A Jump Inversion Theorem for the Degree Spectra

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Abstract. In the present paper we continue the study of the properties of the spectra of structures as sets of degrees initiated in [11]. Here we consider the relationships between the spectra and the jump spectra. Our first result is that every jump spectrum is also a spectrum. The main result sounds like a Jump inversion theorem. Namely, we show that if a spectrum $\mathcal{A}$ is contained in the set of the jumps of the degrees in some spectrum $\mathcal{B}$ then there exists a spectrum $\mathcal{C}$ such that $\mathcal{C} \subseteq \mathcal{B}$ and $\mathcal{A}$ is equal to the set of the jumps of the degrees in $\mathcal{C}$.
Key words: Turing degrees; degree spectra; forcing; Marker's extensions; enumerations.

1 Introduction

Let $\mathfrak{A} = (A; R_1, \dots, R_s)$ be a countable structure, where the set A is infinite, each $R_i \subseteq A^{r_i}$ and the equality $=$ is among $R_1, \dots, R_s$.

The notion of a degree spectrum of a countable structure is introduced by Richter [9] and further studied by Ash, Downey, Jockush and Knight [1, 3, 5].

An enumeration f of $\mathfrak{A}$ is a total mapping of $\mathbb{N}$ onto A.

Given a set $R \subseteq A^a$ and an enumeration f of $\mathfrak{A}$, let

$$f^{-1}(R) = \{\langle x_1, \dots, x_a \rangle \mid (f(x_1), \dots, f(x_a)) \in R\}.$$

Let $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \dots \oplus f^{-1}(R_s)$.

Definition 1. *The degree spectrum of* $\mathfrak{A}$ is the set

$$\mathrm{DS}(\mathfrak{A}) = \{d_{\mathrm{T}}(f^{-1}(\mathfrak{A})) \mid f \text{ is an enumeration of } \mathfrak{A}\} \ .$$

Here by $d_{\mathrm{T}}(B)$ we denote the Turing degree of the set B.

We shall use the following two simple properties of the degree spectra, proved in [11]:

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Proposition 2. *Let f be an arbitrary enumeration of $\mathfrak{A}$. Then there exists an injective enumeration g of $\mathfrak{A}$ such that $g^{-1}(\mathfrak{A}) \leq_{\mathrm{T}} f^{-1}(\mathfrak{A})$.*

Definition 3. A set of Turing degrees $\mathcal{A}$ is *closed upwards* if for all Turing degrees $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \in \mathcal{A}$ & $\mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{b} \in \mathcal{A}$.

Proposition 4. *For every structure $\mathfrak{A}$ the degree spectrum* $\mathrm{DS}(\mathfrak{A})$ *is closed upwards.*

Let us point out that our notion of degree spectrum differs slightly from the one introduced in [9] where the degree spectrum of a structure is defined to be the set of all Turing degrees $\mathrm{d_T}(f^{-1}(\mathfrak{A}))$ for injective enumerations f of $\mathfrak{A}$. The benefit of considering arbitrary enumerations is that in this way we ensure that the degree spectrum of every structure is closed upwards, which is not always true if we consider only injective enumerations.

Definition 5. *The jump spectrum* of $\mathfrak{A}$ is the set $\mathrm{DS}_1(\mathfrak{A}) = \{\mathbf{a}' \mid \mathbf{a} \in \mathrm{DS}(\mathfrak{A})\}$.

In the present paper we prove two results about the relationships between jump spectra and spectra of structures. The first result states that every jump spectrum is a spectrum. Our second result is a jump inversion theorem for the degree spectra. We prove that if $\mathfrak{A}$ and $\mathfrak{B}$ are structures and $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_1(\mathfrak{B})$ then there exists a structure $\mathfrak{C}$ such that $\mathrm{DS}(\mathfrak{C}) \subseteq \mathrm{DS}(\mathfrak{B})$ and $\mathrm{DS}_1(\mathfrak{C}) = \mathrm{DS}(\mathfrak{A})$.

The structure $\mathfrak{C}$ is constructed as a Marker's extension [7] of $\mathfrak{A}$, an idea influenced by the results of Goncharov and Khoussainov [4].

Some applications are presented in the last part of the paper.

2 Every Jump Spectrum is a Spectrum

In this section we show that for every structure $\mathfrak{A}$ there exists a structure $\mathfrak{B}$ such that $\mathrm{DS}_1(\mathfrak{A}) = \mathrm{DS}(\mathfrak{B})$. The structure $\mathfrak{B}$ is constructed in two stages. First, we define the least acceptable extension $\mathfrak{A}^*$ of $\mathfrak{A}$ which we call *Moschovakis' extension* of $\mathfrak{A}$. Roughly speaking $\mathfrak{A}^*$ is an extension of $\mathfrak{A}$ with additional coding machinery. Using this coding machinery we define the set $K_{\mathfrak{A}}$ which is an analogue of Kleene's set K. Finally we set $\mathfrak{B} = (\mathfrak{A}^*, K_{\mathfrak{A}})$.

2.1 Moschovakis' Extension of the Structure $\mathfrak{A}$

Let $\mathfrak{A} = (A; R_1, \ldots, R_s)$ be a countable structure and let equality be among the predicates $R_1, \ldots, R_s$. Following Moschkovakis [8] the least acceptable extension of the structure $\mathfrak{A}$ is defined as follows.

Let 0 be an object which does not belong to A and Π be a pairing operation chosen so that neither 0 nor any element of A is an ordered pair. Let A^* be the least set containing all elements of $A_0 = A \cup \{0\}$ and closed under Π.

We associate an element n^* of A^* with each natural number n by induction:

$$\begin{aligned} 0^* &= 0; \\ (n+1)^* &= \Pi(0, n^*). \end{aligned}$$

The set of all elements n^* defined above will be denoted by N^*.

Let L and R be the functions on A^* satisfying the following conditions:

$$\begin{array}{l} L(0) = R(0) = 0; \\ (\forall t \in A)(L(t) = R(t) = 1^*); \\ (\forall s, t \in A^*)(L(\Pi(s,t)) = s \ \&\ R(\Pi(s,t)) = t). \end{array}$$

The pairing function allows us to code finite sequences of elements: let $\Pi_1(t_1) = t_1$, $\Pi_{n+1}(t_1, t_2, \ldots, t_{n+1}) = \Pi(t_1, \Pi_n(t_2, \ldots, t_{n+1}))$ for every $t_1, t_2, \ldots, t_{n+1} \in A^*$.

For each predicate R_i of the structure $\mathfrak{A}$ define the respective predicate R_i^* on A^* by

$$R_i^*(t) \iff (\exists a_1 \in A) \ldots (\exists a_{r_i} \in A)(t = \Pi_{r_i}(a_1, \ldots, a_{r_i}) \ \&\ R_i(a_1, \ldots, a_{r_i})).$$

Definition 6. *Moschovakis' extension of* $\mathfrak{A}$ is the structure

$$\mathfrak{A}^* = (A^*; A_0, R_1^*, \ldots, R_s^*, G_\Pi, G_L, G_R, =),$$

where G_Π, $G_L$and G_R are the graphs of Π, L and R respectively.

Lemma 7. *Let f be an enumeration of $\mathfrak{A}$. There exists an enumeration f^* of $\mathfrak{A}^*$ such that $(f^*)^{-1}(\mathfrak{A}^*) \equiv_{\mathrm{T}} f^{-1}(\mathfrak{A})$.*

Proof. Let $J(x, y) = 2^{x+1}.(2y + 1)$ be an effective coding of the ordered pairs of natural numbers. Denote by induction $J_1(x_1) = x_1$ and $J_{n+1}(x_1, x_2, \ldots, x_{n+1}) = J(x_1, J_n(x_2, \ldots, x_{n+1}))$ for any $x_1, x_2, \ldots, x_{n+1} \in \mathbb{N}$. And let l and r be computable functions satisfying the equalities:

$$\begin{array}{l} l(0) = r(0) = 0, \\ l(2x + 1) = r(2x + 1) = 2 = J(0, 0), \\ l(J(x, y)) = x \ \&\ r(J(x, y)) = y. \end{array}$$

Define f^* by means of the following inductive definition:

$$\begin{array}{ll} f^*(0) & = 0^*, \\ f^*(2x + 1) & = f(x), \\ f^*(J(x, y)) & = \Pi(f^*(x), f^*(y)). \end{array}$$

Clearly f^* is an enumeration of $\mathfrak{A}^*$. It is easy to see that $(f^*)^{-1}(A_0) = \{2x + 1 \mid x \in \mathbb{N}\} \cup \{0\}$, $(f^*)^{-1}(G_\Pi) = \{\langle x, y\rangle : (x, y) \in G_J\}$, $(f^*)^{-1}(G_L) = \{\langle x, y\rangle : (x, y) \in G_l\}$ and $(f^*)^{-1}(G_R) = \{\langle x, y\rangle : (x, y) \in G_r\}$.

Fix a natural number i, $1 \leq i \leq s$. Then

$$\begin{array}{l} \langle x_1, \ldots, x_{r_i}\rangle \in f^{-1}(R_i) \iff (f(x_1), \ldots, f(x_{r_i})) \in R_i \iff \\ (f^*(2x_1 + 1), \ldots, f^*(2x_{r_i} + 1)) \in R_i \iff \\ \Pi_{r_i}(f^*(2x_1 + 1), \ldots, f^*(2x_{r_i} + 1))) \in R_i^* \iff \\ J_{r_i}(2x_1 + 1, \ldots, 2x_{r_i} + 1) \in (f^*)^{-1}(R_i^*). \end{array}$$

Finally, let R_1 be the equality on A. Then

$$\begin{aligned}\langle x,y\rangle \in (f^*)^{-1}(=) \iff & [\ (x,y \in (f^*)^{-1}(A)\ \&\ \langle x/2, y/2\rangle \in f^{-1}(R_1))\vee \\ & (x=y=0)\vee \\ & (x = J(x_1,x_2)\ \&\ y = J(y_1,y_2)\ \& \\ & \langle x_1,y_1\rangle \in (f^*)^{-1}(=)\ \&\ \langle x_2,y_2\rangle \in (f^*)^{-1}(=))].\end{aligned}$$

Clearly

$$\langle x,y\rangle \in f^{-1}(R_1) \iff \langle 2x+1, 2y+1\rangle \in (f^*)^{-1}(=).$$

So $(f^*)^{-1}(=) \equiv_{\mathrm{T}} f^{-1}(R_1)$.

Combining all above, we get that $(f^*)^{-1}(\mathfrak{A}^*) \equiv_{\mathrm{T}} f^{-1}(\mathfrak{A})$.

From now on given an enumeration f of the structure $\mathfrak{A}$, by f^* we shall denote the enumeration of $\mathfrak{A}^*$ defined in the lemma above.

Proposition 8. $\mathrm{DS}(\mathfrak{A}) = \mathrm{DS}(\mathfrak{A}^*)$.

Proof. Let $\mathbf{a} \in \mathrm{DS}(\mathfrak{A})$ and let f be an enumeration of $\mathfrak{A}$ witnessing this, i.e. $f^{-1}(\mathfrak{A}) \in \mathbf{a}$. Then $(f^*)^{-1}(\mathfrak{A}^*) \equiv_{\mathrm{T}} f^{-1}(\mathfrak{A})$ and hence $\mathbf{a} \in \mathrm{DS}(\mathfrak{A}^*)$.

Now let $\mathbf{a} \in \mathrm{DS}(\mathfrak{A}^*)$ and let h be an enumeration of $\mathfrak{A}^*$ with $h^{-1}(\mathfrak{A}^*) \in \mathbf{a}$. By Proposition 2, there exists an injective enumeration g of $\mathfrak{A}^*$ such that $g^{-1}(\mathfrak{A}^*) \leq_{\mathrm{T}} h^{-1}(\mathfrak{A}^*)$. Our goal is to construct an enumeration f of $\mathfrak{A}$ such that $f^{-1}(\mathfrak{A}) \leq_{\mathrm{T}} g^{-1}(\mathfrak{A}^*)$. Then by Proposition 4 we would get that $\mathbf{a} \in \mathrm{DS}(\mathfrak{A})$.

Let $0^{\#} = g^{-1}(0^*)$. Then the set $g^{-1}(A) = g^{-1}(A_0) \setminus \{0^{\#}\}$ is computable in $g^{-1}(\mathfrak{A}^*)$. Fix an element $z_0 \in g^{-1}(A)$ and let

$$\begin{aligned} & m(0) = z_0; \\ & m(i+1) = \mu z \in g^{-1}(A)[(\forall k \leq i)(m(k) \neq z)]. \end{aligned}$$

Note that $m \leq_{\mathrm{T}} g^{-1}(\mathfrak{A}^*)$ is a bijective enumeration of $g^{-1}(A)$. Let

$$J(x,y) = g^{-1}(\Pi(g(x), g(y))).$$

Clearly J is computable in $g^{-1}(\mathfrak{A}^*)$. As usual set $J_1(x) = x$ and

$$J_{n+1}(x_1,\dots,x_{n+1}) = J(x_1, J_n(x_2,\dots,x_{n+1})).$$

Set $f = \lambda x.g(m(x))$. Clearly f is an injective enumeration of the structure $\mathfrak{A}$. Consider a predicate R_i of $\mathfrak{A}$. Then

$$f^{-1}(R_i) = \{\langle x_1,\dots,x_{r_i}\rangle : J_{r_i}(m(x_1),\dots,m(x_{r_i})) \in g^{-1}(R_i^*)\}$$

and hence $f^{-1}(R_i)$ is computable in $g^{-1}(\mathfrak{A}^*)$.

Thus $f^{-1}(\mathfrak{A}) \leq_{\mathrm{T}} g^{-1}(\mathfrak{A}^*)$.

2.2 The set $K_{\mathfrak{A}}$

Let f be an enumeration of $\mathfrak{A}$. Given natural numbers e and x let

$$f \models F_e(x) \iff x \in W_e^{f^{-1}(\mathfrak{A})}$$

and let

$$f \models \neg F_e(x) \iff f \not\models F_e(x).$$

We shall connect with the modelling relation "$\models$" a forcing with conditions all finite mappings of $\mathbb{N}$ into A ordered in the usual way. We shall call these finite mappings *finite parts*. The finite parts will be denoted by the letters δ, τ.

Given a finite part δ and $\bar{R} \subseteq A^n$, let $\delta^{-1}(\bar{R})$ be the finite function on the natural numbers taking values in $\{0,1\}$ such that

$$\begin{aligned}
&\delta^{-1}(\bar{R})(u) \simeq 1 \iff (\exists x_1, \ldots x_n \in dom(\delta))(u = \langle x_1, \ldots, x_n \rangle \ \& \\
&(\delta(x_1), \ldots, \delta(x_n)) \in \bar{R}) \text{ and} \\
&\delta^{-1}(\bar{R})(u) \simeq 0 \iff (\exists x_1, \ldots x_n \in dom(\delta))(u = \langle x_1, \ldots, x_n \rangle \ \& \\
&(\delta(x_1), \ldots, \delta(x_n)) \notin \bar{R}).
\end{aligned} \tag{1}$$

By $\delta^{-1}(\mathfrak{A})$ we shall denote the finite function $\delta^{-1}(R_1) \oplus \ldots \oplus \delta^{-1}(R_s)$.

If α is a partial function and $e \in \mathbb{N}$, then by W_e^α we shall denote the set of all x such that the computation $\{e\}^\alpha(x)$ halts successfully. We shall assume that if during a computation the oracle α is called with an argument outside it's domain, then the computation halts unsuccessfully.

Definition 9. For any $e, x \in \mathbb{N}$ and for every finite part δ, define *the forcing relations* $\delta \Vdash F_e(x)$ and $\delta \Vdash \neg F_e(x)$ as follows:

$$\delta \Vdash F_e(x) \iff x \in W_e^{\delta^{-1}(\mathfrak{A})}$$

$$\delta \Vdash \neg F_e(x) \iff (\forall \tau \supseteq \delta)(\tau \nVdash F_e(x)).$$

The following two properties of the forcing relation are obvious:

(F1) $\delta \Vdash (\neg)F_e(x) \ \& \ \delta \subseteq \tau \Rightarrow \tau \Vdash (\neg)F_e(x)$.
(F2) For every enumeration f of $\mathfrak{A}$,

$$f \models F_e(x) \iff (\exists \tau \subseteq f)(\tau \Vdash F_e(x)).$$

Definition 10. An enumeration f of $\mathfrak{A}$ is *generic* if for every $e, x \in \mathbb{N}$:

$$(\exists \tau \subseteq f)(\tau \Vdash F_e(x) \vee \tau \Vdash \neg F_e(x)).$$

Clearly for every generic enumeration f of $\mathfrak{A}$ for all $e, x \in N$,

$$f \models \neg F_e(x) \iff (\exists \tau \subseteq f)(\tau \Vdash \neg F_e(x)).$$

With each finite part $\tau \neq \emptyset$ such that $\mathrm{dom}(\tau) = \{x_1, \ldots, x_n\}$ and $\tau(x_1) = s_1, \ldots, \tau(x_n) = s_n$, we associate the element $\tau^* = \Pi_n(\Pi(x_1^*, s_1), \ldots, \Pi(x_n^*, s_n))$ of A^*. Let $\tau^* = 0$ if $\tau = \emptyset$.

Define $K_{\mathfrak{A}} = \{\Pi_3(\delta^*, e^*, x^*) \mid (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\ \&\ e^*, x^* \in N^*\}$.
Let

$$\mathfrak{A}^*_K = (A^*; A_0, R^*_1, \ldots, R^*_s, G_\Pi, G_L, G_R, =, K_{\mathfrak{A}}).$$

The following proposition follows directly from Lemma 7.

Proposition 11. *Let f be an enumeration of $\mathfrak{A}$. Then*

$$(f^*)^{-1}(\mathfrak{A}^*_K) \equiv_{\mathrm{T}} f^{-1}(\mathfrak{A}) \oplus (f^*)^{-1}(K_{\mathfrak{A}}).$$

2.3 Every Jump Spectrum is Spectrum

Theorem 12. *For every structure $\mathfrak{A}$ there exists a structure $\mathfrak{B}$ such that* $\mathrm{DS}_1(\mathfrak{A}) = \mathrm{DS}(\mathfrak{B})$.

Proof. Let $\mathfrak{B} = \mathfrak{A}^*_K$ defined above. We shall prove that $\mathrm{DS}_1(\mathfrak{A}) = \mathrm{DS}(\mathfrak{B})$. We divide the proof into two parts.

Proposition 13. $\mathrm{DS}_1(\mathfrak{A}) \subseteq \mathrm{DS}(\mathfrak{B})$.

Proof. Let $\mathbf{a} \in \mathrm{DS}_1(\mathfrak{A})$ and let g be an enumeration of $\mathfrak{A}$ such that $g^{-1}(\mathfrak{A})' \in \mathbf{a}$. By Proposition 2, there exists an injective enumeration f of $\mathfrak{A}$ such that $f^{-1}(\mathfrak{A}) \leq_{\mathrm{T}} g^{-1}(\mathfrak{A})$. Since $f^{-1}(\mathfrak{A})' \leq_{\mathrm{T}} g^{-1}(\mathfrak{A})'$ and $\mathrm{DS}(\mathfrak{B})$ is closed upwards, it is sufficient to show that $d_{\mathrm{T}}(f^{-1}(\mathfrak{A}))' \in \mathrm{DS}(\mathfrak{B})$. For we shall show that $(f^*)^{-1}(\mathfrak{B}) \leq_T f^{-1}(\mathfrak{A})'$ and use once more the fact that $\mathrm{DS}(\mathfrak{B})$ is closed upwards.

From the construction of the enumeration f^* in the proof of Lemma 7 it follows that f^* is also injective.

Recall the definition of the subset $N^* = \{x^* : x \in N\}$ of A^*. For every natural number x let $x^\# = (f^*)^{-1}(x^*)$ and let $N^\# = \{x^\# : x \in N\} = (f^*)^{-1}(N^*)$. Notice that $0^\# = 0$ and $(x+1)^\# = J(0, x^\#)$ and hence $N^\#$ is a computable set. Clearly there exist computable functions n_1 and n_2 such that for all natural numbers x, $n_1(x^\#) = x$ and $n_2(x) = x^\#$.

Denote by Δ the set of all finite parts in $\mathfrak{A}$. Clearly for every finite part τ, there exists a unique element τ^* of A^* defined as in the previous section and a unique natural number $\tau^\# = (f^*)^{-1}(\tau^*)$.

Let $\Delta^* = \{\tau^* : \tau \in \Delta\}$ and $\Delta^\# = \{\tau^\# : \tau \in \Delta\} = (f^*)^{-1}(\Delta^*)$.

It is easy to see that a number $\tau^\#$ belongs to $\Delta^\#$ if and only if $\tau^\# = 0$ or for some $n \geq 1$ there exist n distinct elements ${x_1}^\#, \ldots, {x_n}^\#$ of $N^\#$ and n odd numbers $y_1, \ldots, y_n$ such that

$$\tau^\# = J_n(J({x_1}^\#, y_1), \ldots, J({x_n}^\#, y_n)).$$

Therefore the set $\Delta^\#$ is also computable.

Given a $\tau^\# = J_n(J({x_1}^\#, y_1), \ldots, J({x_n}^\#, y_n)) \in \Delta^\#$, let

$$dom(\tau^\#) = \{{x_1}^\#, \ldots, {x_n}^\#\}$$

and for every ${x_i}^\# \in dom(\tau^\#)$, set $\tau^\#({x_i}^\#) \simeq y_i$.

We shall assume that $dom(\tau^\#) = \emptyset$ if $\tau^\# = 0$.

Notice that $dom(\tau^\#) = \{x^\# : x \in dom(\tau)\}$ and for every $x \in dom(\tau)$, $f^*(\tau^\#(x^\#)) \simeq f(\tau^\#(x^\#)/2) \simeq \tau(x)$.

Let $\bar{R} \subseteq A^n$ and $\tau \in \Delta$. Recall the definition of the finite function $\tau^{-1}(\bar{R})$ given in the previous section. Clearly

$$\begin{aligned} &\tau^{-1}(\bar{R})(u) \simeq 1 \iff (\exists x_1{}^\#, \dots, x_n{}^\# \in dom(\tau^\#))(u = \langle x_1, \dots, x_n \rangle \ \& \\ &\langle \tau^\#(x_1{}^\#)/2, \dots, \tau^\#(x_n{}^\#)/2 \rangle \in f^{-1}(\bar{R})) \end{aligned} \tag{2}$$

and

$$\begin{aligned} &\tau^{-1}(\bar{R})(u) \simeq 0 \iff (\exists x_1{}^\#, \dots, x_n{}^\# \in dom(\tau^\#))(u = \langle x_1, \dots, x_n \rangle \ \& \\ &\langle \tau^\#(x_1{}^\#)/2, \dots, \tau^\#(x_n{}^\#)/2 \rangle \notin f^{-1}(\bar{R})). \end{aligned} \tag{3}$$

By (2) and (3), there exists a computable function ρ such that for every $\tau \in \Delta$, $\tau^{-1}(\mathfrak{A}) = \{\rho(\tau^\#)\}^{f^{-1}(\mathfrak{A})}$.

It is easy to see that there exists a computable predicate P such that for all $\tau, \delta \in \Delta$, $P(\tau^\#, \delta^\#) \simeq 1 \iff \tau \subseteq \delta$.

Thus we obtain that

$$\begin{aligned} &(f^*)^{-1}(K_{\mathfrak{A}}) = \{J_3(\delta^\#, e^\#, x^\#) : (\exists \tau \in \Delta)(\delta \subseteq \tau \ \& \ \tau \Vdash F_e(x))\} = \\ &\{J_3(\delta^\#, e^\#, x^\#) : (\exists \tau^\# \in \Delta^\#)(P(\delta^\#, \tau^\#) \simeq 1 \ \& \ x \in W_e^{\{\rho(\tau^\#)\}^{f^{-1}(\mathfrak{A})}})\}. \end{aligned}$$

Hence $(f^*)^{-1}(K_{\mathfrak{A}})$ is c.e. in $f^{-1}(\mathfrak{A})$. From here it follows that $(f^*)^{-1}(K_{\mathfrak{A}}) \leq_{\mathrm{T}} f^{-1}(\mathfrak{A})'$. Therefore, by Proposition 11, $(f^*)^{-1}(\mathfrak{B}) \leq_{\mathrm{T}} f^{-1}(\mathfrak{A})'$.

Now we turn to the proof of the reverse inclusion. We shall need the following property of the jump spectrum:

Lemma 14. *Every jump spectrum is closed upwards.*

Proof. Consider a structure $\mathfrak{A}$. Let $\mathbf{b}$ be a degree, $\mathbf{b} \geq \mathbf{a}$ and $\mathbf{a} \in \mathrm{DS}_1(\mathfrak{A})$. Then for some $\mathbf{c} \in \mathrm{DS}(\mathfrak{A})$, $\mathbf{c}' = \mathbf{a}$. By the relativized jump inversion theorem of Friedberg, there is a degree $\mathbf{d} \geq \mathbf{c}$ such that $\mathbf{d}' = \mathbf{b}$. By Proposition 4, $\mathbf{d} \in \mathrm{DS}(\mathfrak{A})$. Thus $\mathbf{b} = \mathbf{d}' \in \mathrm{DS}_1(\mathfrak{A})$.

Proposition 15. $\mathrm{DS}(\mathfrak{B}) \subseteq \mathrm{DS}_1(\mathfrak{A})$.

Proof. Let $\mathbf{a} \in \mathrm{DS}(\mathfrak{B})$ and m be an enumeration of $\mathfrak{B}$ such that $m^{-1}(\mathfrak{B}) \in \mathbf{a}$. By Proposition 2, there exists an injective enumeration f of $\mathfrak{B}$ such that $f^{-1}(\mathfrak{B}) \leq_{\mathrm{T}} m^{-1}(\mathfrak{B})$. We shall construct an enumeration g of the structure $\mathfrak{A}$ such that $g^{-1}(\mathfrak{A})' \leq_{\mathrm{T}} f^{-1}(\mathfrak{B})$. Then, by Lemma 14, $\mathbf{a} \in \mathrm{DS}_1(\mathfrak{A})$.

Recall that $\mathfrak{B} = \mathfrak{A}^*_K$. Let $f^{-1}(A) = A^\#$ and $f^{-1}(K_{\mathfrak{A}}) = K^\#$. Clearly the sets $A^\#$ and $K^\#$ are computable in $f^{-1}(\mathfrak{B})$. Define the computable in $f^{-1}(\mathfrak{B})$ function J by $J(x, y) = f^{-1}(\Pi(f(x), f(y)))$. Clearly there exist computable in $f^{-1}(\mathfrak{B})$ functions l and r such that for all $x, y \in N$,

$$l(J(x, y)) = x \text{ and } r(J(x, y)) = y.$$

Set $J_1(x_1) = x_1$ and $J_{n+1}(x_1, \dots, x_{n+1}) = J(x_1, J_n(x_2, \dots, x_{n+1}))$.

For every natural number x consider the element x^* of A^* and let $x^\# = f^{-1}(x^*)$. Let $N^\# = \{x^\# : x \in N\}$. Now, we have that $N^\#$ is computable in $f^{-1}(\mathfrak{B})$ and that there exist computable in $f^{-1}(\mathfrak{B})$ functions n_1 and n_2 such that for all $x \in N$, $n_1(x) = x^\#$ and $n_2(x^\#) = x$.

Given a partial mapping h of N in A, by $h^\#$ we shall denote the unique mapping of $N^\#$ in $A^\#$ satisfying for all natural numbers x the equality:

$$h^\#(x^\#) \simeq f^{-1}(h(x)).$$

Clearly for all partial mappings h_1 and h_2 of N in A,

$$h_1 \subseteq h_2 \iff {h_1}^\# \subseteq {h_2}^\#.$$

For finite parts τ we shall identify $\tau^\#$ and its code $f^{-1}(\tau^*)$. Denote by Δ the set of all finite parts and let $\Delta^\# = \{\tau^\# : \tau \in \Delta\}$. Notice that the set $\Delta^\#$ is computable in $f^{-1}(\mathfrak{B})$.

As in the proof of the previous proposition one can easily see that there exists a computable in $f^{-1}(\mathfrak{B})$ function ρ such for every finite part τ, $\tau^{-1}(\mathfrak{A}) = \{\rho(\tau^\#)\}^{f^{-1}(\mathfrak{B})}$.

Now we turn to the construction of the enumeration g. We shall construct g as a generic enumeration such that $g^\#$ is computable in $f^{-1}(\mathfrak{B})$.

The enumeration g will be constructed by stages. At each stage s we shall define a finite part τ_s so that $\tau_s \subseteq \tau_{s+1}$ and let $g = \bigcup_s \tau_s$.

From the construction it will follow that the function $\lambda s.{\tau_s}^\#$ is computable in $f^{-1}(\mathfrak{B})$ and hence the mapping $g^\#$ is also computable in $f^{-1}(\mathfrak{B})$.

We shall consider two kinds of stages. On stages $s = 2r$ we shall ensure that the mapping g is total and surjective. On stages $s = 2r + 1$ we shall ensure that g is generic.

Let $\tau_0 = \emptyset$. Suppose that we have already defined τ_s.

(a) Case $s = 2r$. Let x be the least natural number such that $x^\#$ does not belong to $\mathrm{dom}({\tau_s}^\#)$ and let y be the least natural number in $A^\#$ which does not belong to the range of ${\tau_s}^\#$. Set $\tau_{s+1}(x) = f(y)$ and $\tau_{s+1}(z) \simeq \tau_s(z)$ for $z \neq x$.

(b) Case $s = 2\langle e, x\rangle + 1$. Consider the set $X_{\langle e,x\rangle} = \{\delta \mid \delta \Vdash F_e(x)\}$. Check whether there exists a finite part $\delta \in X_{\langle e,x\rangle}$ which extends τ_s. Clearly this is equivalent to $J_3({\tau_s}^\#, e^\#, x^\#) \in K^\#$.

If the answer is negative then $\tau_s \Vdash \neg F_e(x)$. Set $\tau_{s+1} = \tau_s$.

In the case of a positive answer find a $\delta^\#$ such that ${\tau_s}^\# \subseteq \delta^\#$ and

$$x \in W_e^{\{\rho(\delta^\#)\}^{f^{-1}(\mathfrak{B})}}.$$

We can do that effectively in $f^{-1}(\mathfrak{B})$ by enumerating all triples $(\delta^\#, t_1, t_2)$, where ${\tau_s}^\# \subseteq \delta^\#$, $t_1, t_2 \in N$ and checking for every such triple whether

$$x \in W_{e,t_1}^{\{\rho(\delta^\#)\}_{t_2}^{f^{-1}(\mathfrak{B})}}.$$

Set $\tau_{s+1} = \delta$.

End of the construction
By the genericity of g,

$$x \in g^{-1}(\mathfrak{A})' \iff g \models F_x(x) \iff (\exists \tau \subseteq g)(\tau \Vdash F_x(x)) \iff (\exists \tau^{\#} \subseteq g^{\#})(x \in W_x^{\{\rho(\tau^{\#})\}^{f^{-1}(\mathfrak{B})}}).$$

and

$$x \in N \setminus g^{-1}(\mathfrak{A})' \iff g \models \neg F_x(x) \iff (\exists \tau \subseteq g)(\tau \Vdash \neg F_x(x)) \iff (\exists \tau^{\#} \subseteq g^{\#})(J_3(\tau^{\#}, x^{\#}, x^{\#}) \notin K^{\#}).$$

Since $g^{\#}$ is computable in $f^{-1}(\mathfrak{B})$, we get from here that $g^{-1}(\mathfrak{A})'$ and $N \setminus g^{-1}(\mathfrak{A})'$ are c.e. in $f^{-1}(\mathfrak{B})$ and hence $g^{-1}(\mathfrak{A})' \leq_{\mathrm{T}} f^{-1}(\mathfrak{B})$.

The proof of the theorem is concluded.

3 Marker's Extensions

Marker [7] presented a method of constructing for any $n \geq 1$ an $\aleph_0$-categorical almost strongly minimal theory which is not Σ_n-axiomatizable. Further Goncharov and Khoussainov [4] adapted the construction to the general case in order to find for any $n \geq 1$ examples of $\aleph_1$-categorical computable models as well as $\aleph_0$-categorical computable models whose theories are Turing equivalent to $\emptyset^{(n)}$. We shall give the definition of Marker's $\exists$ and $\forall$ extensions following [4].

Let $\mathfrak{A} = (A; R_1, \dots, R_s, =)$ be a countable structure such that each predicate R_i has arity r_i.

Marker's $\exists$-extension of R_i, denoted by $R_i^{\exists}$, is defined as follows. Consider a set X_i with new elements such that $X_i = \{x^i_{\langle a_1,\dots,a_{r_i}\rangle} \mid R_i(a_1,\dots,a_{r_i})\}$. We shall call the set X_i an $\exists$-fellow for R_i. We suppose that all sets A, X_1,..., X_s are pairwise disjoint.

The predicate $R_i^{\exists}$ is a predicate of arity $r_i + 1$ such that

$$R_i^{\exists}(a_1,\dots,a_{r_i},x) \iff a_1,\dots,a_{r_i} \in A \;\&\; x \in X_i \;\&\; x = x^i_{\langle a_1,\dots,a_{r_i}\rangle}.$$

The property of $R_i^{\exists}$ is that for every $a_1,\dots,a_{r_i} \in A$

$$(\exists x \in X_i) R_i^{\exists}(a_1,\dots,a_{r_i},x) \iff R_i(a_1,\dots,a_{r_i}). \tag{4}$$

Definition 16. The structure $\mathfrak{A}^{\exists}$ is defined as follows:

$$(A \cup \bigcup_{i=1}^{s} X_i; R_1^{\exists},\dots,R_s^{\exists}, X_1,\dots,X_s, =),$$

where each $R_i^{\exists}$ is the Marker's $\exists$-extension of R_i with the $\exists$-fellow X_i.

Further, Marker's $\forall$-extension of $R_i^{\exists}$, denoted by $R_i^{\exists\forall}$, is defined as follows. Consider an infinite set Y_i of new elements such that

$$Y_i = \{y^i_{\langle a_1,\dots,a_{r_i},x\rangle} : \neg R_i^{\exists}(a_1,\dots,a_{r_i},x) \ \&\ a_1,\dots,a_{r_i} \in A,\ \&\ x \in X_i\}.$$

We shall call the set Y_i a $\forall$-fellow for $R_i^{\exists}$. We suppose that all sets A, $X_1,\dots$, X_s and $Y_1,\dots$, Y_s are pairwise disjoint.

The predicate $R_i^{\exists\forall}$ is a predicate of arity r_i+2 such that

1. If $R_i^{\exists\forall}(a_1,\dots,a_{r_i},x,y)$ then $a_1,\dots,a_{r_i} \in A$, $x \in X_i$ and $y \in Y_i$;
2. If $a_1,\dots,a_{r_i} \in A,\ \&\ x \in\ X_i\ \&\ y \in Y_i$ then

$$\neg R_i^{\exists\forall}(a_1,\dots,a_{r_i},x,y) \iff y = y^i_{\langle a_1,\dots,a_{r_i},x\rangle}\ .$$

From the definition of $R_i^{\exists\forall}$ it follows that if $a_1,\dots,a_{r_i} \in A$ and $x \in X_i$ then

$$(\forall y \in Y_i) R_i^{\exists\forall}(a_1,\dots,a_{r_i},x,y) \iff R_i^{\exists}(a_1,\dots,a_{r_i},x). \tag{5}$$

Definition 17. The structure $\mathfrak{A}^{\exists\forall}$ is defined as follows

$$(A \cup \bigcup_{i=1}^{s} X_i \cup \bigcup_{i=1}^{s} Y_i; R_1^{\exists\forall},\dots,R_s^{\exists\forall}, X_1,\dots,X_s,Y_1,\dots,Y_s,=),$$

where X_i is the $\exists$-fellow for R_i and Y_i is the $\forall$-fellow for $R_i^{\exists}$.

The structure $\mathfrak{A}^{\exists\forall}$ has the following properties:

Proposition 18. *1. Let $a_1,\dots,a_{r_i} \in A$. Then:*

(a) $R_i(a_1,\dots,a_{r_i}) \iff (\exists x \in X_i)(\forall y \in Y_i) R_i^{\exists\forall}(a_1,\dots,a_{r_i},x,y)$;

(b) If $R_i(a_1,\dots,a_{r_i})$ then there exists a unique $x \in X_i$ such that $(\forall y \in Y_i) R_i^{\exists\forall}(a_1,\dots,a_{r_i},x,y)$;

2. For each sequence $a_1,\dots,a_{r_i} \in A$ and $x \in X_i$ there exists at most one $y \in Y_i$ such that $\neg R_i^{\exists\forall}(a_1,\dots,a_{r_i},x,y)$;

3. For each $y \in Y_i$ there exists a unique sequence $a_1,\dots,a_{r_i} \in A$ and $x \in X_i$ such that $\neg R_i^{\exists\forall}(a_1,\dots,a_{r_i},x,y)$;

4. For each $x \in X_i$ there exists a unique sequence $a_1,\dots,a_{r_i} \in A$ such that for all $y \in Y_i$ the predicate $R_i^{\exists\forall}(a_1,\dots,a_{r_i},x,y)$ is true.

Proof. 1. (a)($\Rightarrow$) Let $R_i(a_1,\dots,a_{r_i})$. Then by (4) there exists $x \in X_i$ such that $R_i^{\exists}(a_1,\dots,a_{r_i},x)$ (in fact $x = x^i_{\langle a_1,\dots,a_{r_i}\rangle}$). By (5) it follows that for every $y \in Y_i$ $R_i^{\exists\forall}(a_1,\dots,a_{r_i},x,y)$.

($\Leftarrow$) Let $x \in X_i$ and $R_i^{\exists\forall}(a_1,\dots,a_{r_i},x,y)$ for all $y \in Y_i$. Then by (5) $R_i^{\exists}(a_1,\dots,a_{r_i},x)$ and hence by (4) $R_i(a_1,\dots,a_{r_i})$.

1. (b) Follows from the definition of X_i and (5).

2. Follows from (5) and the definition of Y_i.

3. Follows from the definition of Y_i.

4. Let $x \in X_i$. Then $x = x^i_{\langle a_1,\dots,a_{r_i}\rangle}$ for some $a_1,\dots,a_{r_i}$ from A such that $R_i(a_1,\dots,a_{r_i})$. Hence $R_i^{\exists}(a_1,\dots,a_{r_i},x)$. Then, by (5), there is no $y \in Y_i$ such that $\neg R_i^{\exists\forall}(a_1,\dots,a_{r_i},x,y)$. Clearly for every sequence $b_1,\dots,b_{r_i} \in A$ not equal to $a_1,\dots,a_{r_i}$, $R_i^{\exists}(b_1,\dots,b_{r_i},x)$ is false and hence for $y = y^i_{\langle b_1,\dots,b_{r_i},x\rangle}$ the predicate $R_i^{\exists\forall}(b_1,\dots,b_{r_i},x,y)$ is false.

4 Join of Two Structures

Let $\mathfrak{A} = (A; R_1, \dots, R_s, =)$ and $\mathfrak{B} = (B; P_1, \dots, P_t, =)$ be countable structures in the languages $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Suppose that $\mathcal{L}_1 \cap \mathcal{L}_2 = \{=\}$ and $A \cap B = \emptyset$. Let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \{A, B\}$, where A and B are unary predicates.

Definition 19. *The join of the structures* $\mathfrak{A}$ *and* $\mathfrak{B}$ is the structure $\mathfrak{A} \oplus \mathfrak{B} = (A \cup B; R_1, \dots, R_s, P_1, \dots, P_t, A, B, =)$ in the language $\mathcal{L}$, where

(a) the predicate A is true only over the elements of A and similarly B is true only over the elements of B;

(b) each predicate R_i is defined on the elements of A as in the structure $\mathfrak{A}$ and false if some of the arguments of R_i are not in A and similarly each predicate P_j is defined as in the structure $\mathfrak{B}$ over the elements of B and false if some of the arguments of P_j are not in B.

Lemma 20. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be countable structures and* $\mathfrak{C} = \mathfrak{A} \oplus \mathfrak{B}$*. Then* $\mathrm{DS}(\mathfrak{C}) \subseteq \mathrm{DS}(\mathfrak{A})$ *and* $\mathrm{DS}(\mathfrak{C}) \subseteq \mathrm{DS}(\mathfrak{B})$*.*

Proof. We shall prove that $\mathrm{DS}(\mathfrak{C}) \subseteq \mathrm{DS}(\mathfrak{A})$. The proof of $\mathrm{DS}(\mathfrak{C}) \subseteq \mathrm{DS}(\mathfrak{B})$ is similar.

Let f be an enumeration of $\mathfrak{C}$. Fix $z_0 \in f^{-1}(A)$. Define

$$\begin{aligned} &m(0) = z_0; \\ &m(i+1) = \mu z \in f^{-1}(A)[(\forall k \le i)(\langle m(k), z\rangle \notin f^{-1}(=))]. \end{aligned}$$

Set $h = \lambda x. f(m(x))$. Note that $m \le_{\mathrm{T}} f^{-1}(\mathfrak{C})$ and the enumeration h of $\mathfrak{A}$ is injective and hence $h^{-1}(=)$ is computable. Moreover

$$\begin{aligned} \langle x_1, \dots, x_{r_i}\rangle \in h^{-1}(R_i) &\iff R_i(f(m(x_1)), \dots, f(m(x_{r_i}))) \\ &\iff \langle m(x_1), \dots, m(x_{r_i})\rangle \in f^{-1}(R_i). \end{aligned}$$

Thus $h^{-1}(R_i) \le_{\mathrm{T}} f^{-1}(\mathfrak{C})$.

Then $h^{-1}(\mathfrak{A}) \le_{\mathrm{T}} f^{-1}(\mathfrak{C})$. Since $\mathrm{DS}(\mathfrak{A})$ is closed upwards, $\mathrm{d_T}(f^{-1}(\mathfrak{C})) \in \mathrm{DS}(\mathfrak{A})$.

5 Representation of $\Sigma^0_2(D)$ Sets

Let $D \subseteq \mathbb{N}$. A set $M \subseteq \mathbb{N}$ is in $\Sigma^0_2(D)$ if there exists a computable in D predicate Q such that

$$n \in M \iff \exists a \forall b Q(n, a, b) \ .$$

Definition 21. [4] If $M \in \Sigma^0_2(D)$ then M is *one-to-one representable* if there exists a computable in D predicate Q with the following properties:

1. $n \in M \iff \exists a \forall b Q(n, a, b)$;
2. $n \in M \iff$ there exists a unique a such that $\forall b Q(n, a, b)$;
3. for every pair $\langle n, a\rangle$ there is at most one b such that $\neg Q(n, a, b)$;
4. for every b there is a unique pair $\langle n, a\rangle$ such that $\neg Q(n, a, b)$;

5. for every a there exists a unique n such that $\forall b Q(n, a, b)$.

The predicate Q from the above definition is called *an one-to-one representation of M*. Goncharov and Khoussainov [4] proved the following lemma:

Lemma 22. *If M is a co-infinite $\Sigma_2^0(D)$ subset of $\mathbb{N}$ and there is an infinite computable in D subset S of M such that $M \setminus S$ is infinite, then M has an one-to-one representation.*

Remark 23. We will use the lemma in the next section in the proof of Theorem 25. In order to satisfy the conditions of the lemma we need the following technical explanations.

Let $\mathfrak{A} = (A; R_1, \ldots, R_s, =)$ be a countable structure. Recall that the set A is infinite. We can easily find a structure $\mathfrak{A}^\#$ with the same degree spectrum as $\mathfrak{A}$ and such that for every injective enumeration $f^\#$ of $\mathfrak{A}^\#$ and for each predicate R of $\mathfrak{A}^\#$ the set ${f^\#}^{-1}(R)$ is co-infinite and there is a computable infinite subset S of ${f^\#}^{-1}(R)$ such that ${f^\#}^{-1}(R) \setminus S$ is infinite.

One way to do this is the following. We add to the domain A of the structure $\mathfrak{A}$ two new elements say "T" and "F". For each r-ary predicate R of $\mathfrak{A}$ define a $(r+1)$-ary predicate $R^\#$ as follows:

$$R^\#(a_1, \ldots, a_r, b) = \begin{cases} true & \text{if } T \in \{a_1, \ldots, a_r, b\}; \\ false & \text{if } F \in \{a_1, \ldots, a_r, b\} \ \&\ T \notin \{a_1, \ldots, a_r, b\}; \\ R(a_1, \ldots, a_r) & \text{if F ,T} \notin \{a_1, \ldots, a_r, b\}. \end{cases}$$

Let $\mathfrak{A}^\# = (A \cup \{T, F\}; {R_1}^\#, \ldots, {R_s}^\#, =)$.

Lemma 24. $\mathrm{DS}(\mathfrak{A}) = \mathrm{DS}(\mathfrak{A}^\#)$ *and for every injective enumeration $f^\#$ of $\mathfrak{A}^\#$ and each nontrivial predicate ${R_i}^\#$ the set ${f^\#}^{-1}({R_i}^\#)$ is co-infinite and there is a computable infinite set $S \subseteq {f^\#}^{-1}({R_i}^\#)$ such that ${f^\#}^{-1}({R_i}^\#) \setminus S$ is infinite.*

Proof. For each injective enumeration f of $\mathfrak{A}$ we construct an enumeration $f^\#$ of $\mathfrak{A}^\#$ as follows: $f^\#(0) = T, f^\#(1) = F$ and $f^\#(x+2) = f(x)$. Then

$$\begin{array}{l} \langle x_1, \ldots, x_{r_i}, z\rangle \in {f^\#}^{-1}({R_i}^\#) \iff (0 \in \{x_1, \ldots, x_{r_i}, z\}) \vee \\ (0, 1 \notin \{x_1, \ldots, x_{r_i}, z\} \ \&\ \langle x_1 - 2, \ldots, x_{r_i} - 2\rangle \in f^{-1}(R_i)). \end{array}$$

It is obvious that ${f^\#}^{-1}({R_i}^\#) \leq_{\mathrm{T}} f^{-1}(R_i)$. Moreover let $c \neq 0, 1$.

$$\langle x_1, \ldots, x_{r_i}\rangle \in f^{-1}(R_i) \iff \langle x_1 + 2, \ldots, x_{r_i} + 2, c\rangle \in {f^\#}^{-1}({R_i}^\#).$$

So ${f^\#}^{-1}({R_i}^\#) \equiv_{\mathrm{T}} f^{-1}(R_i)$.

For each injective enumeration $f^\#$ of $\mathfrak{A}^\#$ we construct an injective enumeration f of $\mathfrak{A}$ as follows. Let $tt = {f^\#}^{-1}(T), ff = {f^\#}^{-1}(F)$ and $a \in {f^\#}^{-1}(A)$.

$$\begin{array}{l} m(0) = a; \\ m(i+1) = \mu z[(\forall k \leq i)(z \neq m(k) \ \&\ z \neq tt \ \&\ z \neq ff]. \end{array}$$

Set $f = \lambda x.f^{\#}(m(x))$. Then

$$\langle x_1,\dots,x_{r_i}\rangle \in f^{-1}(R_i) \iff \langle m(x_1),\dots,m(x_{r_i}),a\rangle \in {f^{\#}}^{-1}({R_i}^{\#}).$$

$$\langle x_1,\dots,x_{r_i},z\rangle \in {f^{\#}}^{-1}({R_i}^{\#}) \iff (tt \in \{x_1,\dots,x_{r_i},z\})\vee$$
$$(tt, ff \notin \{x_1,\dots,x_{r_i},z\} \;\&\; \langle m^{-1}(x_1),\dots,m^{-1}(x_{r_i})\rangle \in f^{-1}(R_i)).$$

So $f^{-1}(R_i) \equiv_{\mathrm{T}} {f^{\#}}^{-1}({R_i}^{\#})$.

In order to see that $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}(\mathfrak{A}^{\#})$ let $\mathbf{a} \in \mathrm{DS}(\mathfrak{A})$ and let h be an enumeration of $\mathfrak{A}$, $h^{-1}(\mathfrak{A}) \in \mathbf{a}$. By Proposition 2, there exists an injective enumeration f of $\mathfrak{A}$ such that $f^{-1}(\mathfrak{A}) \leq_{\mathrm{T}} h^{-1}(\mathfrak{A})$. Then let $f^{\#}$ be the enumeration of $\mathfrak{A}^{\#}$ constructed above and so $f^{-1}(\mathfrak{A}) \equiv_{\mathrm{T}} {f^{\#}}^{-1}(\mathfrak{A}^{\#})$. Then by Proposition 4 we have that $\mathbf{a} \in \mathrm{DS}(\mathfrak{A}^{\#})$. The proof of $\mathrm{DS}(\mathfrak{A}^{\#}) \subseteq \mathrm{DS}(\mathfrak{A})$ is similar.

For each injective enumeration $f^{\#}$ of $\mathfrak{A}^{\#}$ the set ${f^{\#}}^{-1}({R_i}^{\#})$ is co-infinite since the set $\{\langle x_1,\dots,x_{r_i},z\rangle \mid ff \in \{x_1,\dots,x_{r_i},z\} \;\&\; tt \notin \{x_1,\dots,x_{r_i},z\}\}$ is infinite, here $tt = {f^{\#}}^{-1}(T), ff = {f^{\#}}^{-1}(F)$. There is an infinite computable subset $S = \{\langle x_1,\dots,x_{r_i},z\rangle \mid tt \in \{x_1,\dots,x_{r_i},z\}\}$ of ${f^{\#}}^{-1}({R_i}^{\#})$. Moreover ${f^{\#}}^{-1}({R_i}^{\#}) \setminus S$ is infinite. Let $a_1,\dots,a_{r_i} \in A$ such that $R_i(a_1,\dots,a_{r_i})$. The set $\{\langle {f^{\#}}^{-1}(a_1),\dots,{f^{\#}}^{-1}(a_{r_i}),z\rangle \mid z \in \mathbb{N} \;\&\; z \notin \{tt, ff\}\} \subseteq {f^{\#}}^{-1}({R_i}^{\#}) \setminus S$ is infinite.

Note that actually the set ${f^{\#}}^{-1}(\neg {R_i}^{\#})$ is also co-infinite and there is an infinite computable subset P of ${f^{\#}}^{-1}(\neg {R_i}^{\#})$, so that ${f^{\#}}^{-1}(\neg {R_i}^{\#}) \setminus P$ is infinite.

6 The Jump Inversion Theorem

Theorem 25. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures such that $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_1(\mathfrak{B})$. Then there exists a structure $\mathfrak{C}$ such that $\mathrm{DS}(\mathfrak{C}) \subseteq \mathrm{DS}(\mathfrak{B})$ and $\mathrm{DS}_1(\mathfrak{C}) = \mathrm{DS}(\mathfrak{A})$.*

Proof. Let $\mathfrak{A} = (A; R_1,\dots,R_s,=)$. For every predicate R_i consider a new predicate R_i^c which is equal to the negation of R_i, i.e.

$$R_i^c(a_1,\dots,a_{r_i}) \iff \neg R_i(a_1,\dots,a_{r_i}),$$

for every $a_1,\dots,a_{r_i} \in A$.

By Lemma 24 we may suppose that for every injective enumeration f of $\mathfrak{A}$ and each nontrivial predicate R_i the sets $f^{-1}(R_i)$ and $f^{-1}(R_i^c)$ are co-infinite and there are computable infinite sets $S \subseteq f^{-1}(R_i)$ and $P \subseteq f^{-1}(R_i^c)$ such that $f^{-1}(R_i) \setminus S$ and $f^{-1}(R_i^c) \setminus P$ are infinite.

We extend the structure $\mathfrak{A}$ including the negations of the predicates as follows:

$$\bar{\mathfrak{A}} = (A; R_1, R_1^c,\dots,R_s,R_s^c,=).$$

We will denote the new structure by $\bar{\mathfrak{A}} = (A; \bar{R}_1,\bar{R}_2,\dots,\bar{R}_{2s-1},\bar{R}_{2s},=)$, where $\bar{R}_{2i-1} = R_i$ and $\bar{R}_{2i} = R_i^c$ for $i = 1,\dots,s$.

It is clear that $\mathrm{DS}(\mathfrak{A}) = \mathrm{DS}(\bar{\mathfrak{A}})$ since for each enumeration f of $\mathfrak{A}$ we have that $f^{-1}(\mathfrak{A}) \equiv_{\mathrm{T}} f^{-1}(\bar{\mathfrak{A}})$.

Consider now the structure $\bar{\mathfrak{A}}^{\exists\forall}$. Let X_j be the $\exists$-fellow of $\bar{R}_j$ and Y_j be the $\forall$-fellow of $\bar{R}_j^{\exists}$, $j = 1, \ldots, 2s$.

Without loss of generality we may assume that the structures $\mathfrak{B} = (B; P_1, \ldots, P_t, =)$ and $\bar{\mathfrak{A}}^{\exists\forall}$ are disjoint.

Let $\mathfrak{C} = \mathfrak{B} \oplus \bar{\mathfrak{A}}^{\exists\forall}$. By Lemma 20, $\mathrm{DS}(\mathfrak{C}) \subseteq \mathrm{DS}(\mathfrak{B})$. We shall prove that $\mathrm{DS}_1(\mathfrak{C}) = \mathrm{DS}(\bar{\mathfrak{A}})$.

We start with the proof of the inclusion $\mathrm{DS}_1(\mathfrak{C}) \subseteq \mathrm{DS}(\bar{\mathfrak{A}})$.

Let $\mathbf{c} \in \mathrm{DS}_1(\mathfrak{C})$ and let g be an enumeration of $\mathfrak{C}$ such that $\mathbf{c} = \mathrm{d_T}(g^{-1}(\mathfrak{C}))'$. By Proposition 2 there is an injective enumeration h of $\mathfrak{C}$ such that $h^{-1}(\mathfrak{C}) \leq_{\mathrm{T}} g^{-1}(\mathfrak{C})$. We shall construct an enumeration f of $\bar{\mathfrak{A}}$ such that $f^{-1}(\bar{\mathfrak{A}}) \leq_{\mathrm{T}} h^{-1}(\mathfrak{C})'$ and hence $f^{-1}(\bar{\mathfrak{A}}) \leq_{\mathrm{T}} g^{-1}(\mathfrak{C})'$. Then by Proposition 4, $\mathbf{c} \in \mathrm{DS}(\bar{\mathfrak{A}})$.

We have

$$z \in h^{-1}(A) \iff (\forall j \leq 2s)(z \notin h^{-1}(X_j) \ \&\ z \notin h^{-1}(Y_j)) \ \&\ z \notin h^{-1}(B).$$

Thus $h^{-1}(A) \leq_{\mathrm{T}} h^{-1}(\mathfrak{C})$.

Fix $x_0 \in h^{-1}(A)$. Let

$m(0) = x_0$; $m(i+1) = \mu z \in h^{-1}(A)[(\forall k \leq i)(m(k) \neq z)]$.

Clearly $m \leq_{\mathrm{T}} h^{-1}(\mathfrak{C})$.

Set $f = \lambda a.h(m(a))$. Note that the enumeration f is injective.

Let R be an r-ary predicate of $\bar{\mathfrak{A}}$, X be the $\exists$-fellow of R and Y be the $\forall$-fellow of $R^{\exists}$.

By Proposition 18, we have

$$\begin{aligned}
&\langle a_1, \ldots, a_r\rangle \in f^{-1}(R) \iff R(f(a_1), \ldots, f(a_r)) \iff \\
&(\exists a \in X)(\forall b \in Y) R^{\exists\forall}(f(a_1), \ldots, f(a_r), a, b) \iff \\
&(\exists x \in h^{-1}(X))(\forall y \in h^{-1}(Y)) R^{\exists\forall}(h(m(a_1)), \ldots, h(m(a_r)), h(x), h(y)) \iff \\
&(\exists x \in h^{-1}(X))(\forall y \in h^{-1}(Y))(\langle m(a_1), \ldots, m(a_r), x, y\rangle \in h^{-1}(R^{\exists\forall})) \iff \\
&(\exists x)(\forall y)(\langle m(a_1), \ldots, m(a_r), x, y\rangle \in h^{-1}(R^{\exists\forall}) \ \&\ x \in h^{-1}(X) \ \&\ y \in h^{-1}(Y)).
\end{aligned}$$

Hence $f^{-1}(R) \in \Sigma_2^0(h^{-1}(\mathfrak{C}))$.

Consider now the complement predicate R^c and let and X^c be the $\exists$-fellow for R^c and Y^c be the $\forall$-fellow for $(R^c)^{\exists}$. We have again:

$$\begin{aligned}
&\langle a_1, \ldots, a_r\rangle \in f^{-1}(R^c) \iff R^c(f(a_1), \ldots, f(a_r)) \iff \\
&(\exists a \in X^c)(\forall b \in Y^c)(R^c)^{\exists\forall}(f(a_1), \ldots, f(a_r), a, b) \iff \\
&(\exists x \in h^{-1}(X^c))(\forall y \in h^{-1}(Y^c))(\langle m(a_1), \ldots, m(a_r), x, y\rangle \in h^{-1}(R^c)^{\exists\forall}).
\end{aligned}$$

Thus $f^{-1}(R^c) \in \Sigma_2^0(h^{-1}(\mathfrak{C}))$. Therefore $f^{-1}(R) \in \Delta_2^0(h^{-1}(\mathfrak{C}))$ and hence

$$f^{-1}(R) \leq_{\mathrm{T}} h^{-1}(\mathfrak{C})'.$$

So, $f^{-1}(\bar{\mathfrak{A}}) \leq_{\mathrm{T}} h^{-1}(\mathfrak{C})'$.

Now we turn to the proof of the reverse inclusion $\mathrm{DS}(\bar{\mathfrak{A}}) \subseteq \mathrm{DS}_1(\mathfrak{C})$.

Let $\mathbf{a} \in \mathrm{DS}(\bar{\mathfrak{A}})$ and let n be an enumeration of $\bar{\mathfrak{A}}$ such that $\mathbf{a} = \mathrm{d_T}(n^{-1}(\bar{\mathfrak{A}}))$. By Proposition 2, there is an injective enumeration f of $\bar{\mathfrak{A}}$ such that $f^{-1}(\bar{\mathfrak{A}}) \leq_{\mathrm{T}} n^{-1}(\bar{\mathfrak{A}})$. We are going to construct an enumeration h of $\mathfrak{C}$ such that $h^{-1}(\mathfrak{C})' \leq_{\mathrm{T}}$

$f^{-1}(\bar{\mathfrak{A}})$. Since, by Lemma 14, $\mathrm{DS}_1(\mathfrak{C})$ is closed upwards we shall obtain that $\mathbf{a} \in \mathrm{DS}_1(\mathfrak{C})$.

Recall that $\mathrm{DS}(\bar{\mathfrak{A}}) = \mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_1(\mathfrak{B})$ and $\mathrm{d_T}(f^{-1}(\bar{\mathfrak{A}})) \in \mathrm{DS}(\bar{\mathfrak{A}})$. Then there is an enumeration g of $\mathfrak{B}$ such that $f^{-1}(\bar{\mathfrak{A}}) \equiv_{\mathrm{T}} (g^{-1}(\mathfrak{B}))'$. Set $D = g^{-1}(\mathfrak{B})$. Consider the predicate $\bar{R}_j$. Let $\bar{R}_j$ be r-ary. Since $f^{-1}(\bar{\mathfrak{A}}) \leq_{\mathrm{T}} D'$, we have that $f^{-1}(\bar{R}_j) \leq_{\mathrm{T}} D'$. Then $f^{-1}(\bar{R}_j) \in \Sigma^0_2(D)$. Set $M_j = f^{-1}(\bar{R}_j)$. The enumeration f is injective and hence the set M_j is co-infinite and there is a computable infinite set $S \subseteq M_j$ such that $M_j \setminus S$ is infinite. So M_j satisfies all conditions from Lemma 22. Then by Lemma 22 there exists a computable in D predicate Q_j which is a one-to-one representation of M_j. Then

1. $\langle n_1, \dots, n_r\rangle \in M_j \iff$ there exists a unique a such that $(\forall b) Q_j(\langle n_1, \dots, n_r\rangle, a, b)$;
2. for every b let $r(b) = \langle\langle n_1, \dots, n_r\rangle, a\rangle$ be the unique pair such that
$$\neg Q_j(\langle n_1, \dots, n_r\rangle, a, b);$$
3. for every a let $l(a) = \langle n_1, \dots, n_r\rangle$ be the unique $\langle n_1, \dots, n_r\rangle$ such that $\forall b Q_j(\langle n_1, \dots, n_r\rangle, a, b)$.

Let $\mathbb{N}_1 = \{\langle 1, n\rangle \mid n \in \mathbb{N}\}$, $\mathbb{N}_2 = \{\langle 2, j, a\rangle \mid j \leq 2s \ \&\ a \in \mathbb{N}\}$ and $\mathbb{N}_3 = \{\langle 3, j, b\rangle \mid j \leq 2s \ \&\ b \in \mathbb{N}\}$. Set $\mathbb{N}_0 = \mathbb{N} \setminus (\bigcup_{i=1}^3 \mathbb{N}_i)$. Consider a computable bijection m from $\mathbb{N}$ onto $\mathbb{N}_0$.

The definition of the enumeration h of $\mathfrak{C}$ is the following:
$h(m(n)) = g(n)$;
$h(\langle 1, n\rangle) = f(n)$;
$h(\langle 2, j, a\rangle) = x^j_{\langle f(n_1), \dots, f(n_r)\rangle}$, if $l(a) = \langle n_1, \dots, n_r\rangle$;
$h(\langle 3, j, b\rangle) = y^j_{\langle f(n_1), \dots, f(n_r), h(\langle 2, j, a\rangle)\rangle}$, if $r(b) = \langle\langle n_1, \dots, n_r\rangle, a\rangle$.

Recall that $X_j = \{x^j_{\langle a_1, \dots, a_r\rangle} \mid \bar{R}_j(a_1, \dots, a_r)\}$ is the $\exists$-fellow for $\bar{R}_j$ and $Y_j = \{y^j_{\langle a_1, \dots, a_r, x\rangle} \mid \neg \bar{R}^{\exists}_j(a_1, \dots, a_r, x)\}$ is the $\forall$-fellow for $\bar{R}^{\exists}_j$.

From the choice of Y_j it follows that

$$\begin{aligned}\neg Q_j(\langle n_1, \dots, n_r\rangle, a, b) &\iff r(b) = \langle\langle n_1, \dots, n_r\rangle, a\rangle \\ &\iff h(\langle 3, j, b\rangle) = y^j_{\langle f(n_1), \dots, f(n_r), h(\langle 2, j, a\rangle)\rangle} \\ &\iff \neg \bar{R}^{\exists\forall}_j(f(n_1), \dots, f(n_r), h(\langle 2, j, a\rangle), h(\langle 3, j, b\rangle)).\end{aligned}$$

An then

$$Q_j(\langle n_1, \dots, n_r\rangle, a, b) \iff \bar{R}^{\exists\forall}_j(\langle f(n_1), \dots, f(n_r), h(\langle 2, j, a\rangle), h(\langle 3, j, b\rangle)).$$

Define

$$\bar{R}^{\exists\forall, h}_j(\langle\langle 1, n_1\rangle, \dots, \langle 1, n_r\rangle, \langle 2, j, a\rangle, \langle 3, j, b\rangle\rangle) \iff Q_j(\langle n_1, \dots, n_r\rangle, a, b)\ .$$

It follows that $\bar{R}^{\exists\forall, h}_j \leq_{\mathrm{T}} D$. Moreover

$$\begin{aligned}&\bar{R}^{\exists\forall, h}_j(\langle\langle 1, n_1\rangle, \dots, \langle 1, n_r\ \rangle, \langle 2, j, a\rangle, \langle 3, j, b\rangle\rangle) \iff \\ &\qquad \bar{R}^{\exists\forall}_j(h(\langle 1, n_1\rangle), \dots, h(\langle 1, n_r\rangle)), h(\langle 2, j, a\rangle), h(\langle 3, j, b\rangle))\end{aligned}$$

So $\bar{R}_j^{\exists\forall,h} = h^{-1}(\bar{R}_j^{\exists\forall})$ and hence $h^{-1}(\bar{R}_j^{\exists\forall}) \leq_{\mathrm{T}} D$.

The sets $h^{-1}(A) = \mathbb{N}_1$, $h^{-1}(X_j) = \{\langle 2, j, a\rangle \mid a \in \mathbb{N}\}$, $h^{-1}(Y_j) = \{\langle 3, j, b\rangle \mid b \in \mathbb{N}\}$ are computable. Then $h^{-1}(\bar{\mathfrak{A}}^{\exists\forall}) \leq_{\mathrm{T}} D$.

Note that

$$\begin{aligned}\bar{R}_j(f(n_1), \ldots, f(n_r)) &\iff \langle n_1, \ldots, n_r\rangle \in f^{-1}(\bar{R}_j)\\ &\iff (\exists a)(\forall b) Q_j(\langle n_1, \ldots, n_r\rangle, a, b)\\ &\iff (\exists a)(\forall b) \bar{R}_j^{\exists\forall,h}(\langle\langle 1, n_1\rangle \ldots \langle 1, n_r\rangle, \langle 2, j, a\rangle, \langle 3, j, b\rangle\rangle)\\ &\iff (\exists x \in X_j)(\forall y \in Y_j) \bar{R}_j^{\exists\forall}(f(n_1), \ldots, f(n_r), x, y).\end{aligned}$$

For every predicate P_j of $\mathfrak{B}$ it holds that
$h^{-1}(P_j) = \{\langle m(n_1), \ldots, m(n_{p_j})\rangle \mid \langle n_1, \ldots, n_{p_j}\rangle \in g^{-1}(P_j)\}$ and $h^{-1}(B) = \mathbb{N}_0$. It is obvious that $h^{-1}(\mathfrak{B}) \leq_{\mathrm{T}} D = g^{-1}(\mathfrak{B})$.

The pullback of the equality is defined over the elements which are pullbacks of elements of B as $g^{-1}(=)$. Over the other elements the equality is defined in the usual way. So, $h^{-1}(=)$ is the set:

$$\{\langle x, y\rangle \mid (\langle m^{-1}(x), m^{-1}(y)\rangle \in g^{-1}(=) \ \&\ x, y \in \mathbb{N}_0) \vee (x = y \ \&\ x, y \notin \mathbb{N}_0)\}.$$

Then $h^{-1}(=) \leq_{\mathrm{T}} D$. Thus $h^{-1}(\mathfrak{B} \oplus \bar{\mathfrak{A}}^{\exists\forall}) = h^{-1}(\mathfrak{C}) \leq_{\mathrm{T}} D = g^{-1}(\mathfrak{B})$. Using that $g^{-1}(\mathfrak{B})' \equiv_{\mathrm{T}} f^{-1}(\bar{\mathfrak{A}})$, we get from here that $h^{-1}(\mathfrak{C})' \leq_{\mathrm{T}} f^{-1}(\bar{\mathfrak{A}})$.

7 Some Applications

The jump inversion theorem proved in the previous section can be easily generalized in the following way.

Definition 26. Given a structure $\mathfrak{A}$ and $n \geq 0$, let the *nth jump spectrum* $\mathrm{DS}_n(\mathfrak{A})$ be the set $\{\mathbf{a}^{(n)} : \mathbf{a} \in \mathrm{DS}(\mathfrak{A})\}$.

Clearly $\mathrm{DS}_0(\mathfrak{A}) = \mathrm{DS}(\mathfrak{A})$ and $\mathrm{DS}_{n+1}(\mathfrak{A}) = \{\mathbf{a}' : \mathbf{a} \in \mathrm{DS}_n(\mathfrak{A})\}$. Using this and Theorem 12, one can easily see by induction on n that for every n there exists a structure $\mathfrak{A}^{(n)}$ such that $\mathrm{DS}_n(\mathfrak{A}) = \mathrm{DS}(\mathfrak{A}^{(n)})$.

Theorem 27. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures such that $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_n(\mathfrak{B})$. Then there exists a structure $\mathfrak{C}$ such that $\mathrm{DS}(\mathfrak{C}) \subseteq \mathrm{DS}(\mathfrak{B})$ and $\mathrm{DS}_n(\mathfrak{C}) = \mathrm{DS}(\mathfrak{A})$.*

Proof. Induction on n. The assertion is obvious for $n = 0$. Suppose that it is true for some n. Let $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_{n+1}(\mathfrak{B})$. Consider a structure $\mathfrak{B}^{(n)}$ such that $\mathrm{DS}(\mathfrak{B}^{(n)}) = \mathrm{DS}_n(\mathfrak{B})$. Clearly $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_1(\mathfrak{B}^{(n)})$ and hence by Theorem 25 there exists a structure $\mathfrak{C}^*$ such that $\mathrm{DS}(\mathfrak{C}^*) \subseteq \mathrm{DS}(\mathfrak{B}^{(n)})$ and $\mathrm{DS}_1(\mathfrak{C}^*) = \mathrm{DS}(\mathfrak{A})$. By the induction hypothesis, there exists a structure $\mathfrak{C}$ such that $\mathrm{DS}(\mathfrak{C}) \subseteq \mathrm{DS}(\mathfrak{B})$ and $\mathrm{DS}_n(\mathfrak{C}) = \mathrm{DS}(\mathfrak{C}^*)$. Then $\mathrm{DS}_{n+1}(\mathfrak{C}) = \mathrm{DS}_1(\mathfrak{C}^*) = \mathrm{DS}(\mathfrak{A})$.

Definition 28. A degree $\mathbf{a}$ is said to be the *nth jump degree* of a structure $\mathfrak{A}$ if $\mathbf{a}$ is the least element of $\mathrm{DS}_n(\mathfrak{A})$.

Notice that if $\mathbf{a}$ is the nth jump degree of $\mathfrak{A}$ then for all k, $\mathbf{a}^{(k)}$ is the $(n+k)th$ jump degree of $\mathfrak{A}$. Hence if a structure $\mathfrak{A}$ possesses a nth jump degree then it possesses $(n+k)$th jump degrees for all k.

With respect to the jump degrees of $\mathfrak{A}$ it does not matter whether we consider arbitrary enumerations of $\mathfrak{A}$ or only injective enumerations of $\mathfrak{A}$. Indeed, by Proposition 2, if $\mathbf{a}$ is the least element of the spectrum of $\mathfrak{A}$ then $\mathbf{a} = d_{\mathrm{T}}(f^{-1}(\mathfrak{A}))$ for some injective enumeration f.

The definitions above can be naturally generalized for all recursive ordinals α. In [3] Downey and Knight proved with a fairly complicated construction that for every recursive ordinal α there exists a linear order $\mathfrak{A}$ such that $\mathfrak{A}$ has αth jump degree equal to $\mathbf{0}^{\alpha}$ but for all $\beta < \alpha$, there is no βth jump degree of $\mathfrak{A}$.

Here we shall present a construction which allows us to obtain for every natural number n examples of structures which have $(n+1)$th jump degree equal to $\mathbf{0}^{(n+1)}$ but do not have kth jump degree for $k \leq n$.

The idea of this construction is the following. Suppose that we have a structure $\mathfrak{A}$ satisfying the following conditions:

(C1) $\mathrm{DS}(\mathfrak{A}) \subseteq \{\mathbf{a} : \mathbf{0^{(n)}} \leq \mathbf{a}\}$.
(C2) $\mathrm{DS}(\mathfrak{A})$ has no least element.
(C3) $\mathfrak{A}$ has a first jump degree equal to $\mathbf{0}^{(n+1)}$.

Let $\mathfrak{B} = (N; =)$ be a structure such that $\mathrm{DS}(\mathfrak{B})$ is equal to the set of all Turing degrees. Clearly $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_n(\mathfrak{B})$. By Theorem 27, there exists a structure $\mathfrak{C}$ such that $\mathrm{DS}_n(\mathfrak{C}) = \mathrm{DS}(\mathfrak{A})$. Therefore $\mathfrak{C}$ does not have a nth jump degree and hence it has no kth jump degree for $k \leq n$. On the other hand $\mathrm{DS}_{n+1}(\mathfrak{C}) = \mathrm{DS}_1(\mathfrak{A})$ and hence the $(n+1)$th jump degree of $\mathfrak{C}$ is $\mathbf{0}^{(n+1)}$.

Now we provide an example of a structure satisfying the conditions (C1) – (C3).

We shall need the following fact about the degree spectra of the subgroups of the additive group Q of the rational numbers, i.e. of the torsion free Abelian groups of rank 1. Details can be found in [2] and [11].

Given a set A of natural numbers and $z \in N$, let

$$W_z(A) = \{x : (\exists v)(\langle x, v\rangle \in W_z \ \&\ D_v \subseteq A)\},.$$

where D_v denotes the finite set with canonical code v.

We say that $A \leq_e B$ if $A = W_z(B)$ for some z. Notice that if $A \leq_e B$ and B is c.e. in C, then A is c.e. in C. Furthermore if A is c.e. in B, then $A \leq_e B \oplus (N \setminus B)$.

For every set A of natural numbers let $J_e(A) = \{x : x \in W_x(A)\}$. The set $J_e(A)$ is called the *enumeration jump* of A.

Fact.[2, 11] For every set A of natural numbers there exists a group $G_A \subseteq Q$ satisfying the following conditions:

1. $\mathrm{DS}(G_A) = \{d_{\mathrm{T}}(X) : A \text{ is c.e. in } X\}$.
2. $d_{\mathrm{T}}(J_e(A))$ is the first jump degree of G_A.

By a relativization of the Jump inversion theorem of McEvoy [6], we obtain that there exists a set A such that

1. $\emptyset^{(n)} \oplus (N \setminus \emptyset^{(n)}) <_e A$;
2. $(\forall X)(X \oplus (N \setminus X) \leq_e A \Rightarrow X \leq_T \emptyset^{(n)})$;
3. $\emptyset^{(n+1)} \equiv_{\mathrm{T}} J_e(A)$.

Now consider G_A. Let $d_{\mathrm{T}}(X) \in \mathrm{DS}(G_A)$. Then A is c.e. in X and hence $\emptyset^{(n)} \oplus N \setminus \emptyset^{(n)}$ is c.e. in X. Then $\emptyset^{(n)} \leq_{\mathrm{T}} X$. So, G_A satisfies (C1). Clearly G_A satisfies (C3).

Assume that $d_{\mathrm{T}}(X)$ is the least element of $\mathrm{DS}(G_A)$. Then, by Selman's Theorem [10], $X \oplus (N \setminus X) \leq_e A$ and hence $X \leq_T \emptyset^{(n)}$. Thus A is c.e. $\emptyset^{(n)}$. From here it follows that $A \leq_e \emptyset^{(n)} \oplus (N \setminus \emptyset^{(n)})$. A contradiction. So, G_A satisfies (C2).

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