$$

The assignment is defined by induction:

1. If $\sigma = n$ is a length-1 string then $D_\sigma = \{n\}$.
2. If $\sigma = \tau n$. Then we have two cases:
   (a) If $g(n)$ is not an axiom for any member of D_τ then we let $D_\sigma = \{a_0\}$.
   (b) If $g(n) = \langle a, D\rangle$ is an axiom for $a \in D_\tau$ then we let $D_\sigma = D$.

From the definition, it follows that $G \leq_{\mathrm{e}} A$, and from part 1 in particular, that also $A \leq_{\mathrm{e}} G$, as $n \in G$ if and only if $\{n\} \cap A \neq \emptyset$ if and only if $n \in A$. It remains to be shown that $\overline{G}$ is a maximal independent set.

Fix $\tau, \sigma \in \omega^{<\omega}$ such that $\sigma = \tau n$. We must show that either $\sigma \in G$ or $\tau \in G$ to ensure that $\overline{G}$ is independent. If $\tau \notin G$ then $\tau \neq \langle\rangle$ and $D_\tau \subseteq \overline{A}$. If $g(n)$ is not an axiom for any element in D_τ, then $D_\sigma = \{a_0\} \subseteq A$ and hence $\sigma \in G$. Otherwise $g(n) = \langle a, D_\sigma\rangle$ and $a \notin A$. As $A = \Gamma(\overline{A})$ it must be that $D_\sigma \nsubseteq \overline{A}$ and so $D_\sigma \cap A \neq \emptyset$, hence $\sigma \in G$.

Finally, we must show that every $\tau \in G$ has a neighbor σ in $\overline{G}$ to ensure that $\overline{G}$ is maximal. If $\tau = \langle\rangle$, then σ can be chosen as any of its length-1 neighbors corresponding to elements $b \in \overline{A}$. Suppose that $\tau \neq \langle\rangle$ and let $a \in D_\tau \cap A$. Then $a \in \Gamma(\overline{A})$ and hence there is an axiom $\langle a, D\rangle \in \Gamma$ such that $D \subseteq \overline{A}$. Fix n such that $\langle a, D\rangle = g(n)$. We assign the set D to the string $\sigma = \tau n$; it follows that $\sigma \notin G$. $\square$

Note, that the proof above holds even if we restrict ourselves to *singleton degrees*, the degree structure induced by restricting reductions to singleton operators. The singleton degree of a set that is cototal with respect to singleton reduction contains the complement of a maximal independent set for $\omega^{<\omega}$.

2.4. **Complements of maximal antichains in $\omega^{<\omega}$.** A closely related example comes from simply considering maximal antichains in $\omega^{<\omega}$. In this case, the partial ordering on finite sequences of natural numbers is defined by $\sigma \leq \tau$ if and only if $\sigma \preceq \tau$. An *antichain* is a subset of $\omega^{<\omega}$ such that no two elements in it are comparable, and an antichain is *maximal* if it cannot be extended to a proper superset that is also an antichain. Examples of computable maximal antichains are easy to come up with: For any fixed n, the set of all elements of $\omega^{<\omega}$ of length n is a maximal antichain.

If S is a maximal antichain, then $\overline{S} \leq_{\mathrm{e}} S$ as $\sigma \in \overline{S}$ if an only if there is some $\tau \in S$ that is comparable with σ. As in the example above, this reduction does not have the unique axiom property. Consider for example the maximal antichain of all strings of length n. Then every string of length $m < n$ has infinitely many reasons to be enumerated into the complement of this maximal antichain. Ethan McCarthy has shown that complements of maximal antichains are also universal for the cototal enumeration degrees.

Theorem 2.5 (McCarthy [14])**.** *Every cototal degree contains the complement of a maximal antichain in $\omega^{<\omega}$.*

2.5. **The set of words that appear in a minimal subshift.** We will next give a more detailed account of our motivating examples, introduced by Jeandel [10]. The first one requires us to recall some definitions from symbolic dynamics.

Definition 2.6. Let $X \subseteq 2^{\omega}$ be closed in the usual topology on Cantor space.

(a) X is a *subshift* if X is closed under the shift operation, which removes the first bit in a binary sequence, i.e., $a\alpha \in X$ implies $\alpha \in X$.
(b) If X is a subshift then the *language of* X is the set
$$\mathcal{L}_X = \{\sigma \in 2^{<\omega} \ : \ (\exists \alpha \in X)[\sigma \text{ is a subword of } \alpha]\}.$$
The set $\overline{\mathcal{L}_X}$ is called the *set of forbidden words.*
(c) A subshift X is *minimal* if it has no nonempty proper subset that is also a subshift. This is equivalent to saying that every $\sigma \in \mathcal{L}_X$ is a subword of every $\alpha \in X$.

Jeandel discovered an interesting relationship between the enumeration degree of the language of a minimal subshift and the Turing degrees of the elements of the subshift: The Turing degrees of elements in X are exactly the Turing degrees that enumerate $\mathcal{L}_X$. This fact is particularly interesting if one takes into account Selman's characterization of enumeration reducibility. For an arbitrary set A, let $\mathcal{E}_A$ denote the set of all Turing degrees whose elements compute enumerations of A. Selman [21] proved that $A \leq_{\mathrm{e}} B$ if and only if $\mathcal{E}_B \subseteq \mathcal{E}_A$. Thus, the enumeration degree of the set $\mathcal{L}_X$ can be characterized by $\mathcal{E}_{\mathcal{L}_X}$, which turns out to be exactly the set of Turing degrees that compute elements of the minimal subshift X. It is then natural to ask what additional properties an enumeration degree must have in order to be the enumeration degree of the language of a minimal subshift. The following theorem shows that it must be cototal.

Theorem 2.7 (Jeandel [10])**.** $\mathcal{L}_X \leq_{\mathrm{e}} \overline{\mathcal{L}_X}$.

Ethan McCarthy has very recently shown that, in fact, cototality precisely characterizes the enumeration degrees of languages of minimal subshifts.

Theorem 2.8 (McCarthy [14])**.** *If A is cototal, then $A \equiv_{\mathrm{e}} \mathcal{L}_X$ for some minimal subshift X.*

2.6. **The non-identity words in a finitely generated simple group.** The second example from Jeandel [10] is related to group theory.

Definition 2.9. Let G be a group.

(a) G is *finitely generated* if there are finitely many elements in G, called generators, such that every element in G can be expressed as a product of these generators. (For convenience, we will assume that the set of generators is closed under inverses.)
(b) G is *simple* if its only normal subgroups are G and the trivial group.
(c) The set of *identity words* of G is the set $\mathcal{W}_G$ of all words (i.e., finite sequences of generators) that represent the identity element.
(d) A *presentation* of G is a pair $\langle F \mid R\rangle$ such that F is a set of generators and $\mathcal{W}_G$ is the normal closure of $R \subset \mathcal{W}_G$.

The word problem for a group G is the problem of deciding the set $\mathcal{W}_G$. Kuznetsov [12] showed that if G is a finitely generated simple group with a presentation $\langle F \mid R\rangle$ such that R is computable, then it has a decidable word problem. Jeandel considered the collection of all finitely generated simple groups without restricting the complexity of their presentation. He showed that the set of non-identity words in a finitely generated simple group is cototal. This was also independently observed by Thomas and Williams [26].

Theorem 2.10 (Jeandel [10]; Thomas and Williams [26])**.** *If G is a finitely generated simple group then $\overline{\mathcal{W}_G} \leq_{\mathrm{e}} \mathcal{W}_G$.*

This generalizes Kuznetsov's result, as if a group $G = \langle F \mid R\rangle$ has a computable set of relations R, then $\mathcal{W}_G$ is automatically c.e. The fact that $\overline{\mathcal{W}_G} \leq_{\mathrm{e}} \mathcal{W}_G$ shows that $\overline{\mathcal{W}_G}$ is also c.e. and hence $\mathcal{W}_G$ is computable.

2.7. **Joins of nontrivial $\mathcal{K}$-pairs.** Our next example relates to a class of pairs of enumeration degrees that have been recently shown to play an important role when it comes to the first-order definability of relations on $\mathcal{D}_{\mathrm{e}}$.

Definition 2.11. A pairs of sets $\{A, B\}$ form a *$\mathcal{K}$-pair* if there is a c.e. set W such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$. A $\mathcal{K}$-pair is *nontrivial* if neither of its components is c.e.

$\mathcal{K}$-pairs were introduced by Kalimullin [11]. He showed that they are first-order definable in the structure of the enumeration degrees and used them to give a first-order definition of the enumeration jump. Cai, Ganchev, Lempp, Miller, and M. Soskova [3] used $\mathcal{K}$-pairs to define the class of total enumeration degrees. It is therefore reasonable to always keep an eye on the class of $\mathcal{K}$-pairs as it might hold the key to the first-order definability of relations that we are considering in this article as well: cototal enumeration degrees and the skip operator. In the next section, $\mathcal{K}$-pairs will give us a wide variety of examples of sets that do not have cototal degree. When one considers the join $A \oplus B$, however, of a nontrivial $\mathcal{K}$-pair $\{A, B\}$, one always gets a cototal set. To see this, we will need to review an important property of $\mathcal{K}$-pairs.

Proposition 2.12 (Kalimullin [11]). *If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair then*

- $A \leq_{\mathrm{e}} \overline{B}$ *and* $B \leq_{\mathrm{e}} \overline{A}$*;*
- $\overline{B} \leq_{\mathrm{e}} A \oplus \overline{K}$ *and* $\overline{A} \leq_{\mathrm{e}} B \oplus \overline{K}$.

It follows from the first part that if $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair, then $A \oplus B \leq_{\mathrm{e}} \overline{B} \oplus \overline{A} \equiv_{\mathrm{e}} \overline{A \oplus B}$.

We would like to point out that this example generalizes the fact that every total degree is cototal, as by Cai, Ganchev, Lempp, Miller, and M. Soskova [3], the total degrees are exactly the ones that contain the join of a particular kind of a $\mathcal{K}$-pair. The joins of nontrivial $\mathcal{K}$-pairs therefore form a first-order definable class of cototal enumeration degrees that contains the total enumeration degrees. Unfortunately, they do not contain all cototal degrees. Ahmad [1] showed that there are nonsplitting Σ^0_2-enumeration degrees, i.e. degrees that are not the least upper bound of any pair of strictly smaller degrees. So, even though, as we have already seen, all Σ^0_2-enumeration degrees are cototal, the nonsplitting ones cannot be joins of nontrivial $\mathcal{K}$-pairs.

2.8. **Continuous degrees.** Motivated by a question of Pour-El and Lempp from computable analysis, Miller [16] introduced a degree structure that captures the complexity of elements of computable metric spaces, such as $\mathcal{C}[0,1]$ and $[0,1]^\omega$. This structure naturally embeds into the enumeration degrees, and the range of this embedding is strictly between the class of total enumeration degrees and the class of all enumeration degrees.

As an example, consider the metric space $\mathcal{C}[0,1]$ of continuous functions on the unit interval with the standard metric

$$d(f, g) = \max_{x \in [0,1]} |f(x) - g(x)|.$$

A computable presentation of a metric space $\mathcal{M}$ consists of a fixed dense sequence $Q^{\mathcal{M}} = \{q_n\}_{n<\omega}$ on which the metric is computable as a function on indices. For a computable presentation of $\mathcal{C}[0,1]$ we can fix, for example, a reasonable enumeration of the polygonal functions having segments with rational endpoints. A *name* n_f for a continuous function f is a code (say, as an element of ω^ω) that gives a way to approximate f. Specifically, a name n_f should code a function taking a rational number $\varepsilon > 0$ and producing an index $n_f(\varepsilon)$ such that $d(f, q_{n_f(\varepsilon)}) < \varepsilon$. For $f, g \in \mathcal{C}[0,1]$, we say that f is *reducible* to g if every name for g computes a name for f. In the same way, we can compare the complexity of elements from arbitrary metric spaces. This reducibility induces a degree structure, the *continuous degrees*. It turns out that every continuous degree contains an element of $\mathcal{C}[0,1]$.

In order to understand the embedding of the continuous degrees into the enumeration degrees, it is easier to focus on another computable metric space: The *Hilbert cube* is $[0,1]^\omega$ along with the metric

$$d(\alpha, \beta) = \sum_{n \in \omega} 2^{-n} |\alpha(n) - \beta(n)|.$$

A dense set witnessing that $[0,1]^\omega$ is computable is, for example, a reasonable enumeration of the rational sequences with finite support. As was the case with $\mathcal{C}[0,1]$, every continuous degree contains an element of $[0,1]^\omega$.

Miller gave a way to assign to a sequence $\alpha \in [0,1]^\omega$ a set A_α such that $\mathcal{E}_{A_\alpha}$ (defined in Section 2.5) is the set of all Turing degrees that compute names of α. This induces an embedding of the continuous degrees into the enumeration degrees.

Definition 2.13 (Miller [16])**.** For $\alpha \in [0,1]^\omega$, let

$$A_\alpha = \bigoplus_{i<\omega}(\{q \in \mathbb{Q} \colon q <_{\mathbb{Q}} \alpha(i)\} \oplus \{q \in \mathbb{Q} \colon q >_{\mathbb{Q}} \alpha(i)\}).$$

It is not hard to see that A_α has the desired property: Computing a name for α is exactly as hard as enumerating A_α. We say that the enumeration degree of A_α is *continuous*. By showing that there is a nontotal continuous enumeration degree, Miller proved that there are continuous functions that do not have a name of least Turing degree, which answered Pour-El and Lempp's question.

Note that if α does not have any rational entries, then A_α is a total set. If, on the other hand, α does have rational entries, then every component of A_α is nonuniformly equivalent to a total set. The existence of nontotal continuous enumeration degrees shows that this nonuniformity is significant. We are nevertheless able to show that all continuous degrees are cototal.

Proposition 2.14. *Every continuous degree is cototal.*

Proof. Let $\alpha \in [0,1]^\omega$ and $A_\alpha = \bigoplus_{i<\omega}(\{q \in \mathbb{Q} \colon q <_{\mathbb{Q}} \alpha(i)\} \oplus \{q \in \mathbb{Q} \colon q >_{\mathbb{Q}} \alpha(i)\})$. By rearranging the odd and even elements in every column of $\overline{A}_\alpha$, we obtain the set $B_\alpha \equiv_{\mathrm{e}} \overline{A}_\alpha$ defined by

$$B_\alpha = \bigoplus_{i<\omega}(\{q \in \mathbb{Q} \colon q \leq_{\mathbb{Q}} \alpha(i)\} \oplus \{q \in \mathbb{Q} \colon q \geq_{\mathbb{Q}} \alpha(i)\}).$$

It is now easy to see that q is a member of the i-th even column of A_α if and only if there is an $r >_{\mathbb{Q}} q$ such that r is in the i-th even column of B_α. Similarly, q is a member of the i-th odd column of A_α if and only if there is an $r <_{\mathbb{Q}} q$ such that r is in the i-th odd column of B_α. It follows that $A_\alpha \leq_{\mathrm{e}} B_\alpha \equiv_{\mathrm{e}} \overline{A}_\alpha$. □

2.9. **Sets with good approximations have cototal degree.** Lachlan and Shore [13] introduced the following general notion of an approximation to a set.

Definition 2.15. Let A be a set of natural numbers. A uniformly computable sequence of finite sets $\{A_s\}_{s<\omega}$ (given by canonical indices) is a *good approximation to A* if

- for every n, there is a stage s such that $A \upharpoonright n \subseteq A_s \subseteq A$; and
- for every n, there is a stage s such that for every $t > s$, if $A_t \subseteq A$ then $A \upharpoonright n \subseteq A_t$.

This definition can be seen as a generalization of Cooper's notion of a Σ^0_2-approximation with infinitely many thin stages, used to show the density of the Σ^0_2-enumeration degrees [6]. Lachlan and Shore [13] introduced the hierarchy of the n-c.e.a. sets. A set is 1-c.e.a. if it is c.e., and $(n+1)$-c.e.a. if it is the join of an n-c.e.a. set X and a set Y c.e. in X. It is not difficult to see that the enumeration degrees of the 2-c.e.a. sets are exactly the Σ^0_2-enumeration degrees. Lachlan and Shore proved that every set that is n-c.e.a. has a good approximation and then showed that the enumeration degrees of the n-c.e.a. sets are dense. Harris [9] proved that sets that have good approximations always have cototal enumeration degrees. We outline his proof below for completeness.

Proposition 2.16 (Harris [9, Proposition 4.1])**.** *If A has a good approximation, then $K_A \leq_{\mathrm{e}} \overline{K_A}$.*

Proof. Let $\{A_s\}_{s<\omega}$ be a good approximation to A. Consider the set C defined by

$$C = \{\langle x, s\rangle \colon (\exists t > s)[A_t \subseteq A \ \&\ x \notin A_t]\}.$$

It follows from the definition that $C \leq_{\mathrm{e}} A$. Using the fact that $\overline{K}_A = \bigoplus_{e<\omega} \overline{\Gamma_e(A)}$ is a uniform upper bound of the set of complements of all sets that are enumeration reducible to A, we obtain that $\overline{C} \leq_{\mathrm{e}} \overline{K_A}$. Now, let us take a closer look at $\overline{C}$:

$$\overline{C} = \{\langle x, s\rangle \colon (\forall t > s)[A_t \subseteq A \to x \in A_t]\}.$$

Using the second property of good approximations, notice that $x \in A$ if and only if there is a stage s such that $\langle x, s\rangle \in \overline{C}$. It follows that $A \leq_{\mathrm{e}} \overline{C}$. This now gives us that $K_A \equiv_{\mathrm{e}} A \leq_{\mathrm{e}} \overline{C} \leq_{\mathrm{e}} \overline{K_A}$. □

In particular, we obtain that the enumeration degrees of n-c.e.a. sets are cototal.

3. The skip

In the previous section, we saw many examples of cototal sets and enumeration degrees. In this section, we study the skip operator, in part to provide a wide variety of examples of degrees that are not cototal. Recall that the *skip* of a set $A \subseteq \omega$ is $A^\diamond = \overline{K_A}$. As we saw in the introduction, the skip gives us an easy way to determine whether or not a degree is cototal. For the reader's convenience, we restate that result:

Proposition 1.1. *A set $A \subseteq \omega$ has cototal degree if and only if $A \leq_{\mathrm{e}} A^\diamond$.*

In addition to being a tool in our study of cototality, the skip is a natural operator in its own right. As we discussed in the introduction, the enumeration jump fails to have some of the nice properties of the Turing jump. For example, it is well-known that $A \leq_T B$ if and only if $K^A \leq_1 K^B$, where K^A denotes the halting set relative to A. The analogous property does not hold, in general, for the enumeration jump. It is true that $A \leq_{\mathrm{e}} B$ implies $K_A \oplus \overline{K}_A \leq_1 K_B \oplus \overline{K}_B$, but the reverse implication can fail; we will see an example after Proposition 3.22. The skip, on the other hand, gives us an embedding of the enumeration degrees into the 1-degrees.

Proposition 3.1. *$A \leq_{\mathrm{e}} B$ if and only if $A^\diamond \leq_1 B^\diamond$.*

Proof. If $A \leq_{\mathrm{e}} B$, then $K_A \leq_{\mathrm{e}} B$ and hence K_A is a fixed column of $K_B = \bigoplus_{e<\omega} \Gamma_e(B)$, where $\{\Gamma_e\}_{e\in\omega}$ is the standard listing of all enumeration operators. It follows that $\overline{K}_A$ is a fixed column in $\overline{K}_B$ and hence $\overline{K}_A \leq_1 \overline{K}_B$.

If $\overline{K}_A \leq_1 \overline{K}_B$ then $K_A \leq_1 K_B$ and hence $A \equiv_{\mathrm{e}} K_A \leq_{\mathrm{e}} K_B \equiv_{\mathrm{e}} B$. □

This shows that we can define the skip operator on degrees.

Definition 3.2. The *skip* of the enumeration degree $\mathbf{a}$ is $\mathbf{a}^\diamond = \deg_{\mathrm{e}}(A^\diamond)$ for any member $A \in \mathbf{a}$.

3.1. **Skip inversion.** It follows from Proposition 1.1 that an enumeration degree $\mathbf{a}$ is cototal if and only if $\mathbf{a} \leq \mathbf{a}^\Diamond$, if and only if $\mathbf{a}^\Diamond = \mathbf{a}'$. The definition of the enumeration jump operator restricts its range to the total enumeration degrees and by monotonicity to the total enumeration degrees in the cone above $\mathbf{0}'_\mathrm{e}$. By transferring the Friedberg Jump Inversion Theorem through the standard embedding into the enumeration degrees, we see that every total enumeration degree above $\mathbf{0}'_\mathrm{e}$ is in the range of the jump operator. The range of the skip operator is also restricted by monotonicity to enumeration degrees above $\mathbf{0}^\Diamond_\mathrm{e} = \mathbf{0}'_\mathrm{e}$. We show that this is the only restriction on the range of the skip operator, thereby providing a further analogy between the skip and the Turing jump. Recall that $\overline{K}$, the complement of the halting set, is a representative of the degree $\mathbf{0}'_\mathrm{e}$.

Theorem 3.3. *For any set $S \geq_\mathrm{e} \overline{K}$, there is a set A such that $A^\Diamond \equiv_\mathrm{e} S$. (In fact, we also have $S \equiv_\mathrm{e} \overline{A} \equiv_\mathrm{e} \overline{A} \oplus \overline{K}$ and $\overline{S} \leq_\mathrm{e} A \oplus \overline{K}$.)*

Proof. Given a set $S \geq_\mathrm{e} \overline{K}$, we build a set A such that $S \equiv_\mathrm{e} \overline{A} \leq_\mathrm{e} A^\Diamond \leq_\mathrm{e} \overline{A} \oplus \overline{K}$. For a set $X \subseteq \omega$ and a natural number e, let $X^{[e]} = \{\langle e, x\rangle \colon x \in \omega\} \cap X$. We will build A meeting two types of requirements:

$$\mathcal{R}_e \colon e \in S \iff A^{[e]} \neq \omega^{[e]},$$
$$\mathcal{P}_e \colon \text{"force } e \text{ into } K_A \text{ subject to higher-priority restraints".}$$

The $\mathcal{R}_e$-requirements ensure that $S \leq_\mathrm{e} \overline{A}$, as $e \in S$ if and only if there is an $x \in \omega$ such that $\langle e, x\rangle \in \overline{A}$. The basic strategy for $\mathcal{R}_e$ is quite simple: If $e \notin S$ then enumerate all of $\omega^{[e]}$ into A. Otherwise, withhold one number $a_e \in \omega^{[e]}$ from A and enumerate $\omega^{[e]} \smallsetminus \{a_e\}$ into A.

The $\mathcal{P}_e$-requirements will let us prove that $A^\Diamond$ can be enumerated from $\overline{K}$ and S. The basic strategy for $\mathcal{P}_e$ is to try to force e into K_A by adding a finite set to the current version of A so that $e \notin K_A$ can only be caused by the finitely many numbers a_i that higher priority $\mathcal{R}$-requirements use for coding the values of S. We will use the 1-equivalent form of the set K_A, namely, $\{e \colon e \in \Gamma_e(A)\}$, where $\{\Gamma_e\}_{e\in\omega}$ is our fixed listing of all enumeration operators.

We now proceed in stages as follows:

Stage 0: Set $a_e = \langle e, 0\rangle$ for all $e \in \omega$, and set $A_0 = \emptyset$.

Stage $s = e + 1$: For each subset $D \subseteq \{i \colon i < e\}$, check if there are a finite subset $F_D \subseteq \omega \smallsetminus \{a_i \colon 0 \leq i < e\}$ and a stage t such that $e \in \Gamma_{e,t}(F_D \cup \{a_i \colon i \in D\})$. If so, take F_D from the least such pair; otherwise, set $F_D = \emptyset$. Set

$$F = \bigcup_{D \subseteq \{i \colon i < e\}} F_D \text{ and}$$

$$G = \{z \colon z \text{ is the least member of } \omega^{[i]} \smallsetminus (A_s \cup F \cup \{a_i\}) \text{ for some } i < e\}.$$

Enumerate $F \cup G$ into A_{s+1}. For each $j \geq e$ with $a_j \in F$, we reset $a_j \in \omega^{[j]}$ to be a fresh number outside F.

Denote the resulting set after ω many stages by A_ω. Finally, let

$$A = A_\omega \cup \{a_e \colon e \notin S\}.$$

In order to make the proof more compact, we introduce the following definition and prove a lemma about it:

Definition 3.4. For sets $A, B \subseteq \omega$, we say $A \leq_{e'} B$ if there is a "K-c.e. enumeration operator reducing A to B", i.e., a K-c.e. set Φ such that for all x, $x \in A$ if and only if there is a finite set $F \subseteq B$ (given by a canonical index) with $\langle x, F\rangle \in \Phi$.

Lemma 3.5. *For any sets $A, B \subseteq \omega$, we have $A \leq_{e'} B$ if and only if $A \leq_{\mathrm{e}} B \oplus \overline{K}$.*

Proof. If $A \leq_{e'} B$ via a K-c.e. operator $\Phi = W^K$, say, then each axiom $\langle x, F\rangle \in \Phi$ can be rewritten into axioms $\langle x, F, P, N\rangle$ where $\langle x, F\rangle \in W^K$ via computations requiring $P \subseteq K$ and $N \subseteq \overline{K}$, and these axioms $\langle x, F, P, N\rangle$ can be combined into a single c.e. enumeration operator Ψ witnessing $A \leq_{\mathrm{e}} B \oplus K \oplus \overline{K} \equiv_{\mathrm{e}} B \oplus \overline{K}$.

Conversely, suppose $A \leq_{\mathrm{e}} B \oplus K \oplus \overline{K} (\equiv_{\mathrm{e}} B \oplus \overline{K})$ via a c.e. enumeration operator Ψ, then we can define a K-c.e. enumeration operator Φ by enumerating $\langle x, F\rangle$ into Φ for any $\langle x, F \oplus P \oplus N\rangle \in \Psi$ with $P \subseteq K$ and $N \subseteq \overline{K}$. $\square$

From the construction and the definition of A, it is now clear that all $\mathcal{R}_e$-requirements are satisfied, and so $\overline{K} \leq_{\mathrm{e}} S \leq_{\mathrm{e}} \overline{A} \leq_{\mathrm{e}} A^\Diamond$.

We next observe that

$$\{a_e\}_{e\in\omega} \leq_T K. \tag{1}$$

Using (1) and that $e \in S$ if and only if $a_e \notin A$, it is now clear that both $\overline{S} \leq_{e'} A$ and $\overline{A} \leq_{e'} S$, and so by Lemma 3.5, we have both $\overline{S} \leq_{\mathrm{e}} A \oplus \overline{K}$ and $\overline{A} \leq_{\mathrm{e}} S \oplus \overline{K} \equiv_{\mathrm{e}} S$. The last inequality combined with the already established $S \leq_{\mathrm{e}} \overline{A}$ gives us that $\overline{A} \equiv_{\mathrm{e}} S$.

Finally, using (1) and the action of the $\mathcal{P}_e$-requirements, we also have $A^\Diamond \leq_{e'} \overline{A}$. This is because K can figure out for which D we found an F_D at stage $s = e+1$ such that $e \in \Gamma_e(F_D \cup \{a_i : i \in D\})$. Then $e \in A^\Diamond$ if and only if $\overline{A}$ intersects $\{a_i : i \in D\}$ for every such D. So again by Lemma 3.5, we have that $A^\Diamond \leq_{\mathrm{e}} \overline{A} \oplus \overline{K} \equiv_{\mathrm{e}} S$. $\square$

Notice that the proof of Theorem 3.3 directly gives us the following result.

Theorem 3.6. *Let $n \geq 2$. For any Π^0_n-set $S \geq_{\mathrm{e}} \overline{K}$, there is a Σ^0_n-set A such that $A^\Diamond \equiv_{\mathrm{e}} S$. Furthermore, for any Σ^0_n-set $S \geq_{\mathrm{e}} \overline{K}$, there is a Π^0_n-set A such that $A^\Diamond \equiv_{\mathrm{e}} S$.*

Proof. This follows directly from the proof of Theorem 3.3, noting that A as built there is equal to $A_\omega \cup \{\langle e, k\rangle : e \notin S\}$, that A_ω is Δ^0_2, and that $\{\langle e, k\rangle : e \notin S\}$ is of the same complexity as the complement of S. $\square$

Definition 3.7. An enumeration degree $\mathbf{a}$ is *quasiminimal* if it is nonzero and the only total enumeration degree bounded by $\mathbf{a}$ is $\mathbf{0}_{\mathrm{e}}$.

McEvoy [15] proved that the enumeration jump restricted to the quasiminimal degrees has the same range as the unrestricted jump operator. We show that the skip has the same property.

Corollary 3.8. *For any set $S \geq_{\mathrm{e}} \overline{K}$, there is a set A of quasiminimal degree such that $A^\Diamond \equiv_{\mathrm{e}} S$.*

Proof. We modify the construction in Theorem 3.3 slightly. We add additional requirements $\mathcal{Q}_e$ that ensure that A is quasiminimal:

$\mathcal{R}_e \colon e \in S \iff A^{[e]} \neq \omega^{[e]}$,

$\mathcal{P}_e \colon$ "force e into K_A subject to higher-priority restraints",

$\mathcal{Q}_e \colon \Gamma_e(A) = X \oplus \overline{X} \Rightarrow X$ is computable.

At stage $s = e + 1$, after we have defined the set F and G for the sake of the requirement $\mathcal{P}_e$, we will handle the requirement $\mathcal{Q}_e$. The procedure is similar. For any subset $D \subseteq \{i\colon i < e\}$, check if there are a finite subset $E_D \subseteq \omega \setminus \{a_i\colon 0 \le i < e\}$, a number x, and a stage t such that $\{2x, 2x+1\} \subseteq \Gamma_{e,t}(\{a_i\colon i \in D\} \cup E_D)$; for any such D, choose the set E_D from the least such triple; if there is no such triple, set $E_D = \emptyset$. Set

$$E = \bigcup_{D \subseteq \{i\colon i<e\}} E_D.$$

Enumerate $F \cup G \cup E$ into A_{s+1} and then redefine the values of a_j appropriately: If $j \ge e$ and $a_j \in F \cup E$, we reset $a_j \in \omega^{[j]}$ to be a fresh number outside $F \cup E$.

If $\Gamma_e(A)$ turns out to be a total set $X \oplus \overline{X}$, then we can compute X: Let $A^* = (\omega^{[<e]} \cap A) \cup \omega^{[\ge e]}$. As every column of A is finitely different from ω, it follows that A^* is a computable set and $A \subseteq A^*$. Now, $x \in X$ if and only if $2x \in \Gamma_e(A^*)$ and $x \notin X$ if and only if $2x+1 \in \Gamma_e(A^*)$. □

3.2. **Further properties of the skip operator and examples.** We will now investigate the possible behavior of the iterated skip operator.

Definition 3.9. Fix $A \subseteq \omega$. We inductively define $A^{\langle n\rangle}$, the *n-th skip of* A.

- $A^{\langle 0\rangle} = A$,
- $A^{\langle n+1\rangle} = (A^{\langle n\rangle})^{\Diamond}$.

The *n-th skip of* $\deg_e(A)$ is $\deg_e(A)^{\langle n\rangle} = \deg_e(A^{\langle n\rangle})$.

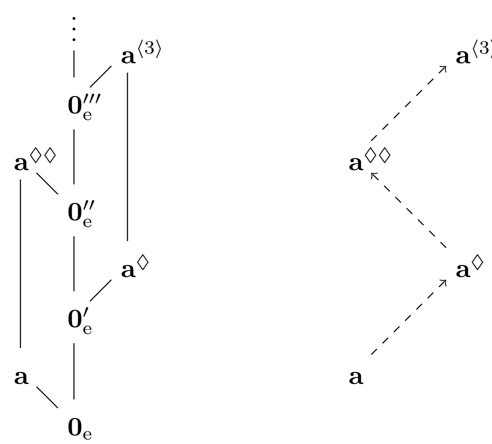

FIGURE 2. Iterated skips of a degree: the zig-zag

If $\mathbf{a}$ is a cototal enumeration degree, then every iteration of the skip of $\mathbf{a}$ agrees with the corresponding iteration of the jump of $\mathbf{a}$, i.e., for all $n < \omega$, we have that $\mathbf{a}^{\langle n\rangle} = \mathbf{a}^{(n)}$. Theorem 3.3 proves that there are non-cototal enumeration degrees, e.g., the skip invert of a nontotal enumeration degree. It is natural to ask what we can say in general about the sequence $\{\mathbf{a}^{\langle n\rangle}\}_{n\in\omega}$. One immediate observation is that even though the skip of A need not be above A, its double skip always is: For any set A, we know that $\overline{A} \le_1 A^{\Diamond}$. Applying this twice, we have $A \le_1 \overline{A^{\Diamond}} \le_1 A^{\Diamond\Diamond}$, so a fortiori $A \le_e A^{\Diamond\Diamond}$. It follows that $\mathbf{a}^{\langle n\rangle} \le \mathbf{a}^{\langle n+2\rangle}$ for all n. In addition, by monotonicity, we have that for every n, $\mathbf{0}_{\mathrm{e}}^{(n)} \leq \mathbf{a}^{\langle n \rangle}$. If $\mathbf{a}^{\langle n \rangle}$ is not cototal for every natural number n, then we have a form of *zig-zag* behavior of the skip, illustrated in Figure 2. We will search for examples of degrees whose skips have this general behavior.

3.2.1. *Skips of generic sets.* We will start by investigating the skip for the class of enumeration degrees of 1-generic sets. We will define a relativized form of 1-genericity, suitable for the context of the enumeration degrees. We use the notation "relative to $\langle X \rangle$" to denote "relative to the enumeration degree of X".

Definition 3.10. Let G and X be sets of natural numbers. G is 1-*generic relative to* $\langle X \rangle$ if and only if for every $W \subseteq 2^{<\omega}$ such that $W \leq_{\mathrm{e}} X$:

$$(\exists \sigma \preceq G)[\sigma \in W \vee (\forall \tau \succeq \sigma)[\tau \notin W]].$$

If $X = \emptyset$, then we call G simply 1-*generic* and if $X = \overline{K}$, then G is 2-*generic*.

Note that G is 1-generic relative to X in the usual sense if and only if G is 1-generic relative to $\langle X \oplus \overline{X} \rangle$ in the sense of the definition above.

Relativizing the notion of quasiminimality, we get the following two notions:

Definition 3.11. An enumeration degree $\mathbf{a}$ is a *quasiminimal cover* of an enumeration degree $\mathbf{b}$ if $\mathbf{b} < \mathbf{a}$ and there is no total enumeration degree $\mathbf{x}$ such that $\mathbf{b} < \mathbf{x} \leq \mathbf{a}$. The degree $\mathbf{a}$ is a *strong quasiminimal cover* of $\mathbf{b}$ if $\mathbf{b} < \mathbf{a}$ and every total enumeration degree $\mathbf{x}$ bounded by $\mathbf{a}$ is below $\mathbf{b}$.

The next proposition exhibits two important properties of generic enumeration degrees.

Proposition 3.12. *Let G be 1-generic relative to $\langle X \rangle$.*

(a) $\deg_{\mathrm{e}}(G \oplus X)$ *is a strong quasiminimal cover of* $\deg_{\mathrm{e}}(X)$.
(b) $\overline{G}$ *is 1-generic relative to* $\langle X \rangle$.

Proof. To see that $G \not\leq_e X$, note that G must be infinite and for every enumeration operator Γ the set $\{\sigma \colon (\exists n)[\sigma(n) = 0 \ \wedge n \in \Gamma(X)]\}$ is enumeration reducible to X . Let Y be a set of natural numbers and assume that $Y \oplus \overline{Y} \leq_{\mathrm{e}} G \oplus X$ via the enumeration operator Γ. We will show that $Y \oplus \overline{Y} \leq_{\mathrm{e}} X$. Consider the set

$$Q = \{\sigma \colon (\exists x)[\{2x, 2x+1\} \subseteq \Gamma(\sigma \oplus X)]\},$$

where we write $\sigma \oplus X$ to mean $\{n \colon \sigma(n) = 1\} \oplus X$. Note that Q is enumeration reducible to X and so, by our assumptions, G must avoid it, i.e., no $\sigma \in Q$ is an initial segment of G. Let $\sigma \preceq G$ be a string with no extension in Q. Then $z \in Y \oplus \overline{Y}$ if and only if there is an extension $\tau \succeq \sigma$ such that $z \in \Gamma(\tau \oplus X)$.

For the second part of this proposition, we introduce the following notation. If $\sigma \in 2^{<\omega}$, then let $\bar{\sigma}$ be the string obtained by inverting every bit of σ. For $W \subseteq 2^{<\omega}$, let $W^- = \{\bar{\sigma} \colon \sigma \in W\}$. Note that $\bar{\sigma} \preceq G$ if and only if $\sigma \preceq \overline{G}$. So if G meets W^- then $\overline{G}$ meets W, and if G avoids W^- then $\overline{G}$ avoids W. Finally, note that $W \leq_{\mathrm{e}} X$ implies that $W^- \leq_{\mathrm{e}} X$. □

It is well known that the Turing jump of a 1-generic set has a nice characterization: $K^G \equiv_T G \oplus K$, or, in other words, G is generalized low. This property relativizes: If G is 1-generic relative X, then $K^{G \oplus X} \equiv_T G \oplus K^X$. A similar property is true of the skip of a 1-generic set G relative to $\langle X \rangle$.

Proposition 3.13. *If G is 1-generic relative to $\langle X\rangle$, then $(G\oplus X)^\Diamond \equiv_{\mathrm{e}} \overline{G}\oplus X^\Diamond$.*

Proof. Note that we always have $\overline{G}\oplus X^\Diamond \leq_{\mathrm{e}} (G\oplus X)^\Diamond$, no matter what the sets G and X are, simply from the monotonicity of the skip operator. The nontrivial reduction is the reverse one. Suppose $\langle e,x\rangle \in (G\oplus X)^\Diamond$, i.e., $x\notin \Gamma_e(G\oplus X)$. Consider the set

$$D_{e,x} = \{\sigma\in 2^{<\omega}\colon x\in\Gamma_e(\sigma\oplus X)\}.$$

This set is enumeration reducible to X uniformly in e and x, and so there must be a string $\sigma\prec G$ such that no extension of σ is in $D_{e,x}$. The set

$$E_{e,x} = \{\sigma\colon (\exists\tau\succeq\sigma)[\tau\in D_{e,x}]\}$$

is also uniformly enumeration reducible to X, and so its complement is uniformly enumeration reducible to $X^\Diamond$. We claim that:

$$\langle e,x\rangle\in(G\oplus X)^\Diamond \text{ if and only if } (\exists\sigma)[\{n\colon \sigma(n)=0\}\subseteq\overline{G}\ \&\ \sigma\in\overline{E}_{e,x}].$$

The implication from left to right has already been established: If $\langle e,x\rangle\in(G\oplus X)^\Diamond$, then the initial segment of G with no extension in $D_{e,x}$ witnesses that the statement on the right is true. So let $\langle e,x\rangle$ be such that there is a σ with $\{n\colon\sigma(n)=0\}\subseteq\overline{G}$ and such that $\sigma\in\overline{E}_{e,x}$. Towards a contradiction, suppose that $\langle e,x\rangle\notin(G\oplus X)^\Diamond$, i.e., $x\in\Gamma_e(G\oplus X)$. Let $\tau\prec G$ be such that $x\in\Gamma_e(\tau\oplus X)$ and define σ^* of length $\max(|\sigma|,|\tau|)$ as follows

$$\sigma^*(n) = \begin{cases} \sigma(n) & \text{if } n<|\sigma|,\\ \tau(n) & \text{if } |\sigma|\leq n<|\tau|.\end{cases}$$

Then σ^* is an extension of σ. Furthermore, if $\tau(n)=1$ then $\sigma^*(n)=1$. Indeed, this is obvious for $n\geq|\sigma|$, and for $n<|\sigma|$, this follows from the fact that $\{n\colon\sigma(n)=0\}\subseteq\overline{G}$ and $\tau\prec G$. Thus $\sigma^*\in D_{e,x}$, contradicting our assumption that σ has no extension in $D_{e,x}$. $\square$

Now, we can easily give an example of a set G whose iterated skips form a zig-zag. Consider G to be a set that is arithmetically generic, i.e., G is 1-generic relative to $\langle\emptyset^{(n)}\rangle$ for every natural number n. Note that $\overline{G}$ has the same property. Then by induction using the characterization above we can show that for all $n<\omega$:

- If n is odd then $G^{\langle n\rangle}\equiv_{\mathrm{e}}\overline{G}\oplus\emptyset^{(n)}$ and $(\overline{G})^{\langle n\rangle}\equiv_{\mathrm{e}} G\oplus\emptyset^{(n)}$.
- If n is even then $G^{\langle n\rangle}\equiv_{\mathrm{e}} G\oplus\emptyset^{(n)}$ and $(\overline{G})^{\langle n\rangle}\equiv_{\mathrm{e}}\overline{G}\oplus\emptyset^{(n)}$.

Furthermore, all iterates of the skip for both sets G and $\overline{G}$ are not total, as their degrees are quasiminimal covers of the corresponding iterate of the jump of $\mathbf{0}_{\mathrm{e}}$. It follows that they also do not have cototal degree, as by Proposition 1.1 sets H of cototal degree have total skips: $K_H\equiv_{\mathrm{e}} H\leq_{\mathrm{e}} H^\Diamond=\overline{K_H}$. This gives an example of a double zig-zag as in Figure 3. It is worth noting that only the reductions implied by the diagram occur. For example, $G\not\leq_{\mathrm{e}} G^{\langle 3\rangle}$; otherwise $G^{\langle 3\rangle}\equiv_{\mathrm{e}} G\oplus G^{\langle 3\rangle}\equiv_{\mathrm{e}} G\oplus\overline{G}\oplus\emptyset^{\langle 3\rangle}$ would be total.

3.2.2. *Skips of nontrivial $\mathcal{K}$-pairs.* Kalimullin [11] relativized the notion of a $\mathcal{K}$-pair in a way similar to how we relativized the notion of 1-genericity.

Definition 3.14. A pair of sets of natural numbers $\{A,B\}$ forms a *$\mathcal{K}$-pair relative to $\langle X\rangle$* if there is a set $W\leq_{\mathrm{e}} X$ such that $A\times B\subseteq W$ and $\overline{A}\times\overline{B}\subseteq\overline{W}$. The pair $\{A,B\}$ is a *nontrivial $\mathcal{K}$-pair relative to $\langle X\rangle$* if, in addition, $A\not\leq_{\mathrm{e}} X$ and $B\not\leq_{\mathrm{e}} X$.

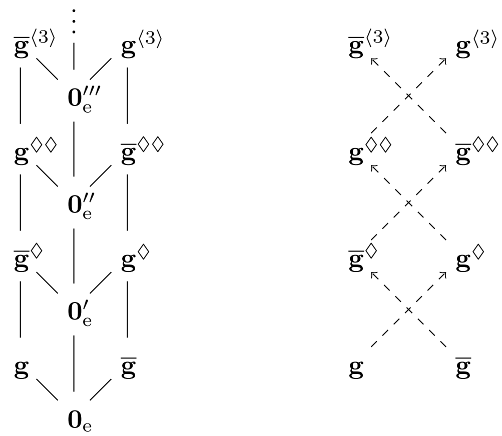

FIGURE 3. The iterated skips of the degrees of an arithmetically generic set and its complement: double zig-zag

Note that if $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair, then $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair relative to every $\langle X \rangle$ such that $A, B \not\leq_e X$. We summarize some properties of relativized $\mathcal{K}$-pairs below.

Proposition 3.15 (Kalimullin [11])**.** *Let $A, B, X \subseteq \omega$ and suppose that $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair relative to $\langle X \rangle$.*

(a) *If $C \leq_e B$ then $\{A, C\}$ forms a $\mathcal{K}$-pair relative to $\langle X \rangle$.*
(b) *$A \leq_e \overline{B} \oplus X$.*
(c) *$\overline{A} \leq_e B \oplus X^\Diamond$.*
(d) *$\deg_e(A \oplus X)$ and $\deg_e(B \oplus X)$ are strong quasiminimal covers of $\deg_e(X)$.*
(e) *For every $Z \subseteq \omega$, the degrees $\deg_e(A \oplus X \oplus Z)$ and $\deg_e(B \oplus X \oplus Z)$ have a greatest lower bound, and it is $\deg_e(X \oplus Z)$.*

Note that items (a), (b) and (c) are symmetrically true if we swap A and B.

The skip of a nontrivial $\mathcal{K}$-pair relative to $\langle X \rangle$ has the following properties:

Proposition 3.16. *If $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair relative to $\langle X \rangle$, then*

$$(A \oplus X)^\Diamond \leq_e B \oplus X^\Diamond \quad \text{and} \quad (B \oplus X)^\Diamond \leq_e A \oplus X^\Diamond.$$

The oracle set X is of cototal degree if and only if for every nontrivial $\mathcal{K}$-pair $\{A, B\}$ relative to $\langle X \rangle$,

$$(A \oplus X)^\Diamond \equiv_e B \oplus X^\Diamond \quad \text{and} \quad (B \oplus X)^\Diamond \equiv_e A \oplus X^\Diamond.$$

Proof. If $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair relative to $\langle X \rangle$, then $\{A \oplus X, B\}$ also forms a nontrivial $\mathcal{K}$-pair relative to $\langle X \rangle$: Replace the witnessing set W by

$$W^* = \{\langle 2a, b \rangle : \langle a, b \rangle \in W\} \cup \{\langle 2a+1, b \rangle : a \in X\}.$$

As $K_{A \oplus X} \equiv_e A \oplus X$, it follows that $\{K_{A \oplus X}, B\}$ forms a nontrivial $\mathcal{K}$-pair relative to $\langle X \rangle$, and so

$$(A \oplus X)^\Diamond = \overline{K_{A \oplus X}} \leq_e B \oplus X^\Diamond.$$

On the other hand, if X is of cototal degree, then using the monotonicity of the skip operator we get that $B \leq_e \overline{A} \oplus X \leq_e A^\Diamond \oplus X^\Diamond \leq_e (A \oplus X)^\Diamond$, and hence $B \oplus X^\Diamond \leq_e (A \oplus X)^\Diamond$.

Finally, consider the oracle set X and let $\{A, B\}$ be a nontrivial (unrelativized) $\mathcal{K}$-pair such that $A, B \not\leq_{\mathrm{e}} X$. Note that both $\{A, B\}$ and $\{A \oplus X, B \oplus X\}$ are nontrivial $\mathcal{K}$-pairs relative to $\langle X \rangle$. If the characterization of the skip operator holds for both pairs, then we have that

$$(A \oplus X)^{\diamond} \equiv_{\mathrm{e}} B \oplus X^{\diamond} \equiv_{\mathrm{e}} B \oplus X \oplus X^{\diamond}, \text{ and}$$
$$(B \oplus X)^{\diamond} \equiv_{\mathrm{e}} A \oplus X^{\diamond} \equiv_{\mathrm{e}} A \oplus X \oplus X^{\diamond}.$$

Now, using the last property from Proposition 3.15, we have that

$$\begin{aligned}\deg_{\mathrm{e}}(X^{\diamond}) = {} & \deg_{\mathrm{e}}(A \oplus X^{\diamond}) \wedge \deg_{\mathrm{e}}(B \oplus X^{\diamond}) = \\ & \deg_{\mathrm{e}}(A \oplus X \oplus X^{\diamond}) \wedge \deg_{\mathrm{e}}(B \oplus X \oplus X^{\diamond}) = \deg_{\mathrm{e}}(X \oplus X^{\diamond}).\end{aligned}$$

It follows that X is of cototal degree. $\square$

If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair and both A and B are not arithmetical, then $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair relative to $\langle \emptyset^{(n)} \rangle$ for every natural number n. As every set $\emptyset^{(n)}$ is of (co)total enumeration degree, it follows by Proposition 3.16 that the iterated skips of A and B also form a double zigzag: For all $n < \omega$,

- if n is odd then $A^{\langle n \rangle} \equiv_{\mathrm{e}} B \oplus \emptyset^{(n)}$ and $B^{\langle n \rangle} \equiv_{\mathrm{e}} A \oplus \emptyset^{(n)}$, and
- if n is even then $A^{\langle n \rangle} \equiv_{\mathrm{e}} A \oplus \emptyset^{(n)}$ and $B^{\langle n \rangle} \equiv_{\mathrm{e}} B \oplus \emptyset^{(n)}$.

Furthermore, by Proposition 3.15, for every natural number n, $\{\deg_{\mathrm{e}}(A)^{\langle n \rangle}, \deg_{\mathrm{e}}(B)^{\langle n \rangle}\}$ forms a minimal pair of quasiminimal degrees above $\mathbf{0}_{\mathrm{e}}^{(n)}$.

A pair of enumeration degrees $\{\mathbf{a}, \mathbf{b}\}$ forms a *$\mathcal{K}$-pair* (relative to $\mathbf{x}$) if there are representatives $A \in \mathbf{a}$ and $B \in \mathbf{b}$ that form a $\mathcal{K}$-pair (relative to $\mathbf{x}$). We will use the characterization of the skips of $\mathcal{K}$-pairs along with the following theorem of Ganchev and Sorbi [7] to give an example of degrees whose iterated skips behave quite differently.

Theorem 3.17 (Ganchev, Sorbi [7]). *For every enumeration degree $\mathbf{x} > \mathbf{0}_{\mathrm{e}}$, there is a degree $\mathbf{a} \leq \mathbf{x}$ such that $\mathbf{a}$ is half of a nontrivial $\mathcal{K}$-pair and such that $\mathbf{a}' = \mathbf{x}'$.*

One of the main ingredients in the proof of the theorem above is the following observation, which follows easily from Proposition 3.16. If $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair, then A and B have equivalent enumeration jumps:

$$J_{\mathrm{e}}(A) \equiv_{\mathrm{e}} A \oplus A^{\diamond} \equiv_{\mathrm{e}} A \oplus B \oplus \emptyset' \equiv_{\mathrm{e}} B \oplus B^{\diamond} \equiv_{\mathrm{e}} J_{\mathrm{e}}(B).$$

Now consider a nonzero enumeration degree $\mathbf{x} \leq_{\mathrm{e}} \mathbf{0}_{\mathrm{e}}'$, and let $\mathbf{a} \leq \mathbf{x}$ be half of a nontrivial $\mathcal{K}$-pair such that $\mathbf{a}' = \mathbf{x}'$. Let $\mathbf{b}$ be such that $\{\mathbf{a}, \mathbf{b}\}$ forms a nontrivial $\mathcal{K}$-pair. Then $\mathbf{b}^{\diamond} = \mathbf{a} \vee \mathbf{0}_{\mathrm{e}}' = \mathbf{0}_{\mathrm{e}}'$ and $\mathbf{b}' = \mathbf{a}' = \mathbf{x}'$. In particular, if we take $\mathbf{x}$ to be high, i.e., such that $\mathbf{x}' = \mathbf{0}_{\mathrm{e}}''$, then we have an example of an enumeration degree such that all iterations of its skip are total enumeration degrees, but mismatch its iterations of the jump by one iteration:

$$\mathbf{b}^{\diamond} < \mathbf{b}' = \mathbf{b}^{\diamond\diamond} < \mathbf{b}'' = \mathbf{b}^{\langle 3 \rangle} < \cdots < \mathbf{b}^{(n)} = \mathbf{b}^{\langle n+1 \rangle} < \cdots.$$

If we take $\mathbf{x}$ to be an intermediate degree, i.e., a degree such that for all n, $\mathbf{0}_{\mathrm{e}}^{(n)} < \mathbf{x}^{(n)} < \mathbf{0}_{\mathrm{e}}^{(n+1)}$ then we get the following:

$$\mathbf{b}^{\diamond} < \mathbf{b}' < \mathbf{b}^{\diamond\diamond} < \mathbf{b}'' < \mathbf{b}^{\langle 3 \rangle} < \cdots < \mathbf{b}^{(n)} < \mathbf{b}^{\langle n+1 \rangle} < \cdots.$$

We end this discussion with some thoughts about the definability of the skip operator. Kalimullin [11] proved that the relation "$\{\mathbf{a}, \mathbf{b}\}$ forms a $\mathcal{K}$-pair relative to $\mathbf{x}$" is first-order definable with parameter $\mathbf{x}$. Using this result, he showed that the enumeration jump operator is first-order definable. Combining these results with the characterization of the skip operator for nontrivial $\mathcal{K}$-pairs, we immediately obtain the following result.

Corollary 3.18. *The relation*

$$SK = \{(\mathbf{a}, \mathbf{a}^{\Diamond}) \colon \mathbf{a} \text{ is half of a nontrivial } \mathcal{K}\text{-pair } \}$$

is first-order definable in $\mathcal{D}_{\mathrm{e}}$.

Proof. If $\mathbf{a}$ is half of a nontrivial pair, then $\mathbf{a}^{\Diamond} = \mathbf{0}'_{\mathrm{e}} \vee \mathbf{b}$ where $\mathbf{b}$ is some nonzero degree that forms a $\mathcal{K}$-pair with $\mathbf{a}$. □

It remains an open question whether or not the skip operator is first-order definable in $\mathcal{D}_{\mathrm{e}}$.

3.2.3. *A skip* 2*-cycle.* As seen above, the skip can exhibit a form of *zig-zag* behavior. We now show that there is another extreme case that could occur: The double skip $\mathbf{a}^{\Diamond\Diamond}$ of an enumeration degree $\mathbf{a}$ could be equal to $\mathbf{a}$ itself. Perhaps surprisingly, this degree is not constructed in a way that is common in computability theory. Instead, we use the following theorem due to Knaster and Tarski.

Theorem 3.19 (Knaster–Tarski Fixed Point Theorem)**.** *Let L be a complete lattice and let $f \colon L \to L$ be monotone, i.e., for all $x, y \in L$, we have that $x \leq y$ implies that $f(x) \leq f(y)$. Then f has a fixed point. In fact, the fixed points of f form a complete lattice.*

We apply the Knaster–Tarski theorem to a function on 2^{ω}, which we view as the power set lattice of ω, ordered by subset inclusion.

Theorem 3.20. *There is a set A such that $A^{\Diamond\Diamond} = A$.*

Proof. Let $f \colon 2^{\omega} \to 2^{\omega}$ be the double skip operator, i.e., $f(A) = A^{\Diamond\Diamond}$. Note that if $A \subseteq B$, then $K_A \subseteq K_B$, so $A^{\Diamond} \supseteq B^{\Diamond}$. Applied twice, we obtain $A^{\Diamond\Diamond} \subseteq B^{\Diamond\Diamond}$, so f is monotone. Hence, by the Knaster–Tarski Fixed Point Theorem, there is an A such that $A^{\Diamond\Diamond} = A$. □

Note that we do not just have that A and $A^{\Diamond\Diamond}$ are enumeration equivalent, but they are equal as sets. However, we will mainly be interested in the fact that the enumeration degree $\mathbf{a}$ of A satisfies $\mathbf{a}^{\Diamond\Diamond} = \mathbf{a}$. If we have such a degree $\mathbf{a}$, then we will say that $\mathbf{a}$ and $\mathbf{a}^{\Diamond}$ form a *skip* 2*-cycle*.

As we show next, skip 2-cycles are computationally very complicated; namely, they compute all hyperarithmetic sets.

Proposition 3.21. *Let $\mathbf{a}$ and $\mathbf{a}^{\Diamond}$ form a skip* 2*-cycle. Then $\mathbf{a} \geq \mathbf{b}$ for every total hyperarithmetic degree $\mathbf{b}$.*

Proof. Let A be a set of degree $\mathbf{a}$. We build an enumeration operator Φ such that $\Phi(A, p) = H(p) \oplus \overline{H(p)}$ for every ordinal notation p, where $H(p)$ is defined as in [19, Chapter 2]. By the Recursion Theorem, we may assume that we know an index for Φ. We let $\Phi(A, 1) = H(1) \oplus \overline{H(1)} = \emptyset \oplus \omega$.

Now assume that $|p|$ is a successor, say, $p = 2^q$ and so $|p| = |q| + 1$. Note that if $C \geq_{\mathrm{e}} D \oplus \overline{D}$, then $C^{\Diamond} \geq_{\mathrm{e}} K^D \oplus \overline{K^D}$ uniformly in an index for the first reduction. Furthermore, we inductively assume that $\Phi(A, q) = H(q) \oplus \overline{H(q)}$. Combining these facts,

$$A \equiv_{\mathrm{e}} A^{\Diamond\Diamond} \geq_{\mathrm{e}} H(2^{2^q}) \oplus \overline{H(2^{2^q})} \geq_{\mathrm{e}} H(p) \oplus \overline{H(p)}$$

uniformly, so let $\Phi(A,p) = H(p) \oplus \overline{H(p)}$.

Finally, assume that $|p|$ is a limit ordinal, say, $p = 3 \cdot 5^e$ and so $|p|$ is the limit of $|q_0|, |q_1|, \dots$, where $q_i = \varphi_e(i)$. Using the inductive assumption that $\Phi(A, q_i) = H(q_i) \oplus \overline{H(q_i)}$, we can set $\Phi(A,p) = H(p) \oplus \overline{H(p)}$, where $H(p) = \bigoplus_{i\in\omega} H(q_i)$. $\square$

Given the fact that we have shown the existence of a skip 2-cycle, it is only natural to consider whether (proper) skip n-cycles exist for any other natural number $n \geq 1$. This turns out to be false.

Proposition 3.22. *Let $n \in \omega$ be nonzero such that $\mathbf{a}^{\langle n\rangle} = \mathbf{a}$. Then $\mathbf{a}^{\Diamond\Diamond} = \mathbf{a}$.*

Proof. First, observe that $\mathbf{a}^{\langle 2n\rangle} = \mathbf{a}^{\langle n\rangle} = \mathbf{a}$, so without loss of generality we may assume that n is even. By monotonicity of the double skip, we then have that

$$\mathbf{a} \leq \mathbf{a}^{\langle 2\rangle} \leq \cdots \leq \mathbf{a}^{\langle n-2\rangle} \leq \mathbf{a}^{\langle n\rangle} = \mathbf{a},$$

so

$$\mathbf{a} = \mathbf{a}^{\langle 2\rangle} = \cdots = \mathbf{a}^{\langle n-2\rangle} = \mathbf{a}^{\langle n\rangle}. \qquad \square$$

The set A we obtained in Theorem 3.20 allows us to give the example of a pair of sets A and B that illustrate the flaw in the enumeration jump mentioned in the last paragraph of Section 1.2.

Proposition 3.23. *$J_{\mathrm{e}}(A) \equiv_1 J_{\mathrm{e}}(B)$ does not necessarily imply $A \equiv_{\mathrm{e}} B$.*

Proof. Let A be the set we obtained in Theorem 3.20, and let $B = A^{\Diamond} = \overline{K_A}$, so $A = B^{\Diamond}$. Then $K_A = \overline{B} \leq_1 \overline{K_B}$ since B is a column of K_B, and similarly, $K_B = \overline{A} \leq_1 \overline{K_A}$. It follows that

$$K_A \oplus \overline{K_A} \leq_1 \overline{K_B} \oplus B \leq_1 \overline{K_B} \oplus K_B \equiv_1 K_B \oplus \overline{K_B},$$

and similarly

$$K_B \oplus \overline{K_B} \leq_1 \overline{K_A} \oplus A \leq_1 K_A \oplus \overline{K_A}.$$

Thus $J_{\mathrm{e}}(A) = K_A \oplus \overline{K_A} \equiv_1 K_B \oplus \overline{K_B} = J_{\mathrm{e}}(B)$, but clearly A is not enumeration equivalent to B. $\square$

4. Separating cototality properties

4.1. **Degrees that are not weakly cototal.** Let us begin by showing that the weakest cototality property we introduced, aptly named *weakly cototal*, is nontrivial, i.e., that there are degrees that are not weakly cototal. We will present three different examples in this section. First, we note that sufficiently generic sets are not weakly cototal.

Proposition 4.1. *If $\mathbf{a}$ is a 2-generic enumeration degree, then $\mathbf{a}$ is not weakly cototal.*

Proof. Let G be 2-generic and let $A \equiv_{\mathrm{e}} G$. Towards a contradiction, let us assume that $\overline{A}$ has total enumeration degree. Then by Proposition 3.13 with $X = \emptyset$, we have that

$$\overline{G} \oplus \overline{K} \equiv_{\mathrm{e}} G^{\Diamond} \geq_{\mathrm{e}} \overline{A}.$$

By Proposition 3.12(b) with $X = \overline{K}$, $\overline{G}$ is 2-generic, so by Proposition 3.12(a) with $X = \overline{K}$ and the totality of $\deg_{\mathrm{e}}(\overline{A})$, we obtain that $\overline{A} \leq_{\mathrm{e}} \overline{K}$. It follows that $G \oplus \overline{K} \equiv_{\mathrm{e}} A \oplus \overline{K} \geq_{\mathrm{e}} \overline{A} \oplus K$, and so $\deg_{\mathrm{e}}(G \oplus \overline{K})$ is total. This contradicts Proposition 3.12(a), that $\deg_{\mathrm{e}}(G \oplus \overline{K})$ is a quasiminimal cover for $\deg_{\mathrm{e}}(\overline{K})$ and so cannot be a total enumeration degree. $\square$

Next, we show that we can also get such examples using $\mathcal{K}$-pairs.

Proposition 4.2. *Let* $\mathbf{a}, \mathbf{b} \not\leq_{\mathrm{e}} \mathbf{0}'_{\mathrm{e}}$ *form a nontrivial* $\mathcal{K}$*-pair. Then* $\mathbf{a}$ *is not weakly cototal.*

Proof. Let $\{A, B\}$ form a nontrivial $\mathcal{K}$-pair with $A, B \not\leq_{\mathrm{e}} \overline{K}$. It follows that $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair relative to $\langle\overline{K}\rangle$, and so by Proposition 3.15, the degree of $B \oplus \overline{K}$ is a strong quasiminimal cover of $\mathbf{0}'_{\mathrm{e}}$. Towards a contradiction, suppose that A has weakly cototal degree. As $\mathcal{K}$-pairs are closed with respect to enumeration equivalence, we may assume that $\overline{A}$ is of total enumeration degree. By the same Proposition 3.15, we have, on the one hand, that $\overline{A} \leq_{\mathrm{e}} B \oplus \overline{K}$ and so $\overline{A} \leq_{\mathrm{e}} \overline{K}$, and on the other hand, that $B \leq_{\mathrm{e}} \overline{A}$. It follows that $B \leq_{\mathrm{e}} \overline{K}$, contradicting our choice of B. ☐

For our final example of a degree that is not weakly cototal, recall from Theorem 3.20 that there is a degree $\mathbf{a}$ such that $\mathbf{a}^{\diamond\diamond} = \mathbf{a}$. Such a degree is not weakly cototal.

Proposition 4.3. *Let* $\mathbf{a}$ *be such that* $\mathbf{a}^{\diamond\diamond} = \mathbf{a}$*. Then* $\mathbf{a}$ *is not weakly cototal.*

Proof. Towards a contradiction, assume that A in the degree $\mathbf{a}$ is such that $\overline{A}$ has total enumeration degree. Then $A^{\diamond} \geq_{\mathrm{e}} \overline{A}$ implies that

$$A^{\diamond\diamond} \geq_{\mathrm{e}} \left(\overline{A}\right)^{\diamond} \geq_{\mathrm{e}} A,$$

so $\deg_{\mathrm{e}}(A)$ is the skip of the total degree $\deg_{\mathrm{e}}(\overline{A})$ and hence total. But then $A^{\diamond\diamond} >_{\mathrm{e}} A$, which is a contradiction. ☐

4.2. **Weakly cototal degrees that are not cototal.** We will prove the next separation using the skip inversion we proved in Theorem 3.3 above.

Proposition 4.4. *There is a degree* $\mathbf{a}$ *that is weakly cototal, but not cototal.*

Proof. Let $B \geq_{\mathrm{e}} \overline{K}$ be any total set, and let $S = K_B$. Then note that $S \equiv_{\mathrm{e}} B$, so the *degree* of S is total, but S is not total *as a set*. Now apply Theorem 3.3 to obtain an A such that $A^{\diamond} \equiv_{\mathrm{e}} S$ and $\overline{S} \leq_{\mathrm{e}} A \oplus \overline{K}$.

Then A is weakly cototal since $A \equiv_{\mathrm{e}} K_A$ and $\overline{K_A} = A^{\diamond} \equiv_{\mathrm{e}} S$, which has total degree. Let $\mathbf{a}$ be the degree of A. We claim that $\mathbf{a}$ is not cototal. By Proposition 1.1, it suffices to show that $A \not\leq_{\mathrm{e}} A^{\diamond}$. Towards a contradiction, assume that $A \leq_{\mathrm{e}} A^{\diamond}$. Since $A^{\diamond} \geq_{\mathrm{e}} \overline{K}$ always holds, we now see that

$$S \equiv_{\mathrm{e}} A^{\diamond} \geq_{\mathrm{e}} A \oplus \overline{K} \geq_{\mathrm{e}} \overline{S}$$

so S would be a total set, which is a contradiction. ☐

The proof above combined with Theorem 3.6 yields the promised Π^0_2 degree that is not cototal. Of course, as noted earlier, such a degree can be obtained using a theorem of Badillo and Harris [2] proving the existence of a Π^0_2-enumeration degree that contains only properly Π^0_2-sets. As all Π^0_2 enumeration degrees are weakly cototal, this gives us a more concrete separation result.

An alternative way to separate the weakly cototal degrees from the cototal degrees is given by the following proposition.

Proposition 4.5. *If* $\mathbf{b} \not\leq \mathbf{0}'_{\mathrm{e}}$ *but forms a nontrivial* $\mathcal{K}$*-pair with* $\mathbf{a} \leq \mathbf{0}'_{\mathrm{e}}$*, then* $\mathbf{b}$ *forms a minimal pair with* $\mathbf{b}^{\diamond}$*.*

Proof. Towards a contradiction, assume there is a nonzero degree $\mathbf{c}$ such that $\mathbf{c} \leq \mathbf{b}$ and $\mathbf{c} \leq \mathbf{b}^\Diamond$. The fact that $\mathbf{c} \leq \mathbf{b}$ gives us that $\mathbf{a}$ and $\mathbf{c}$ form a $\mathcal{K}$-pair by Proposition 3.15(a). Using this, Proposition 3.15(b), and Proposition 3.16 twice, we have

$$\mathbf{b} \leq \mathbf{a}^\Diamond = \mathbf{c} \oplus \mathbf{0}'_\mathrm{e} \leq \mathbf{b}^\Diamond = \mathbf{a} \oplus \mathbf{0}'_\mathrm{e} = \mathbf{0}'_\mathrm{e}.$$

So $\mathbf{b} \leq \mathbf{0}'_\mathrm{e}$, which is a contradiction. □

Corollary 4.6. *If* $\mathbf{b}$ *is as in the previous proposition, then* $\mathbf{b}$ *is weakly cototal, but not cototal.*

Proof. By the previous proposition combined with Proposition 1.1, $\mathbf{b}$ is not cototal. On the other hand, from Proposition 3.16 we know that $B^\Diamond \equiv_\mathrm{e} A \oplus \overline{K} \equiv_\mathrm{e} \overline{K}$, since $A \leq_\mathrm{e} \overline{K}$. Also as $\overline{K}$ has total degree, as in the proof of Proposition 4.4, this implies that B is weakly cototal. □

The only separation left to prove is the separation of the cototal degrees from the graph-cototal degrees. We will prove this result in the next section.

5. There is a cototal degree that is not graph-cototal

Theorem 5.1. *There is a cototal enumeration degree that is not graph-cototal.*

Proof. We fix the undirected graph $\mathcal{G} = (\omega^{<\omega}, E)$, where the edge relation is given by $E(a, b)$ if and only if $a^- = b$ or $a = b^-$ (i.e., a is an immediate successor of b or the immediate predecessor of b). We will build the complement of a maximal independent set for the graph $\mathcal{G}$. Recall that this is a subset $A \subseteq \omega^{<\omega}$ with the property that every element $a \in \omega^{<\omega}$ is either outside A or is connected by an edge to an element outside A, but not both.

Our other condition on the set A will be that it is not enumeration equivalent to a graph-cototal set. We construct A as such using a construction in the framework of a $0'''$-priority construction over $0'$. We start by listing an infinite sequence of requirements that collectively ensure that we meet our goal. We then make use of a tree of strategies. Strategies on the tree inherit the standard ordering of nodes: We use $\alpha \preceq \beta$ to denote that α is a prefix of β and $\alpha <_L \beta$ to denote that α is to the left of β in the tree. Every strategy is assigned one of the requirements. At every stage we build a finite path through this tree, activating strategies along it and injuring all strategies to the right of it. Activated strategies perform actions towards satisfaction of their requirements. Injured strategies are *initialized*—they must start over as if they were never activated before. The intention is that there will be a *true path*, a leftmost infinite path of nodes visited at infinitely many stages, such that every strategy along this path succeeds in satisfying the requirement that is assigned to it. We refer the reader to Soare [22] for a more detailed introduction to priority arguments and the tree method. We warn the reader that our argument differs from standard infinite-injury arguments in a couple of ways: There will be some strategies α which intentionally injure other strategies β with $\alpha \prec \beta$, and this will cause injury along the true path. Also, we will have strategies β which cause strategies $\alpha \prec \beta$ to revert to a previous state in α's construction, though for every α each state in α's construction will only be susceptible to reversion by finitely many $\beta \succ \alpha$. Finally, we will make use of the notion of *moment* to refer to substages in the construction. We assume that actions that strategies make, such as injury and initialization, have immediate effect during moments in the construction, rather than at the end of a stage.

At every moment in the construction, we will say some strategies restrain elements in A and some restrain elements out of A. When we refer to the set A at any given moment in the dynamic construction, we mean

$$\omega^{<\omega} \setminus \{a \colon \text{some strategy } \beta \text{ currently restrains } a \text{ out of } A\}.$$

Our set $A \subseteq \omega^{<\omega}$ now needs to satisfy the following requirements, for all $a \in \omega^{<\omega}$ and all enumeration operators Φ and Ψ.

Requirements:

$$\begin{aligned}
\text{global} &\colon (\forall x, y \in \omega^{<\omega} \setminus A)[\neg\, xEy] \\
\mathcal{N}_a &\colon a \notin A \text{ or } (\exists x)[xEa \wedge x \notin A] \\
\mathcal{R}_{\Phi,\Psi} &\colon A = \Psi(\Phi(A)) \implies \Phi(A) \neq \overline{G_f} \text{ for any total function } f \colon \omega \to \omega
\end{aligned}$$

Clearly, our global requirement and the $\mathcal{N}_a$-requirements and $\mathcal{R}_{\Phi,\Psi}$-requirements will ensure that A is of cototal (see Section 2.3) but not of graph-cototal enumeration degree.

Construction: We define a priority tree as follows: Each $\mathcal{N}_a$-strategy has only one outcome, d. Each $\mathcal{R}_{\Phi,\Psi}$-strategy has infinitely many possible outcomes: stop $<$ $\infty < \cdots < \text{wait}_1 < \text{wait}_0$. We assign all nodes on a given level of the tree to the same requirement, and every non-global requirement is associated to some level. Finally, if the last coordinate in $a \in \omega^{<\omega}$ is k, then we ensure that the $\mathcal{N}_a$-strategy does not appear in the first k levels of the tree.

The main difficulty in this construction is in performing the strategy for an $\mathcal{R}_{\Phi,\Psi}$-requirement while allowing lower-priority requirements to succeed. As we will see, one $\mathcal{R}$-requirement may restrain infinitely many elements into A, while lower-priority requirements may need to extract some of these elements from A.

Let us describe the $\mathcal{R}_{\Phi,\Psi}$-strategy for a node α on the priority tree. The strategy has parameters x_α, F^n_α, H^n_α, y^n_α, z^n_α, and D^n_α, whose meaning we now explain. The goal of the strategy is to ensure that some column of $\Phi(A)$ is either complete or misses two elements and thus $\Phi(A)$ cannot be the complement of the graph of a total function. The parameter x is the column that the strategy uses. The superscript n on the parameters F, H, y, z, and D refers to the values of these parameters under the assumption that the true outcome of α is the outcome wait_n. The parameter F is a set that the strategy restrains in A, i.e., the strategy makes sure that no lower-priority strategy removes any element of F from A. The set $H \cup D$ is a finite set that, if we remove it from A and all higher-priority restraints remain, will cause the element $\langle x, y\rangle$ (for our parameter y) to be removed from $\Phi(A)$. The set $H \cup D$ is partitioned into two pieces as the two pieces will relate to other strategies in different ways. The set H is comprised of elements that at some prior stage were restrained out of A by strategies below some outcome wait_m of α (except when α is activated for the first time after initialization, when H contains a single fresh element), and D is comprised of elements that at some prior stage were restrained out of A by a strategy below the outcome ∞ of α. We only extract the set $H \cup D$ if we see some other number w such that we can also ensure that $\langle x, w\rangle \notin \Phi(A)$. The number z is the least number other than y for which we currently do not know that $\langle x, z\rangle \in \Phi(A)$. We try to ensure that z increases infinitely often, thus making the entire column contained in $\Phi(A)$, ensuring that $\Phi(A)$ cannot be the complement of the graph of a total function. The general idea is that if it ever happens that we cannot increase z, i.e., we cannot put $\langle x, z\rangle$ into $\Phi(A)$, then by removing $H \cup D$ from A, we can ensure that two elements are missing from the x-th column of $\Phi(A)$. In this case, as before, $\Phi(A)$ is not the complement of the graph of a total function.

Step -1: When first activated (or after initialization) the strategy starts from Step -1. Pick a large a_0 with $a_0^-, a_0 \in A$ and $a_0{}^\frown\langle m\rangle \in A$ for all m. (Here by *large* we mean that neither the string a_0, nor any of its components have been mentioned in the construction so far.) Check, using oracle $\mathbf{0}'$, if there are finite sets F and G (given by canonical indices) such that

$$a_0 \in \Psi(G) \text{ and } G \subseteq \Phi(F) \text{ and } F \subseteq \tilde{A}, \tag{2}$$

where $\tilde{A}$ is the set of those a for which there is no strategy $\gamma \preceq \alpha$, $\gamma <_L \alpha$, or $\gamma \succeq \alpha^\frown\langle\infty\rangle$ that currently restrains a out of A. If no such F and G exist, then the $\mathcal{R}_{\Phi,\Psi}$-requirement is trivially satisfied since $a_0 \in A$ but $a_0 \notin \Psi(\Phi(A))$. In this case, place a restraint keeping $a_0 \in A$, place a restraint $b \notin A$ for any b for which some strategy $\gamma \succeq \alpha^\frown\langle\infty\rangle$ is currently restraining $b \notin A$, and take outcome stop. As long as it is not initialized, the strategy will never act again and when visited from now on will take outcome stop. If we can find such F and G with $a_0 \notin F$, then we again take the outcome stop and satisfy the $\mathcal{R}_{\Phi,\Psi}$-requirement by restraining a_0 out of A while ensuring $F \subseteq A$ and thus $a_0 \in \Psi(\Phi(A))$ by restraining F in A. Otherwise, possibly enlarge the finite set F so as to maximize $|G \cap \Phi(F \setminus \{a_0\})|$ for this fixed G. Fix any pair $\langle x, y^0\rangle \in G \setminus \Phi(F \setminus \{a_0\})$ (which is a nonempty set by our assumption) such that for this fixed x, we have that y^0 is least such that $\langle x, y^0\rangle \in G \setminus \Phi(F \setminus \{a_0\})$. Fix x from now on as the parameter x. Let s be the current stage. (Note, that we may assume that $y^0 < s$ by speeding up the construction if necessary.) Let $F^0 = F \setminus \{a_0\}$ and let $H^0 = \{a_0\}$, let D^0 be the set of all elements restrained out by some strategy extending $\alpha^\frown\langle\infty\rangle$, and let z^0 be the least number z other than y^0 that is $\leq s$ such that $\langle x, z\rangle \notin \Phi(F^0)$, if such a number exists; let z^0 be s otherwise. We define $F^i = F^0$, $H^i = H^0$, $D^i = D^0$, $z^i = z^0$ for all $i \leq s$. Go to Step s. (This is to ensure that if α is initialized infinitely often, then it visits each outcome wait_n only finitely often. We will design the construction so that from now on, α cannot be reverted to Step i for $i < s$.) We take outcome ∞.

Regardless of which outcome we took, we initialize all nodes that are strictly to the right of the outcome we took.

Step n: Being in Step n means that n is largest such that F^n, H^n, D^n, y^n, and z^n are defined. We say a node β is on the *n-subtree* if the length of β is strictly less than n and for all $\gamma \preceq \beta$, $\gamma^\frown\langle\mathrm{wait}_k\rangle \preceq \beta$ implies that $k < n$. Note that the n-subtree is finite for every n. If $\beta \succeq \alpha^\frown\langle\infty\rangle$ and β is not on the n-subtree and (when last visited) β did not take outcome stop, then we initialize β. (Strategies $\beta \succeq \alpha^\frown\langle\infty\rangle$ can revert α to a previous step. This action ensures that only finitely many strategies can revert α to a step smaller than n.)

Let W be the set of elements restrained out of A by some strategy γ, such that $\gamma \preceq \alpha$ or $\gamma <_L \alpha$. Let B_∞ be the set of elements restrained out of A by nodes extending $\alpha^\frown\langle\infty\rangle$ and let B_n be the set of elements restrained out of A by nodes extending $\alpha^\frown\langle\mathrm{wait}_n\rangle$ along with the elements that are in $H^k_\beta \cup D^k_\beta$ for some k and some $\beta \succeq \alpha^\frown\langle\mathrm{wait}_n\rangle$.

Let us say that a set $Y \subseteq \omega^{<\omega}$ is *consistent* if it does not contain any pair of elements which are connected to each other. For example, it will follow inductively that $W \cup B_\infty$ is consistent. Furthermore, since B_n consists of elements that are introduced in the construction after W and B_∞ are defined, it will follow that if $Y \subseteq B_n$ is consistent then so is $W \cup B_\infty \cup Y$. Using oracle $\mathbf{0}'$, we check if $\langle x, z^n \rangle \in \Phi(\omega \setminus (W \cup B_\infty \cup Y))$ for all consistent subsets $Y \subseteq B_n$. If so, then we let X_0 be a finite set such that X_0 is disjoint from $W \cup B_\infty \cup B_n$ and $\langle x, z^n \rangle \in \Phi(X_0 \cup (B_n \setminus Y))$ for every consistent $Y \subseteq B_n$. We then redefine F^n to be $F^n \cup X_0$, redefine z^n to be the least number z other than y^n which is $\leq s$ (where s is the current stage) such that $\langle x, z \rangle \notin \Phi(F^n)$, if such a number exists, and let z^n be s otherwise. Leave all other parameters the same, and take outcome wait_n.

If $\langle x, z^n \rangle \notin \Phi(\omega \setminus (W \cup B_\infty \cup Y))$ for some consistent set $Y \subseteq B_n$, then fix any such set Y. We now check whether $\langle x, z^n \rangle \in \Phi(\omega \setminus (W \cup B_\infty))$. If so, then we let X_1 be some set disjoint from $W \cup B_\infty$ such that $\langle x, z^n \rangle \in \Phi(X_1)$. Define $H^{n+1} = Y$, $D^{n+1} = B_\infty$, $y^{n+1} = z^n$, define $F^{n+1} = F^n \cup (X_1 \setminus Y) \cup H^n$, and let z^{n+1} be the least number z other than y^{n+1} that is $\leq s$ (where s is the current stage) such that $\langle x, z \rangle \notin \Phi(F^{n+1})$, if such a number exists, and let z^{n+1} be s otherwise. (This means that next time α is visited, it will be in Step $n+1$, unless it is reverted back to a Step $\leq n$.) We take outcome ∞.

If $\langle x, z^n \rangle \notin \Phi(\omega \setminus (W \cup B_\infty))$, then we place a negative restraint so that $(H^n \cup D^n \cup B_\infty) \cap A = \emptyset$. For each γ such that $\gamma^\frown\langle\infty\rangle \preceq \alpha$ and such that $(H^n \cup D^n \cup B_\infty) \cap F^k_\gamma \neq \emptyset$ or such that $H^n \cup D^n \cup B_\infty$ contains an element a that is the predecessor of an element $b \in H^k_\gamma \cup D^k_\gamma$ (i.e., $b = a^\frown\langle m\rangle$ for some m), we undefine all of γ's parameters with superscript $\geq k$. Note that γ now reverts to being in a smaller step, say Step ℓ; we say we have *reverted* γ *to Step* ℓ. (This action is necessary to ensure that there are no conflicting restraints on a and that the set $H^k_\gamma \cup D^k_\gamma \cup B^\gamma_\infty$ is consistent, should γ later on need to restrain it out of A. It will follow from the proof that we need not worry about the possibility of a being the successor of an element $b \in H^k_\gamma \cup D^k_\gamma$. We also point out that if $\alpha \succeq \beta^\frown\langle \mathrm{wait}_k\rangle$, then we do not need to revert β to a previous step as all of β's parameters that can potentially be restrained out of A are defined before α was accessible.) We also initialize all strategies below $\gamma^\frown\langle\infty\rangle$ which are not on the ℓ-subtree except for α and which (when last visited) did not take outcome stop. We undefine all of α's parameters (since α will not be reverted back to any Step k, unless it is initialized) and take the outcome stop; unless initialized, we will forever take the outcome stop from now on with no further action.

Regardless of which outcome we took, we initialize all nodes which are strictly to the right of the outcome we took.

The behavior of an $\mathcal{N}_a$-strategy β is simple: First it tries to assess whether a will end up an element of A or not. If a is not mentioned by any strategy of higher priority, then β safely assumes that $a \in A$. If $a \in (D^n_\alpha \cup H^n_\alpha) \cap A$ and $\beta \succeq \alpha^\frown\langle \mathrm{wait}_n\rangle$, then β can assume that a will remain in A (unless β is initialized). If $a \in H^n_\alpha$ for a node α with $\alpha^\frown\langle\infty\rangle \preceq \beta$ that is in Step n (i.e., n is largest so that H^n_α is defined), then end the stage. (Here β believes that a will not remain in H^n_α and so simply waits.) Otherwise, if $a \in D^n_\alpha$ for a node α with $\alpha^\frown\langle\infty\rangle \preceq \beta$ that is in Step n, or if $a \notin A$, then do nothing and take the outcome d. (It will follow from the construction that if β is on the true path and $a \in D^n_\alpha$ for a node α with $\alpha^\frown\langle\infty\rangle \preceq \beta$ at infinitely many stages at which β is visited, then $a \notin A$.) If $a \in A$, then we pick a fresh number m and place a restraint to prevent $a^\frown\langle m\rangle$ from being in A. In that case, if any $\gamma \succeq \beta$ is an $\mathcal{R}_{\Phi,\Psi}$-strategy that (when last visited) did not take outcome stop, then initialize γ. We do this to ensure that the global requirement is met, i.e., no strategy can restrain a out of A unless it initializes β.

Verification: We now show that our construction ensures the satisfaction of all requirements.

Lemma 5.2. *If α is a strategy that is reverted to Step n, it is because a node $\beta \succeq \alpha^\frown\langle\infty\rangle$ that is on the $(n+1)$-subtree takes the outcome stop.*

Proof. If α was in Step -1 when the parameters F_α^{n+1}, H_α^{n+1}, and D_α^{n+1} were last modified before the current stage then all strategies extending $\alpha^\frown\langle\infty\rangle$ that had not yet stopped were in their initial state. Otherwise, at the moment when F_α^{n+1} was last modified before the current moment, α was in Step at most $n+1$. At the moment when H_α^{n+1} and D_α^{n+1} were defined, α was at most in Step n. In any case, all strategies extending $\alpha^\frown\langle\infty\rangle$ that were not on the $(n+1)$-subtree and had not yet taken outcome stop were initialized or in initial state. We claim that if $\beta \succeq \alpha^\frown\langle\infty\rangle$ is not on the $(n+1)$-subtree, then (unless α is initialized) at no later stage will β have $(H_\beta^k \cup D_\beta^k \cup B_\infty) \cap F_\alpha^{n+1} \neq \emptyset$ or will $H_\beta^k \cup D_\beta^k \cup B_\infty$ contain a predecessor of an element in $H_\alpha^{n+1} \cup D_\alpha^{n+1}$. We prove this by induction on moments of the construction. At no later stage will an $\mathcal{N}_c$-strategy place a restraint that takes an element out of A that is in F_α^{n+1} or that is the predecessor of an element in $H_\alpha^{n+1} \cup D_\alpha^{n+1}$. This is because the $\mathcal{N}_c$-strategy extracts an element of the form $c^\frown\langle m\rangle$ where m is new, so it could not have appeared in F_α^{n+1} or be a predecessor of an element in $H_\alpha^{n+1} \cup D_\alpha^{n+1}$. The same is true for $\mathcal{R}_{\Phi,\Psi}$-strategies that extract a new element because they take the outcome stop in Step -1, because this is also a new element. Now, any H_β^k or D_β^k is formed from elements that are in $H_\gamma^\ell \cup D_\gamma^\ell$ for $\gamma \succ \beta$, from elements restrained out by $\mathcal{N}_c$-strategies extending β, or from some $\{a_0\}$ introduced by an $\mathcal{R}_{\Phi,\Psi}$-strategy in Step -1 extending β (or equal to β if β is in Step -1). So, by induction, no $H_\gamma^\ell \cup D_\gamma^\ell$ can contain an element in F_α^{n+1} or an element that is the predecessor of an element in $H_\alpha^{n+1} \cup D_\alpha^{n+1}$, thus neither can $H_\beta^k \cup D_\beta^k$. $\square$

Lemma 5.3. *If α is along the true path and is an $\mathcal{R}_{\Phi,\Psi}$-strategy, and $n \in \omega$, then there are only finitely many stages at which α is reverted to Step n.*

Proof. We prove the result by simultaneous induction on n for all strategies combined. Suppose towards a contradiction that some strategy α on the true path is reverted to Step n infinitely often. Since there are only finitely many elements in the $(n+1)$-subtree below $\alpha^\frown\langle\infty\rangle$, there must be some β doing this infinitely often. In particular, α is on the $(n+1)$-subtree. Suppose in addition that α is the longest strategy on the true path and in the $(n+1)$-subtree that is reverted to Step n infinitely often.

Every time β reverts α to Step n, it must take the outcome stop. To be initialized infinitely often after taking outcome stop, it must be that we visit a node left of β infinitely often, as this is the only way in which we initialize stopped strategies. Since α is on the true path, there must be some shortest $\gamma \succeq \alpha$ along the true path which takes an outcome left of β infinitely often.

Case 1: γ is an $\mathcal{R}$-strategy and β is below the outcome wait$_k$ of γ for some $k < n+1$ (as β is on the $(n+1)$-subtree). It follows that $\alpha \neq \gamma$, because $\beta \succeq \alpha^\frown\langle\infty\rangle$.

Then to visit β again, γ must be reverted to a step $\leq n$ infinitely often. By our choice of α and by the induction hypothesis, this is impossible.

Case 2: γ is an $\mathcal{R}$-strategy and β is below the outcome ∞ of γ. Then γ infinitely often visits the outcome stop, but is initialized. Since we initialize a strategy that has taken outcome stop only by visiting a node to the left of it, this contradicts our choice of γ as a strategy on the true path.

Case 3: γ is an $\mathcal{R}$-strategy and β is below the outcome stop of γ. Then γ cannot take an outcome left of the outcome stop.

Case 4: γ is an $\mathcal{N}$-strategy. This is impossible because $\mathcal{N}$-strategies only have one outcome. $\square$

Lemma 5.4. *Every strategy along the true path is initialized only finitely often.*

Proof. For α to be initialized, either some node to its left is visited, or it is not in the k-subtree and some node $\beta^\frown\langle\infty\rangle \preceq \alpha$ is in Step k, or some $\mathcal{N}$-node above it places a restraint. The first case happens only finitely often, since α is along the true path. The second case happens only finitely often since, if α is on the ℓ-subtree for the least number l, and along the true path, then each β with $\beta^\frown\langle\infty\rangle \preceq \alpha$ eventually is not reverted to a step $< \ell$. The last case can only happen finitely often since, by the inductive hypothesis, every node $\beta \prec \alpha$ is initialized only finitely often. $\square$

Lemma 5.5. *For each node α, if H^k_α becomes defined at stage s, then for every $a \in H^{k+1}_\alpha$, the last number in the string a is $> k, s$.*

Proof. For H^k_α to become defined at stage s, this requires α to take outcome ∞ at stage s. Thus every strategy below $\alpha^\frown\langle\text{wait}_k\rangle$ is initialized. Since before an element can enter H^{k+1}_α, it must first be in $H^\ell_\beta \cup D^\ell_\beta$ for some $\beta \succeq \alpha^\frown\langle\text{wait}_k\rangle$ and some $\ell \in \omega$, be restrained out of A by some $\mathcal{N}$-strategy extending $\alpha^\frown\langle\text{wait}_k\rangle$, or be in some $\{a_0\}$ introduced by an $\mathcal{R}_{\Phi,\Psi}$-strategy β in Step -1 with $\beta \succeq \alpha^\frown\langle\text{wait}_k\rangle$, we see that it must be introduced by such a strategy at a stage $\geq s$. Furthermore, the number k has been mentioned. So, when β restrains an element a out of A, its last number is new, thus greater than k and s. $\square$

We will say that a node is *active* at the current moment m if it has been visited at a moment $n \leq m$ and has not been initialized at any moment in the interval $[n, m]$.

Lemma 5.6. *If α is an $\mathcal{R}$-strategy that first takes the outcome stop (since its last initialization) at stage s, then there is no active node below $\alpha^\frown\langle stop\rangle$ at the beginning of stage s.*

Proof. The statement is clearly true if this is the first time when α takes outcome stop. Suppose that this is not the case and consider the stage s at which α last took the outcome stop. Let $t > s$ be the first time that α was initialized after stage s. Then this initialization must be the result of visiting a node to the left of α, since this is the only way we initialize stopped strategies. Thus any $\gamma \succeq \alpha^\frown\langle\text{stop}\rangle$ would also be initialized. $\square$

The following lemma ensures that at no stage of the construction do we take any contradictory actions.

Lemma 5.7. *At every moment of the construction, the following hold:*

(a) *No two different strategies place conflicting restraints on a string (i.e., it is impossible that one restrains it in A and the other restrains it out of A, where being restrained into A means being restrained in by Step -1 or being in the current set F).*
(b) *It is not the case that any H^n_α contains an element that is restrained out of A.*
(c) *If $a \in D^n_\alpha$, and either a is restrained out of A by β or $a \in H^k_\beta$, then $\alpha^\frown\langle stop\rangle <_L \beta$. (So, if α ever restrains a out of A, then β is initialized.)*
(d) *If $a \in H^n_\alpha \cup D^n_\alpha$ and a is restrained in A by β, and $\alpha \neq \beta$, then either $\alpha^\frown\langle stop\rangle <_L \beta$ or $\beta^\frown\langle\infty\rangle \preceq \alpha$. (So, if α restrains a out of A, then β is initialized or reverted to a smaller step.)*
(e) *If $a \in D^n_\alpha$ and $a \in D^m_\beta$ for $\alpha \neq \beta$, then $\alpha \succeq \beta^\frown\langle\infty\rangle$ or $\beta \succeq \alpha^\frown\langle\infty\rangle$. (So, $D^n_\alpha \subseteq D^n_\beta$ or $D^n_\beta \subseteq D^n_\alpha$.)*
(f) *If $(n,\alpha) \neq (m,\beta)$, then $H^n_\alpha \cap H^m_\beta = \emptyset$.*
(g) *It is not the case that two different strategies restrain a out of A at the same time.*

Proof. We will show that all statements hold at all times. Consider the first moment when one of the claims fails, and suppose that it is (a). Let α restrain a into A, while β restrains a out of A.

Case 1: α placed its restraint first. Then β places its negative restraint in one of two ways: Either β is an $\mathcal{N}_c$-strategy for some c, or β is an $\mathcal{R}_{\Phi,\Psi}$-strategy which takes the outcome stop. In the first case, the $\mathcal{N}_c$-strategy restrains an element of the form $c^\frown\langle m\rangle$ where m is fresh, in particular ensuring that $c^\frown\langle m\rangle$ is not already restrained in A. In the second case, the only new element restrained out by β in Step -1 is a fresh element a_0 which cannot happen for the same reason, or β restrains out elements in $H^n_\beta \cup D^n_\beta \cup B_\infty$ (where β is in Step n). But a cannot be in B_∞, as otherwise, at the previous moment, we would have restrained a both in and out of A. If a is in $H^n_\beta \cup D^n_\beta$, then this must have happened at a previous moment. So by the inductive hypothesis, by (d), either $\beta^\frown\langle\text{stop}\rangle <_L \alpha$ or $\beta \succeq \alpha^\frown\langle\infty\rangle$. In the former case, we have that α is initialized as β decides to restrain a out of A, thus there is no conflict. In the second case, α is reverted to a previous Step k at which $a \notin F^k_\alpha$, again showing there is no conflict.

Case 2: β placed its restraint first. Then α is an $\mathcal{R}_{\Phi,\Psi}$-strategy that adds a to F, while β already restrains a out of A. We have three cases where this can happen. First, if α is in Step -1, then by the definition of F^0 it can only be that $\beta \succeq \alpha^\frown\langle\text{wait}_k\rangle$ for some $k \in \omega$. (Note that this is nontrivially possible, because it could be that α was initialized for not being on some subtree, while β is left active, as it had already reached its outcome stop.) However, then β is initialized, because α has outcome stop or ∞, both to the left of wait_k, so there is no conflict. Otherwise, either α remains in Step n and increases F by including X_0, or α moves to Step $n+1$ and defines F^{n+1} to be $F^n \cup (X_1 \smallsetminus Y) \cup H^n$. In the former case, since X_0 is explicitly chosen to be comprised of elements that are not restrained out of A by any strategy at all, we see that a cannot be added to F^n at this moment. In the latter case, H^n is disjoint from anything restrained out of A by (b) (and our assumption of this being the first moment when any of the conditions is violated). Suppose we then have $a \in X_1 \smallsetminus Y$. Then it is either in $X_1 \smallsetminus B_n$ and thus not restrained out of A by any strategy, or it is in B_n. If it is in B_n, then it is either in H^k_γ or D^k_γ for some γ extending $\alpha^\frown\langle\text{wait}_n\rangle$ or is restrained out of A by such a γ. In the first case, this contradicts (b). In the second case, we would have $\gamma^\frown\langle\text{stop}\rangle <_L \beta$ by (c), thus $\alpha^\frown\langle\infty\rangle <_L \beta$ and β is initialized when α places a into F^{n+1}_α. In the third case, either a is restrained out of A by two different strategies at a previous moment, contradicting (g), or $\gamma = \beta$ and β is initialized when α takes outcome ∞.

Suppose the first moment where any of the claims fails is one where (b) fails, i.e., a appears both in H^n_α and is restrained out of A by a node β.

Case 1: α placed a into H^n_α first. Then β places its negative restraint in one of two ways: Either β is an $\mathcal{N}_c$-strategy for some c, or β is an $\mathcal{R}_{\Phi,\Psi}$-strategy that takes the outcome stop. In the first case, the $\mathcal{N}_c$-strategy restrains an element of the form $c^\frown\langle m\rangle$ where m is fresh, in particular ensuring that $c^\frown\langle m\rangle$ is not already in H^n_α. In the second case, the only new element restrained out in Step -1 is a fresh element, which we have just mentioned does not cause a conflict, or we have $a \in H^m_\beta \cup D^m_\beta \cup B_\infty$ where β is in Step m, and β takes the outcome stop. If $a \in B_\infty$, then a was previously restrained out of A, which is a contradiction. If $a \in H^m_\beta$, then we would have $H^m_\beta \cap H^n_\alpha \neq \emptyset$ at a previous moment, contradicting (f). If $a \in D^m_\beta$, then by (c), $\beta^\frown\langle\text{stop}\rangle <_L \alpha$, and so α is initialized when β restrains a out of A; thus there is no conflict.

Case 2: β restrains a out of A first and α places a into H^n_α at the current moment. If α is in Step -1 then H^0 contains only one element, chosen as a fresh number by α and hence not restrained by β. It follows that α is in Step $n-1 \geq 0$ and a must have either been restrained out of A by some strategy $\gamma \succeq \alpha^\frown\langle\text{wait}_{n-1}\rangle$ or must have been in $H^k_\gamma \cup D^k_\gamma$ for some $\gamma \succeq \alpha^\frown\langle\text{wait}_{n-1}\rangle$. It would violate (b) for a to be in H^k_γ at the previous moment, and if $a \in D^k_\gamma$ at the previous moment, then by (c), we would have $\gamma^\frown\langle\text{stop}\rangle <_L \beta$, so $\alpha^\frown\langle\infty\rangle <_L \beta$ and β is initialized when a is added into H^n_α. So suppose γ restrains a out of A. When we redefine $H^n_\alpha = Y$ (for some $Y \subseteq B_n$), we take the outcome $\alpha^\frown\langle\infty\rangle$, thus injuring this γ. Thus if $\beta = \gamma$, then we have that β has relinquished its restraint, and if $\beta \neq \gamma$, then at the previous moment, β and γ restrained the same element out of A, violating (g) at the previous moment.

Suppose the first moment where any of the claims fails is one where (c) fails, i.e., suppose $a \in D^n_\alpha$ and a is restrained out of A by β or is contained in H^k_β while $\alpha^\frown\langle\text{stop}\rangle \not<_L \beta$.

Case 1: α places a into D^n_α first. Suppose a is restrained out of A by β. Again, this cannot happen if β is an $\mathcal{N}$-strategy. Suppose β is an $\mathcal{R}$-strategy that takes the outcome stop in Step -1. Then the only new element restrained out is fresh, so a must have already been restrained out of A by some $\gamma \succeq \beta^\frown\langle\infty\rangle$. Then (c) applied at a previous moment implies that $\alpha^\frown\langle\text{stop}\rangle <_L \gamma$, so either $\alpha^\frown\langle\text{stop}\rangle <_L \beta$ or $\beta \preceq \alpha$. Furthermore, α cannot be below $\beta^\frown\langle\text{stop}\rangle$, because the last time β was initialized, so were all non-stopped $\mathcal{R}$-strategies below it, and hence D^n_α for such an α is empty. So, if $\beta \prec \alpha$, then when β takes the outcome stop, α is initialized. If $\alpha = \beta$, then D^n_α becomes undefined when β takes the outcome stop. In both cases there is no conflict.

Now, suppose we have $a \in H^k \cup D^k \cup B_\infty$ for β. If $a \in H^k$, we have that $\alpha^\frown\langle\text{stop}\rangle <_L \beta$, by (c) at the previous moment. If $a \in D^k$, then by (e) at the previous moment, either $\alpha \succeq \beta^\frown\langle\infty\rangle$ or $\beta \succeq \alpha^\frown\langle\infty\rangle$. In the first case, when β restrains a out of A, it takes the outcome stop, injuring α, while in the second case, $\alpha^\frown\langle\text{stop}\rangle <_L \beta$. If $a \in B_\infty$, then at the previous moment, we had a restrained out of A by some $\gamma \succeq \beta^\frown\langle\infty\rangle$. Thus $\alpha^\frown\langle\text{stop}\rangle <_L \gamma$, by (c) at the previous moment.

Either $\alpha^\frown\langle \text{stop}\rangle <_L \beta$ or $\beta^\frown\langle\infty\rangle \preceq \alpha$ (noting that, if $\alpha = \beta$, we would undefine D^n_α when β takes the outcome stop and that, as before, $\alpha \succeq \beta^\frown\langle \text{stop}\rangle$ is not possible). In the latter case, when β restrains a out of A, it takes the outcome stop, injuring α.

Now, suppose α places a into D^n_α first and at the current moment a enters H^k_β. It cannot be that β is in Step -1 as the only element that enters H^0 is a fresh number, hence different from a. It follows that β is in step $k-1 \geq 0$ and at the previous moment, a was either in $H^\ell_\gamma \cup D^\ell_\gamma$ for some $\gamma \succeq \beta^\frown\langle \text{wait}_{k-1}\rangle$ or was restrained out of A by some $\gamma \succeq \beta^\frown\langle \text{wait}_{k-1}\rangle$. If it was in H^ℓ_γ or restrained out of A by γ, then by (c) at the previous moment, $\alpha^\frown\langle \text{stop}\rangle <_L \gamma$. Thus either $\alpha^\frown\langle \text{stop}\rangle <_L \beta$, $\beta^\frown\langle \text{wait}_{k-1}\rangle \preceq \alpha$, or $\alpha = \beta$. Note that $\beta^\frown\langle\infty\rangle \preceq \alpha$ is not a possible case, because all elements in H^k_β have been introduced into the construction by strategies extending outcome wait_{k-1} after α placed $a \in D^n_\alpha$. In the second case, when β adds a to H^k, it initializes α, avoiding conflict. If $\alpha = \beta$ and $n < k$, then a must have been introduced by some strategy $\delta \succeq \alpha$ which is an $\mathcal{N}$-strategy or an $\mathcal{R}$-strategy which took the outcome stop at Step -1 at some stage after a entered D^n_α, which contradicts freshness. If $\alpha = \beta$ and $n = k$, then both events really happen at the same time, and by definition of D^n_α, a was restrained out at the previous moment by a strategy $\delta \succeq \alpha^\frown\langle\infty\rangle$, which cannot be true by (b). Lastly, suppose a was in D^ℓ_γ for $\gamma \succeq \beta^\frown\langle \text{wait}_{k-1}\rangle$. Then by (e) (at the previous moment), $\alpha \succeq \gamma^\frown\langle\infty\rangle$ or $\gamma \succeq \alpha^\frown\langle\infty\rangle$. If $\alpha \succeq \beta^\frown\langle \text{wait}_{k-1}\rangle$, then when β takes the outcome ∞ to place a into H^k_β, α is initialized. So, we may suppose rather that $\alpha^\frown\langle\infty\rangle \preceq \beta \preceq \gamma$. Thus $\alpha^\frown\langle \text{stop}\rangle <_L \beta$.

Case 2: β restrains a out of A or places a into H^k_β first. If $\alpha = \beta$ these events really occur at the same time, and we already discussed this situation in the previous case. So, let us assume $\alpha \neq \beta$. When a joins D^n_α, D^n_α is defined to be the set of elements restrained out of A by nodes extending $\alpha^\frown\langle\infty\rangle$. Thus at the previous moment, β restrains a out of A, and hence β is below $\alpha^\frown\langle\infty\rangle$ by (g) or a was in H^k_β, and it was restrained out of A by some node $\gamma \succeq \alpha^\frown\langle\infty\rangle$, contradicting (b). It follows that $\alpha^\frown\langle \text{stop}\rangle <_L \beta$.

Consider the first moment when one of the claims fails, and suppose that it is (d). Let $a \in H^n_\alpha \cup D^n_\alpha$ be such that a is restrained in A by some node β.

Case 1: a is placed into $H^n_\alpha \cup D^n_\alpha$ first. Then β cannot be left of α, as otherwise α would be initialized. Furthermore, α cannot currently be in the outcome stop, otherwise H^n_α would be undefined. Thus, $\alpha^\frown\langle \text{stop}\rangle <_L \beta$ or $\beta \preceq \alpha$. If $\beta^\frown\langle \text{wait}_m\rangle \preceq \alpha$, then it must be that β is in step m and expands the definition by adding X_0 to F^m. Indeed, if β is in Step -1 it has outcome stop or ∞, initializing α; if β is in Step $k < m$ then β was reverted to a smaller step $l \leq k$ after α placed a in $H^n_\alpha \cup D^n_\alpha$ by a strategy $\gamma \succeq \beta^\frown\langle\infty\rangle$ and at that moment α must have been initialized; if β is in Step $k > m$ or if β is in Step m and defines F^{m+1} then it will have outcome to the left of α at this moment and so initialize α. The set X_0 added to F^m is disjoint from $H^n_\alpha \cup D^n_\alpha$ for any $\alpha \succeq \beta^\frown\langle \text{wait}_m\rangle$, so a is not in X_0. The strategy α cannot be below β's outcome stop, as β cannot restrain any elements in A after α was first accessible unless it is initialized and then α would be initialized as well. Thus, the only possibility left is $\beta^\frown\langle\infty\rangle \preceq \alpha$, as desired.

Case 2: a is restrained in A by β first. For a to enter $H^n_\alpha \cup D^n_\alpha$, it must have already been either restrained out of A by some strategy below α, or α is in Step -1 and $a = a_0$, or it must have been in $H^k_\gamma \cup D^k_\gamma$ for some γ below $\alpha^\frown\langle \text{wait}_n\rangle$. In the first case, this contradicts (a). The second case contradicts the freshness of a_0. So suppose a was in $H^k_\gamma \cup D^k_\gamma$, and we have either $\gamma^\frown\langle\text{stop}\rangle <_L \beta$ or $\gamma \succeq \beta^\frown\langle\infty\rangle$ by (d) at a previous moment. In the first case, we then also have $\alpha^\frown\langle\text{stop}\rangle <_L \beta$. In the second case, either $\alpha \succeq \beta^\frown\langle\infty\rangle$ or $\alpha^\frown\langle\text{wait}_n\rangle \preceq \beta^\frown\langle\infty\rangle \preceq \gamma$. In the former case, we have what the claim allows, and in the latter case, again $\alpha^\frown\langle\text{stop}\rangle <_L \beta$.

Consider the first moment when one of the claims fails, and suppose that it is (e). Let $a \in D^n_\alpha \cap D^m_\beta$ for $\alpha \neq \beta$. For a to be in D^n_α, some strategy γ_1 below $\alpha^\frown\langle\infty\rangle$ that is an $\mathcal{N}$-strategy or $\mathcal{R}$-strategy in Step -1 must have at some point proposed a into the construction. Similarly, for a to be in D^m_β, some strategy γ_2 below $\beta^\frown\langle\infty\rangle$ that is an $\mathcal{N}$-strategy or $\mathcal{R}$-strategy in Step -1 must have, at some point proposed a in the construction. Now, since in both cases the strategy proposes a fresh element, it is impossible that $\gamma_1 \neq \gamma_2$. Thus $\gamma = \gamma_1 = \gamma_2$ is below both $\alpha^\frown\langle\infty\rangle$ and $\beta^\frown\langle\infty\rangle$, showing that either $\alpha \succeq \beta^\frown\langle\infty\rangle$ or $\beta \succeq \alpha^\frown\langle\infty\rangle$.

Consider the first moment when one of the claims fails, and suppose that it is (f). Let $a \in H^n_\alpha \cap H^m_\beta$. Without loss of generality, H^m_β is the one defined at this moment. Again, it cannot be that β is in Step -1 by freshness of a_0, so β is in Step $m-1 \geq 0$. Then some strategy γ below $\beta^\frown\langle\text{wait}_{m-1}\rangle$ either previously had $a \in H^k_\gamma \cup D^k_\gamma$, or was previously restraining a out of A. If a was in H^k_γ or γ was restraining a out of A, we would already contradict (f) or (b), unless $(n, \alpha) = (k, \gamma)$. However, if $\alpha = \gamma$, then we have that $\alpha \succeq \beta^\frown\langle\text{wait}_{m-1}\rangle$, and hence when β defines H^m_β and takes the outcome ∞, it initializes α. Finally, assume $a \in D^k_\gamma$. Then, by (c), $\gamma^\frown\langle\text{stop}\rangle <_L \alpha$. But $\beta^\frown\langle\infty\rangle <_L \gamma^\frown\langle\text{stop}\rangle <_L \alpha$, so again, when β defines H^m_β and takes the outcome ∞, it initializes α.

Consider the first moment when one of the claims fails, and suppose that it is (g). Let α and β both restrain a out of A, and let us assume that α places its restraint first.

Case 1: β is an $\mathcal{N}_b$-strategy. Then since β restrains an element out of A of the form $b^\frown\langle m\rangle$ where m is fresh, it cannot restrain a out of A if it is already restrained out of A.

Case 2: β is an $\mathcal{R}_{\Phi,\Psi}$-strategy. Suppose a is restrained out of A by β in Step -1. Then since a is not fresh, it must have been restrained out of A by some $\gamma \succeq \beta^\frown\langle\infty\rangle$ not below the outcome stop. By (g) at the previous moment, $\gamma = \alpha$. Thus, when β takes the outcome stop, α is initialized, so there is no conflict. Thus, at the previous moment, $a \in H^n_\beta \cup D^n_\beta \cup B_\infty$. If $a \in D^n_\beta$, then by (c) at the previous moment, we have $\beta^\frown\langle\text{stop}\rangle <_L \alpha$. Thus, when β takes the outcome stop, it initializes α, showing that there is no conflict. It is impossible that $a \in H^n_\beta$ by (b) at the previous moment. If a is in B_∞, then some strategy γ below $\beta^\frown\langle\infty\rangle$ restrains a out of A. By (g) at the previous moment, we have that $\alpha = \gamma \succeq \beta^\frown\langle\infty\rangle$. So again, when β takes the outcome stop, α is initialized, showing that there is no conflict. □

Lemma 5.8. *The set A is $\Delta^0_2(\mathbf{0}')$. That is, for every a, there is a stage t such that $a \in A$ at every moment after stage t or $a \notin A$ at every moment after stage t.*

Proof. Every restraint of a number a being out of A begins with the introduction of a by an $\mathcal{N}$-strategy or an $\mathcal{R}$-strategy in Step -1. If this never happens for a, then we have $a \in A$ at every moment of the construction.

Suppose some γ introduces a. If γ is never subsequently initialized and it restrains a out of A, then $a \notin A$ for every moment after this restraint is placed. If γ is an $\mathcal{R}$-strategy and $a \in H^0$, but γ is never subsequently initialized and never restrains a out of A, then by Lemma 5.7(b) no other strategy does either, so $a \in A$ at every moment after a is introduced.

Otherwise, let s be a stage such that γ has introduced a and is initialized before stage s. After γ introduces a, no $\mathcal{N}$-strategy or $\mathcal{R}$-strategy in Step -1 can re-introduce a by freshness. At stage $t \geq s$, a may be in $H^n_\beta \cup D^n_\beta$ for some $\beta \preceq \gamma$ and some $n \in \omega$. Let Z_t be the set of such β. Furthermore, let U_t be the set of $\beta \preceq \gamma$ that restrain a out at stage t. It follows by Lemma 5.7(g) that at all times, U_t contains at most one element. If at stage $t \geq s$, the set $U_t = \{\beta_t\}$, then by Lemma 5.7(b) and (c), Z_t contains only strategies α that would initialize β_t if they had outcome stop, in particular, only strategies above β_t (as all strategies in $Z_t \cup U_t$ are initial segments of γ, hence comparable). If at a stage $t \geq s$, a enters $H^n_\alpha \cup D^n_\alpha$ for $\alpha \notin Z_{t-1}$ then it does so in one of two ways: by joining H^n_α if there is a $\beta \in Z_{t-1} \cup U_{t-1}$ such that $\alpha^\frown\langle \text{wait}_{n-1}\rangle \preceq \beta$, in which case this β is initialized and hence leaves $Z_t \cup U_t$, or by joining D^n_α if there is a $\beta \in U_{t-1}$ with $\beta \succeq \alpha^\frown\langle\infty\rangle$. In both cases, the minimum priority of strategies in $Z_{t-1} \cup U_{t-1}$ does not decrease, while $Z_t \cup U_t$ only contains nodes that are (possibly non-proper) initial segments of nodes in $Z_{t-1} \cup U_{t-1}$. If at stage t, the strategy β enters U_t, then it must be that β takes the outcome stop and $\beta \in Z_{t-1}$.

As we argued above, at every stage t we have that $Z_t \cup U_t$ only contains nodes that are initial segments of nodes in $Z_{t-1} \cup U_{t-1}$. So, there are only finitely many possibilities for U_t. Furthermore, it also follows from our argument above that, if the strategy that restrains a out of A changes between stages t_1 and t_2, then the strategy restraining a out at the later stage t_2 has higher priority than the strategy restraining a out at stage t_1. Therefore, let $t > s$ be a stage at which $U_r = U_t$ for all $r > t$. If U_r is empty, then $a \in A$ at every moment after t, and if it is not, then $a \notin A$ at every moment after t. □

Lemma 5.9. *If β is an $\mathcal{N}_a$-strategy along the true path, then β ensures that $\mathcal{N}_a$ is satisfied.*

Proof. Let β be an $\mathcal{N}_a$-strategy along the true path. By Lemma 5.4, let s be a stage at which β is visited and such that no $\gamma \preceq \beta$ is ever initialized at a stage $t \geq s$. Furthermore, by Lemma 5.3, we can suppose that s is large enough such that, if $\alpha^\frown\langle\infty\rangle \preceq \beta$, then α will never be in Step k for $k \leq a(|a|-1)$. Then it follows from Lemma 5.5 that at any stage $t \geq s$, $a \notin H^\ell_\alpha$ for any node α such that $\alpha^\frown\langle\infty\rangle \preceq \beta$ with α in Step ℓ. If, at some stage $t > s$ when β is visited, we have both $a \notin D^\ell_\alpha$ for all nodes α such that $\alpha^\frown\langle\infty\rangle \preceq \beta$ with α in Step ℓ, and $a \in A$, then β will place a permanent restraint, ensuring that $a^\frown\langle m\rangle \notin A$ for some fresh m. Otherwise, at any stage $t > s$ we have $a \notin A$ or $a \in D^\ell_\alpha$ for some node α as in the previous sentence, in which case at the last stage $r < t$ when we visited α, we defined D^ℓ_α to only contain elements that are restrained out at that moment. Thus, if β fails to place a permanent restraint, then $a \notin A$ at infinitely many moments when β is visited; thus $a \notin A$ follows by Lemma 5.8. □

Lemma 5.10. *Let a be a string restrained out of A at stage t. Then all successors of a that are introduced by stage t can never again be restrained out of A.*

Proof. Towards a contradiction, suppose that some successor b of a that was already introduced at stage t is restrained out of A at some later stage $s > t$. When we introduce a number, we always select it as a new number. It follows that a cannot be introduced after b and that no $\mathcal{R}$-strategy can introduce b, so b is introduced by an $\mathcal{N}_a$-strategy δ after a is introduced and at a stage $r < t$ when a is in A. Since a has been introduced and will eventually be restrained out of A and since δ initialized all lower priority strategies at stage r, the element a must be hiding in $H^n_\gamma \cup D^n_\gamma$ for some strategy γ of higher priority than δ. At stage t, the string a is restrained out of A by some $\mathcal{R}$-strategy $\beta \preceq \gamma$ that takes the outcome stop. At this stage, the strategy δ is initialized, so the only way in which b can be restrained out of A at stage $s > t$ is if it, in turn, is hiding at stage t in $H^k_\alpha \cup D^k_\alpha$ for some $\mathcal{R}$-strategy α of higher priority than β. Note that strategies of lower priority than β are either in initial state or initialized at stage t. Furthermore, as β is not initialized at any stage in the interval $(r, t]$ (or else every possible strategy that could be keeping a as part of its parameters, such as γ, would also be initialized) and α was visited at a stage q in that interval (when it defined $H^k_\alpha \cup D^k_\alpha$), β must be an extension of α. If $\beta \succeq \alpha^\frown\langle\infty\rangle$, then at stage t the strategy β reverts α to a previous Step l such that $H^l_\alpha \cup D^l_\alpha$ does not contain any successors of a. Otherwise, $\beta \succeq \alpha^\frown\langle \mathrm{wait}_i\rangle$, for some i. But then at the stage $q < t$ at which b entered $H^k_\alpha \cup D^k_\alpha$ the strategy α had outcome ∞, initializing β, γ and all other strategies that could be protecting a, so a could not be restrained out of A at stage t. This gives us the desired contradiction. $\square$

Lemma 5.11. *At no moment are there any two strings that are edge-related and both restrained out of A. Furthermore, if α is an $\mathcal{R}_{\Phi,\Psi}$-strategy that is currently in Step $n \geq 0$, then at any moment we have that, if $a, b \in B_\infty \cup H^n_\alpha \cup D^n_\alpha \cup B_n$ and a and b are edge-related, then $a, b \notin B_\infty$ and $a \notin H^n_\alpha \cup D^n_\alpha$ or $b \notin H^n_\alpha \cup D^n_\alpha$.*

Proof. We prove these two facts simultaneously by induction on moments. Let us first consider the first claim. Suppose towards a contradiction that two strings a and b are edge-related and both restrained out of A. Let α restrain a and β restrain b out of A, and suppose α places its restraint first (or $\alpha = \beta$ thus α and β place their restraint simultaneously).

Case 1: β is an $\mathcal{N}_c$-strategy. Then β restrains an element out of the form $c^\frown\langle m\rangle$ where m is fresh. Thus, it is impossible that a is a successor of $c^\frown\langle m\rangle$. So, the only possibility is that $a = c$, but then β would not restrain any element out of A at all, since $c \notin A$ when β is visited.

Case 2: β is an $\mathcal{R}_{\Phi,\Psi}$-strategy that takes the outcome stop. If β restrains $b = a_0$ out in Step -1, then a and b are not edge-related by construction. If β restrains b out of A, because b is restrained by some strategy $\gamma \succeq \beta^\frown\langle\infty\rangle$, then the claim follows inductively as a and b would both be restrained out of A at a previous moment. So, we can assume that β is in Step $n \geq 0$ and, at the previous moment, a was restrained out of A by α and b was either in H^n_β, D^n_β or B_∞. If $\alpha = \beta$, then a and b were restrained out at the same time, since an $\mathcal{R}_{\Phi,\Psi}$-strategy only places a negative restraint when it takes the outcome stop. Thus from the second claim at the previous moment it now follows that a and b are not edge-related.

Next, let us assume that $\alpha \neq \beta$. It follows by the inductive hypothesis that $b \notin B^\infty$. Suppose $b \in D^n_\beta \cup H^n_\beta$. Then since α is not initialized by β taking outcome stop, we can conclude that either $\alpha \prec \beta$ or $\alpha <_L \beta$. Either way, α has not acted since β and all $\mathcal{R}$-strategies below β that have not yet stopped were initialized. (If α is an $\mathcal{N}$-strategy, then when it acted, it would have initialized β unless it had already stopped, in which case it must have been initialized since then in order to stop again. If α is an $\mathcal{R}$-strategy, then it places a negative restraint by taking the outcome stop, so β was either initialized then for being right of the outcome stop or is below the outcome stop and was first visited then.) Consider the moment t

at which α acted to take a out of A. At that moment, no $\mathcal{R}$-strategy below β had any parameters, unless already in outcome stop. Thus, for b to be in $D^n_\beta \cup H^n_\beta$ now and not out of A at stage t (which is excluded by our inductive hypothesis), it was, at some stage $r > t$, restrained out of A by an $\mathcal{N}$-strategy or introduced by an $\mathcal{R}$-strategy in Step -1 below β. The second case cannot happen, because we explicitly pick a b which is not connected to a. Thus, at some stage, we had an $\mathcal{N}$-strategy restraining an element that is edge-related to an element not in A, which is impossible by Case 1.

Next, let us consider the second claim. Assume that $a, b \in Z_\alpha = B_\infty \cup H^n_\alpha \cup D^n_\alpha \cup B_n$ and that a and b are edge-related. Let us assume that a entered Z_α first, and that b just entered Z_α. We now have several cases.

First, if b enters $H^n_\alpha \cup D^n_\alpha$, then the strategy α is currently moving from Step $n-1$ to Step n, b is a member of $B_\infty \cup B_{n-1}$ at the previous moment, and B_n is empty. In particular, Z_α at the current moment contains only elements from $B_\infty \cup B_{n-1}$, as $D^n_\alpha = B_\infty$ and $H^n_\alpha \subseteq B_{n-1}$. Since $a \in Z_\alpha$ at the current moment, it follows that $a \in B_\infty \cup B_{n-1}$ at the previous moment as well. Thus, by the induction hypothesis, $a, b \in B_{n-1}$, and therefore by construction, we can only have that $a, b \in H^n_\alpha$. However, we explicitly defined H^n_α as a consistent subset of B_{n-1}, hence not containing any edge-related elements.

Next, if b enters B_∞, it means that at the previous stage $s^- < s$ when α was visited, we took the outcome ∞ and some strategy extending this outcome restrained b out of A. It follows that currently $B_n = \emptyset$ once again. It cannot be that $a \in B_\infty$, by the first claim at the current moment, which we have just proven. So $a \in H^n_\alpha \cup D^n_\alpha$ from stage s^- onwards until the current moment at stage s. It follows that at stage s^-, no $\mathcal{N}_a$-strategy $\beta \succeq \alpha^\frown\langle\infty\rangle$ acts, and hence b cannot have been restrained out for this reason. Suppose that b is restrained out by an $\mathcal{R}$-strategy $\beta \succeq \alpha^\frown\langle\infty\rangle$ taking outcome stop at stage s^-. If a is the successor of b, then β would revert the strategy α to a Step l such that $a \notin H^l_\alpha \cup D^l_\alpha$ and so $a \notin Z_\alpha$ at the current moment. It follows that b is the successor of a and b was introduced by an $\mathcal{N}_a$-strategy δ extending β at an earlier stage $r < s$ when a was already introduced but not restrained out of A. At stage r, the strategy α was in a previous step $k < n$. (Otherwise the $\mathcal{N}_a$-strategy δ would not impose a restraint on b at stage r.) This means that $a \in D^n_\alpha$, as the elements of H^n_α are introduced after stage r by strategies extending $\alpha^\frown\langle\mathrm{wait}_{n-1}\rangle$ and all these strategies are initialized at stage r (as $\delta \succeq \alpha^\frown\langle\infty\rangle$ and so α took outcome ∞ at stage r). In order for a to enter D^n_α, it must be restrained out of A when D^n_α is defined at stage t such that $r < t < s$. But then by Lemma 5.10 it follows that b cannot be restrained out of A at stage s.

Finally, let us assume b enters B_n. Then it has to be introduced by some node $\beta \succeq \alpha^\frown\langle\mathrm{wait}_n\rangle$, since otherwise it was already in B_n at the previous moment. Now, by freshness, the only reason it can be connected to a is if $b^- = a$ and a is currently not restrained out. Thus, $a \notin B^\infty$. $\square$

Lemma 5.12. *Let α be an $\mathcal{R}_{\Phi,\Psi}$-strategy along the true path. For all $n \in \omega$ and at any moment, for all $w < z^n_\alpha$, either $w = y^n_\alpha$ or $\langle x, w\rangle \in \Phi(F^n_\alpha)$.*

Proof. Note that this is true when we first enter Step 0, since z^0 is selected as the least z other than y^0 such that $\langle x, z\rangle \notin \Phi(F^0_\alpha)$. This statement is preserved when we revert back to a previous step (since it holds for the previous step). This statement is also preserved when we stay in the same step and increase F and z: We have one new value of w to consider, the old value of z. But we add X_0 to F to ensure that $\langle x, w\rangle \in \Phi(F)$. Similarly, this statement is preserved when we move to a new step; this is guaranteed by the choice of y^{n+1}, z^{n+1}, and the inclusion of H^n in F^{n+1}. $\square$

Lemma 5.13. *If α is an $\mathcal{R}_{\Phi,\Psi}$-strategy along the true path, then α ensures that $\mathcal{R}_{\Phi,\Psi}$ is satisfied.*

Proof. Let s be large enough so that α is never initialized after stage s and let s be least with that property. After stage s, at the first time at which α is visited, if it fails to find sets F and G in Step -1, then it places a finite restraint $a_0 \in A$ and it restrains out of A elements that were already restrained out of A by nodes below α that are not below the outcome stop of α, which are never injured. Note that strategies below the outcome stop of α are in their initial state and so any node $\preceq \alpha$ or $<_L \alpha$ that is currently restraining an element out of A will permanently do so, since the only way to injure this node is to visit some node to its left, which would injure α again as well. Thus the true set A is a subset of the set A that α looks at when it sees that $a_0 \notin \Psi(\Phi(A))$. Thus $a_0 \in A \setminus \Psi(\Phi(A))$, and the requirement is satisfied. If it finds sets F and G with $a_0 \notin F$, then again, it places a finite restraint $a_0 \notin A$, $F \subseteq A$, and the requirement is permanently satisfied since $a_0 \in \Psi(\Phi(A)) \setminus A$.

Now, suppose that the true outcome of α is wait_n for some n. By Lemma 5.3 we can also assume that s is large enough such that α is in Step n at stage s and after stage s, α is never reverted to any Step k with $k \leq n$. It follows that α has outcome wait_n at every stage after stage s. (Note that if α has any other outcome after stage s, it must either be initialized or be reverted to a Step $k \leq n$ in order to get to outcome wait_n again). Thus, in the algorithm, we always have $\langle x, z^n\rangle \in \Phi(\omega \setminus (W \cup B_\infty \cup Y))$ for every consistent set $Y \subseteq B_n$. (Note that we define B_∞ and B_n at the moment when z^n is first defined.) In particular, this is true for $Y = (\omega \setminus A) \cap B_n$ because Y is consistent by Lemma 5.11 (using Lemma 5.8 for the definition of A). Furthermore, the algorithm then replaces F^n by $F^n \cup X_0$, which ensures that $\langle x, z^n\rangle \in \Phi(X_0 \cup (B_n \setminus Y)) \subseteq \Phi(A)$. Thus, for all $z \neq y^n$ (which never changes after stage s), we have that $\langle x, z\rangle \in \Phi(A)$. Next, by Lemma 5.7(b), we see that no element of H^n (which never changes after stage s) is ever restrained out of A by any strategy, thus $F^n \cup H^n \subseteq A$, so $\langle x, y^n\rangle \in \Phi(A)$, showing that the entire x-th column is contained in $\Phi(A)$, so $\mathcal{R}_{\Phi,\Psi}$ is satisfied.

Next, suppose that the true outcome of α is ∞. Fix any $k \in \omega$; we argue that $\{\langle x, z\rangle \colon z \in [0, k]\} \subseteq \Phi(A)^{[x]}$. Assume s is large enough so that α has been visited by stage s after its final initialization. Let $t > s$ be a stage at which $\alpha^\frown\langle\infty\rangle$ is visited and which is large enough so that α is never reverted to Step n for $n \leq k + s + 1$ after stage t. Then, since every time the step is increased, y is increased as well, it follows that at every stage after t, if α is in Step n, then $z^n > y^n \geq n - s$. By Lemma 5.12, and since at every future stage, α restrains F^n_α into A, we see that $[0, k] \subseteq \Phi(A)^{[x]}$. Since this holds for every k, we have that every element of the x-th column is contained in $\Phi(A)$, thus $\mathcal{R}_{\Phi,\Psi}$ is satisfied.

Finally, suppose that the true outcome of α is stop and it is achieved via Step n. Let $t \geq s$ be a stage at which α takes outcome stop and no higher-priority strategy is ever initialized after stage t. Then α places a restraint keeping $H^n \cup D^n \cup B_\infty$ out of A which is never lifted. Since no higher-priority strategy is initialized, every element of W is permanently restrained out of A and W has remained the same since α's last initialization. Thus $A \subseteq \omega \setminus (W \cup B_\infty)$. This outcome means that

$\langle x, z^n \rangle \notin \Phi(\omega \smallsetminus (W \cup B_\infty))$, thus $\langle x, z^n \rangle \notin \Phi(A)$. Furthermore, since $H^n \cup D^n$ is restrained out of A, we have $\langle x, y^n \rangle \notin \Phi(A)$. Thus, $\mathcal{R}_{\Phi,\Psi}$ is satisfied. $\square$

This completes the proof of the theorem. $\square$

6. Open questions

In this section, we collect the open questions arising from this paper, some of which have already been asked.

6.1. **Definability.** As mentioned above, Kalimullin [11] showed that the enumeration jump is first-order definable. Is this also true for the skip?

Question 6.1. Is the skip first-order definable in the enumeration degrees?

Furthermore, we have discussed several cototality notions in this paper. Which of these are definable?

Question 6.2. Which cototality notions are first-order definable in the enumeration degrees?

Note that a positive answer to the first question would imply, by Proposition 1.1, that the cototal degrees are definable.

6.2. **Arithmetical zigzag.** In Section 3.2, we have shown that the skip can exhibit a form of *zigzag behavior*: There are degrees $\mathbf{a}$ such that none of the finite skips of $\mathbf{a}$ are total. However, the examples constructed there are not arithmetical. We suspect that this is not a coincidence.

Conjecture 6.3. If $\mathbf{a}$ is an arithmetical enumeration degree, then $\mathbf{a}^{\langle n \rangle}$ is total for some $n \in \omega$.

6.3. **Graph-cototal degrees.** Theorem 5.1 constructed a cototal Δ^0_3-degree that is not graph-cototal. On the other hand, Proposition 2.1 proves that every Σ^0_2-degree is graph-cototal. This leaves the following open:

Question 6.4. Is every Π^0_2 cototal enumeration degree graph-cototal?

We do not know of a simpler proof of the existence of a cototal enumeration degree that is not graph-cototal. A more informative separation result would be derived from a positive answer to the following question:

Question 6.5. Is there a continuous enumeration degree that is not graph-cototal?

6.4. **Skip cototality.** Let us say that a degree $\mathbf{a}$ is *skip cototal* if $\mathbf{a}^\diamond$ is total. Notice that every skip cototal degree $\mathbf{a}$ is weakly cototal, and that every cototal degree is skip cototal. Furthermore, note that in the proofs of Proposition 4.4 and Corollary 4.6, we in fact constructed a degree $\mathbf{a}$ that is skip cototal but not cototal. Even the alternative example of a weakly cototal degree given by Badillo and Harris [2]—the degree that is entirely composed of properly Π^0_2-sets—is also a skip cototal degree.

Conjecture 6.6. Every weakly cototal degree $\mathbf{a}$ is skip cototal.

As mentioned above, every Π^0_2-degree is weakly cototal. Therefore, a proof of our conjecture would in particular imply that the skip of every Π^0_2-degree is total, which is also open.

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