QUASI-MINIMAL DEGREES FOR DEGREE SPECTRA

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Abstract. We prove the following properties of quasi-minimal degrees for the degree spectrum of a structure. There are uncountably many quasi-minimal degrees for every degree spectrum. The first jump spectrum of every structure consists exactly of the enumeration jumps of the quasi-minimal degrees for the degree spectrum. Every element of the first jump spectrum could be represented as the join of two quasi-minimal degrees for the degree spectrum.

Key words: enumeration degrees; forcing; degree spectra; quasi-minimal degrees.

1. Introduction

Let $\mathfrak{A} = (A; R_1, \ldots, R_k)$ be a countable structure. *The degree spectrum* $\mathrm{DS}(\mathfrak{A})$ of $\mathfrak{A}$ is the set of all Turing degrees which compute the diagram of an isomorphic copy of $\mathfrak{A}$ on the natural numbers.

The notion of degree spectrum of a structure is introduced by Richter [10] as the set of all Turing degrees of the diagrams of the presentations of the structure. Here we consider a modification of her notion, with the benefit that every degree spectrum is closed upwards. *The first jump spectrum* $\mathrm{DS}_1(\mathfrak{A})$ of $\mathfrak{A}$ is the set of all Turing degrees which compute the Turing jumps of the elements of the degree spectrum $\mathrm{DS}(\mathfrak{A})$ of $\mathfrak{A}$.

Soskov [15] initiated the study of the properties of the degree spectra as sets of enumeration degrees. He introduced the notion of *co-spectrum* $\mathrm{CS}(\mathfrak{A})$ of a structure $\mathfrak{A}$ as the set of all enumeration degrees which are lower bounds of the elements of the degree spectrum of $\mathfrak{A}$ and proved several properties which show that the degree spectra behave with respect to their co-spectra very much like the cones of the enumeration degrees $\{\mathbf{x} \mid \mathbf{x} \geq_{\mathrm{e}} \mathbf{a}\}$ behave with respect to the intervals $\{\mathbf{x} \mid \mathbf{x} \leq_{\mathrm{e}} \mathbf{a}\}$. He showed that every countable ideal of enumeration degrees is a co-spectrum of a structure.

Some typical properties of degree spectra and their co-spectra are the existence of minimal pairs in every degree spectrum of a structure and the existence of quasi-minimal degree with respect to the degree spectrum. But there are examples of upwards closed sets of degrees which do not have minimal pairs and for which there are no quasi-minimal degrees. More precisely, for every degree spectrum $\mathrm{DS}(\mathfrak{A})$, there exist degrees $\mathbf{f_0}$ and $\mathbf{f_1}$ in $\mathrm{DS}(\mathfrak{A})$ such that the set of all enumeration degrees less than or equal to both $\mathbf{f_0}$ and $\mathbf{f_1}$ is equal to $\mathrm{CS}(\mathfrak{A})$. Every such pair of degrees is called *a minimal pair for* $\mathrm{DS}(\mathfrak{A})$. For each degree spectrum $\mathrm{DS}(\mathfrak{A})$ there is an enumeration degree $\mathbf{q} \notin \mathrm{CS}(\mathfrak{A})$, called *quasi-minimal for* $\mathrm{DS}(\mathfrak{A})$ such that every

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Turing degree $\mathbf{a} \geq_{\mathrm{e}} \mathbf{q}$ belongs to $\mathrm{DS}(\mathfrak{A})$ and every Turing degree $\mathbf{a} \leq_{\mathrm{e}} \mathbf{q}$ belongs to $\mathrm{CS}(\mathfrak{A})$.

In this paper we shall present some properties of the quasi-minimal degrees for the degree spectrum of a structure. We shall see that there are uncountably many quasi-minimal degrees for every degree spectrum. The first jump spectrum of every structure consists exactly of the enumeration jumps of the quasi-minimal degrees of the degree spectrum. And every element of the first jump spectrum could be represent as the join of two quasi-minimal degrees for the degree spectrum. These properties of the quasi-minimal degrees for a degree spectrum are analogues of the classical results of Jockush [8], Slaman and Sorbi [13] and McEvoy [9] in the enumeration degrees.

2. Preliminaries

2.1. **Enumeration Degrees.** Intuitively a set A of natural numbers is enumeration reducible to a set of natural numbers B, denoted by $A \leq_{\mathrm{e}} B$, if there is an effective procedure to enumerate A given any enumeration of B. More precisely, A is *enumeration reducible to* B if there is an enumeration operator Γ_a such that $A = \Gamma_a(B)$, i.e.

$$(\forall x)(x \in A \iff (\exists v)(\langle v, x\rangle \in W_a \ \&\ D_v \subseteq B))$$

where D_v is the finite set with canonical code v and W_a is the computably enumerable set with index a with respect to the Gödel numbering of all c.e. sets.

The relation $\leq_{\mathrm{e}}$ is reflexive and transitive and induces an equivalence relation $\equiv_{\mathrm{e}}$ on all sets of natural numbers. The enumeration degree of the set A, denoted by $d_{\mathrm{e}}(A)$, is the equivalence class relatively $\equiv_{\mathrm{e}}$. The degree structure $\langle \mathcal{D}_{\mathrm{e}}, \leq_{\mathrm{e}}\rangle$ is defined by setting $\mathcal{D}_{\mathrm{e}} = \{d_{\mathrm{e}}(A) \mid A \subseteq \mathbb{N}\}$, and $d_{\mathrm{e}}(A) \leq_{\mathrm{e}} d_{\mathrm{e}}(B)$ if and only if $A \leq_{\mathrm{e}} B$. The structure $\mathcal{D}_{\mathrm{e}}$ is an upper semilattice with least element $\mathbf{0}_{\mathrm{e}} = d_{\mathrm{e}}(A)$ where A is any computably enumerable set. The operation of least upper bound is given by $d_{\mathrm{e}}(A) \vee d_{\mathrm{e}}(B) = d_{\mathrm{e}}(A \oplus B)$, where $A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$.

By identifying partial functions with their graphs, where $\langle \varphi \rangle = \{\langle x, y\rangle \mid \varphi(x) = y\}$ we shall write $\varphi \leq_{\mathrm{e}} g$ to mean $\langle \varphi \rangle \leq_{\mathrm{e}} \langle g \rangle$. This reducibility coincides with the reducibility between partial functions introduced by Kleene (1952). Similarly, given a set A and a partial function φ, $\varphi \leq_{\mathrm{e}} A$ means $\langle \varphi \rangle \leq_{\mathrm{e}} A$.

Define $A^+ = A \oplus (\mathbb{N} \setminus A)$. A set A is *total* if $A \equiv_{\mathrm{e}} A^+$. In other words $A \equiv_{\mathrm{e}} c_A$, where c_A is the characteristic function of A. Thus a set A is total when it is e-equivalent to the graph of a total function. An enumeration degree $\mathbf{a}$ is total if $\mathbf{a}$ contains a total set or equivalently the graph of a total function. A set A is c.e. in a set B if $A \leq_{\mathrm{e}} B^+$.

Selman [11] proved that $A \leq_{\mathrm{e}} B$ if and only if for every set X if B is c.e. in X then A is c.e in X.

Cooper [3] introduced the jump operation "$'$" for enumeration degrees. Given a set A, let $L_A = \{\langle x, z\rangle \mid x \in \Gamma_z(A)\}$.

The *e-jump* A' of A is the set $(L_A)^+$. We define:

(1) $A^{(0)} = A$;
(2) $A^{(n+1)} = (A^{(n)})'$.

Let $\langle \mathcal{D}_T, \leq_{\mathrm{T}}, \vee, \mathbf{0}_{\mathrm{T}}\rangle$ denote the upper semilattice of Turing degrees (T-degrees), with partial ordering the relation $\leq_{\mathrm{T}}$. For any sets A and B: $A \leq_{\mathrm{T}} B$ if and only if $A^+ \leq_{\mathrm{e}} B^+$ and equivalently if $c_A \leq_{\mathrm{e}} c_B$, where c_A and c_B are the characteristic functions of A and B. This gives an order preserving embedding $\iota : \mathcal{D}_{\mathrm{T}} \to \mathcal{D}_{\mathrm{T}}$ namely $\iota(d_{\mathrm{T}}(A)) = d_{\mathrm{e}}(A^+)$.

The enumeration jump is always a total degree and agrees with the Turing jump under the embedding ι. So, the upper semilattice of the Turing degrees with jump operation can be viewed as a substructure of the enumeration degrees. We shall identify the Turing degree $\mathbf{a} = d_{\mathrm{T}}(A)$ with the total e-degree $\iota(\mathbf{a}) = d_{\mathrm{e}}(A^+)$.

Furthermore McEvoy [9] showed that a set A is Σ^B_{n+1} if $A \leq_{\mathrm{e}} (B^+)^{(n)}$.

By Soskov's Jump Inversion Theorem [14] for every $\mathbf{x} \in \mathcal{D}_e$ there exists a total e-degree $\mathbf{a} \geq_{\mathrm{e}} \mathbf{x}$, such that $\mathbf{a}' = \mathbf{x}'$.

The existence of nontotal e-degrees is an easy consequence of the existence of quasi-minimal e-degrees, first shown by Medvedev (1955).

Definition 2.1.1. *(Medvedev) An e-degree* $\mathbf{a}$ *is said to be quasi-minimal if*

(1) $\mathbf{a} \neq \mathbf{0}_e$,
(2) $\forall$ *total* $\mathbf{b}[\mathbf{b} \leq_{\mathrm{e}} \mathbf{a} \to \mathbf{b} = \mathbf{0}_e]$.

It is easy to see that a nonzero e-degree $\mathbf{a}$ is quasi-minimal if and only if $(\forall A \in \mathbf{a})(\forall$ total function $f)[f \leq_{\mathrm{e}} A \to f$ is computable$]$.

Let $\mathcal{A} \subseteq \mathcal{D}_{\mathrm{T}}$. The *co-set of* $\mathcal{A}$ is the set $co(\mathcal{A})$ of all enumeration degrees which are lower bounds of $\mathcal{A}$:

$$co(\mathcal{A}) = \{\mathbf{b} \mid \mathbf{b} \in \mathcal{D}_{\mathrm{e}} \;\&\; (\forall \mathbf{a} \in \mathcal{A})(\mathbf{b} \leq_{\mathrm{e}} \mathbf{a})\}\ .$$

2.2. **Degree Spectra and Co-Spectra.** Let $\mathfrak{A} = (A; R_1, \ldots, R_k)$ be a countable structure.

An enumeration of $\mathfrak{A}$ is every one-to-one mapping of $\mathbb{N}$ onto A.

Given an enumeration f of $\mathfrak{A}$ and a subset B of A^a, let

$$f^{-1}(B) = \{\langle x_1, \ldots, x_a\rangle \mid (f(x_1), \ldots, f(x_a)) \in B\}.$$

Denote by $f^{-1}(\mathfrak{A}) = f^{-1}(R_1)^+ \oplus \cdots \oplus f^{-1}(R_k)^+$.

Definition 2.2.1. The degree spectrum of $\mathfrak{A}$ *is the set*

$$\mathrm{DS}(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \in \mathcal{D}_{\mathrm{T}} \;\&\; (\exists f)(d_{\mathrm{T}}(f^{-1}(\mathfrak{A})) \leq_{\mathrm{T}} \mathbf{a})\}\ .$$

From the definition of the degree spectrum of a structure it follows that it is always an upwards closed set of Turing degrees.

Definition 2.2.2. *The co-spectrum of* $\mathfrak{A}$ *is the set of all enumeration degrees which are lower bounds of* $\mathrm{DS}(\mathfrak{A})$*, i.e. the co-set of* $\mathrm{DS}(\mathfrak{A})$*:*

$$\mathrm{CS}(\mathfrak{A}) = \{\mathbf{b} \mid \mathbf{b} \in \mathcal{D}_e \;\&\; (\forall \mathbf{a} \in \mathrm{DS}(\mathfrak{A}))(\mathbf{b} \leq_{\mathrm{e}} \mathbf{a})\}\ .$$

Notice that two isomorphic structures $\mathfrak{A}$ and $\mathfrak{B}$ always have the same degree spectrum and hence $\mathrm{CS}(\mathfrak{A}) = \mathrm{CS}(\mathfrak{B})$. So we could suppose that the domain of $\mathfrak{A}$ is the set of the natural numbers $\mathbb{N}$.

For each natural number n, *the nth jump spectrum of* $\mathfrak{A}$ is the set

$$\mathrm{DS}_n(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \in \mathcal{D}_{\mathrm{T}} \;\&\; (\exists \mathbf{b} \in \mathrm{DS}(\mathfrak{A}))(\mathbf{b}^{(n)} \leq_{\mathrm{T}} \mathbf{a})\}\ .$$

The nth co-spectrum of $\mathfrak{A}$ is the set $\mathrm{CS}_n(\mathfrak{A}) = co(\mathrm{DS}_n(\mathfrak{A}))$.

2.3. **The co-spectrum of a structure.** Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_k)$ be a countable structure on the natural numbers.

In this section we shall consider a notion of partial (1-generic) enumeration of $\mathfrak{A}$ and some properties of the elements of the co-spectrum of the structure $\mathfrak{A}$ proved by Soskov [15].

We need first a characterization of the elements of the co-spectrum of $\mathfrak{A}$.

2.4. **Partial 1-generic enumerations.** Let $\perp \notin \mathbb{N}$.

Definition 2.4.1. *A* partial enumeration of $\mathfrak{A}$ *is a partial one-to-one mapping of* $\mathbb{N}$ *onto* $\mathbb{N}$. *A* partial finite part *is a finite mapping of* $\mathbb{N}$ *into* $\mathbb{N} \cup \{\perp\}$.

By δ,ρ,τ we shall denote partial finite parts. Given a partial finite part τ and a partial enumeration φ of $\mathfrak{A}$, by $\tau \subseteq \varphi$ we shall denote that for all x in $\mathrm{dom}(\tau)$ either $\tau(x) = \perp$ and $\varphi(x)$ is not defined or $\tau(x) \in \mathbb{N}$ and $\varphi(x) = \tau(x)$. So $\tau(x) = \perp$ means that for any partial enumeration $\varphi \supseteq \tau$, $\varphi(x)$ is undefined. If φ is a partial enumeration of $\mathfrak{A}$ and B is a subset of $\mathbb{N}^a$, let

$$\varphi^{-1}(B) = \{\langle x_1, \ldots, x_a\rangle : x_1, \ldots, x_a \in \mathrm{dom}(\varphi)\ \&\ (\varphi(x_1), \ldots, \varphi(x_a)) \in B\} \ .$$

Denote by

$$\varphi^{-1}(\mathfrak{A}) = \mathrm{dom}(\varphi) \oplus \varphi^{-1}(R_1)^+ \oplus \cdots \oplus \varphi^{-1}(R_k)^+ \ .$$

Notice that it could happen that $d_{\mathrm{e}}(\varphi^{-1}(\mathfrak{A})) \notin \mathrm{DS}(\mathfrak{A})$. For instance it is possible that $d_{\mathrm{e}}(\varphi^{-1}(\mathfrak{A}))$ is not a total enumeration degree. On the other hand Soskov [15] shows that for every partial enumeration φ of $\mathfrak{A}$ the enumeration degree of $\varphi^{-1}(\mathfrak{A})$ is "almost" in $\mathrm{DS}(\mathfrak{A})$.

Lemma 2.4.1. [15] *Let X be a total set, let φ be a partial enumeration of $\mathfrak{A}$ and $\varphi^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} X$. Then $d_{\mathrm{e}}(X) \in \mathrm{DS}(\mathfrak{A})$.*

Proof. Since X is a total set we have that $\varphi^{-1}(\mathfrak{A})$ is c.e. in X and hence $\mathrm{dom}(\varphi)$ is c.e. in X. Let $\rho : \mathbb{N} \to \mathrm{dom}(\varphi)$ be an enumeration of the domain of φ which is computable in X and one-to-one. Then we can easily construct a total enumeration $g(n) = \varphi(\rho(n))$ of $\mathfrak{A}$. For every basic predicate $R(x_1, \ldots, x_r)$ of $\mathfrak{A}$ we have:

$$g^{-1}(R) = \{\langle n_1, \ldots, n_r\rangle \mid (\rho(n_1), \ldots, \rho(n_r)) \in \varphi^{-1}(R)\} \leq_{\mathrm{c.e.}} X,$$

$$g^{-1}(\neg R) = \{\langle n_1, \ldots, n_r\rangle \mid (\rho(n_1), \ldots, \rho(n_r)) \in \varphi^{-1}(\neg R)\} \leq_{\mathrm{c.e.}} X \ .$$

Then $g^{-1}(R) \leq_{\mathrm{T}} X$ and hence $g^{-1}(\mathfrak{A}) \leq_{\mathrm{T}} X$. So, $d_{\mathrm{e}}(X) \in \mathrm{DS}(\mathfrak{A})$.

□

For every partial enumeration φ of $\mathfrak{A}$, the set $\varphi^{-1}(\mathfrak{A})$ is an upper bound of the elements of the co-spectrum $\mathrm{CS}(\mathfrak{A})$. This is true since if $B \in \mathbf{b}$ and $\mathbf{b}$ is an element of $\mathrm{CS}(\mathfrak{A})$ then for every total set X such that $\varphi^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} X$ we have $d_{\mathrm{e}}(X) \in \mathrm{DS}(\mathfrak{A})$ and hence $B \leq_{\mathrm{e}} X$. Now applying Selman's theorem [11] $B \leq_{\mathrm{e}} \varphi^{-1}(\mathfrak{A})$. Moreover for every partial enumeration φ of $\mathfrak{A}$, the enumeration degree of $\varphi^{-1}(\mathfrak{A})'$ belongs to $\mathrm{DS}_1(\mathfrak{A})$ since by the Jump Inversion Theorem of Soskov [14] there exists a total set F such that $\varphi^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} F$ and $F' \equiv_{\mathrm{e}} \varphi^{-1}(\mathfrak{A})'$. Then by Lemma 2.4.1 $d_{\mathrm{e}}(F) \in \mathrm{DS}(\mathfrak{A})$ and hence, $d_{\mathrm{e}}(F') \in \mathrm{DS}_1(\mathfrak{A})$. So we have the following properties of the partial enumerations of $\mathfrak{A}$:

Proposition 2.4.2. *For every partial enumeration φ of the structure $\mathfrak{A}$ if $\mathbf{a} = \mathbf{d}_e(\varphi^{-1}(\mathfrak{A}))$ then*

- $\mathbf{a} \geq_{\mathrm{e}} \mathbf{b}$ *for every* $\mathbf{b} \in \mathrm{CS}(\mathfrak{A})$*;*

- $\mathbf{a}' \in \mathrm{DS}_1(\mathfrak{A})$.

Let φ be a partial enumeration of $\mathfrak{A}$ and $e, x \in \mathbb{N}$. Then the modeling relation "$\models$" is defined as follows:

(i) $\varphi \models F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(\exists i)(1 \leq i \leq k\ \&$

$(u = \langle 0, i, x_1^u, \dots, x_{r_i}^u\rangle \ \&\ x_1^u, \dots, x_{r_i}^u \in \mathrm{dom}(\varphi)\ \&\ (\varphi(x_1^u), \dots, \varphi(x_{r_i}^u)) \in R_i) \vee$
$(u = \langle 1, i, x_1^u, \dots, x_{r_i}^u\rangle \ \&\ x_1^u, \dots, x_{r_i}^u \in \mathrm{dom}(\varphi)\ \&\ (\varphi(x_1^u), \dots, \varphi(x_{r_i}^u)) \in \neg R_i))).$

(ii) $\varphi \models \neg F_e(x) \iff \varphi \not\models F_e(x)$.

Notice that for every $e \in \mathbb{N}$ the following equivalence is true for all $x \in \mathbb{N}$:

$$\varphi \models F_e(x) \iff x \in \Gamma_e(\varphi^{-1}(\mathfrak{A}))\ .$$

Let τ be a partial finite part and $e, x \in \mathbb{N}$. The forcing relation is defined as follows:

(i) $\tau \Vdash F_e(x)$ if and only if there exists a v such that $\langle v, x\rangle \in W_e$ and for all $u \in D_v$, there is $1 \leq i \leq k$ such that

$(u = \langle 0, i, x_1^u, \dots, x_{r_i}^u\rangle,\ \&\ x_1^u, \dots, x_{r_i}^u \in \mathrm{dom}(\tau)\ \&\ (\tau(x_1^u), \dots, \tau(x_{r_i}^u)) \in R_i) \vee$
$(u = \langle 1, i, x_1^u, \dots, x_{r_i}^u\rangle,\ \&\ x_1^u, \dots, x_{r_i}^u \in \mathrm{dom}(\tau)\ \&\ (\tau(x_1^u), \dots, \tau(x_{r_i}^u)) \in \neg R_i)\ .$

(ii) $\tau \Vdash \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \nVdash F_e(x))$.

The forcing relation $\Vdash$ is monotone, i.e. if $\tau \subseteq \rho$ then $\tau \Vdash F_e(x) \Rightarrow \rho \Vdash F_e(x)$.

For every partial enumeration φ of $\mathfrak{A}$, $e, x \in \mathbb{N}$ the following equivalence is true:

$$\varphi \models F_e(x) \iff (\exists \tau \subseteq \varphi)(\tau \Vdash F_e(x))\ .$$

Definition 2.4.2. *A subset B of $\mathbb{N}$ is* partially forcing definable *on $\mathfrak{A}$ if there exist a natural number e and a partial finite part δ such that for all natural numbers x,*

$$x \in B \iff (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\ .$$

Denote by $D(\mathfrak{A})$ the atomic diagram of $\mathfrak{A}$, i.e. $D(\mathfrak{A}) = i^{-1}(\mathfrak{A})$, where i is the identity function.

Let $B \subseteq \mathbb{N}$ be a partially forcing definable on $\mathfrak{A}$. From the definition of the forcing relation we have that $B \leq_{\mathrm{e}} D(\mathfrak{A})$. And moreover $d_{\mathrm{e}}(B) \in \mathrm{CS}(\mathfrak{A})$. To see this consider an arbitrary total enumeration g of $\mathfrak{A}$. Let $\mathfrak{B} = (\mathbb{N}; g^{-1}(R_1), \dots, g^{-1}(R_k))$ be the structure isomorphic to $\mathfrak{A}$. Then the atomic diagram of $\mathfrak{B}$ is $D(\mathfrak{B}) = g^{-1}(\mathfrak{A})$. The set B is partially forcing definable on $\mathfrak{B}$ also and hence $B \leq_{\mathrm{e}} D(\mathfrak{B}) = g^{-1}(\mathfrak{A})$. Thus $d_{\mathrm{e}}(B) \in \mathrm{CS}(\mathfrak{A})$. So we have the following property of the partially forcing definable sets on $\mathfrak{A}$:

Proposition 2.4.3. *The enumeration degree of every partially forcing definable on $\mathfrak{A}$ set is in* $\mathrm{CS}(\mathfrak{A})$.

Definition 2.4.3. *A partial enumeration φ of $\mathfrak{A}$ is* 1-generic *if for every $e, x \in \mathbb{N}$, there exists a partial finite part $\tau \subseteq \varphi$ such that $\tau \Vdash F_e(x)$ or $\tau \Vdash \neg F_e(x)$.*

The proof of the following properties is standard.

Proposition 2.4.4. [15] *Let φ be a partial 1-generic enumeration. Then*

$$\varphi \models F_e(x) \iff (\exists \tau \subseteq \varphi)(\tau \Vdash F_e(x))\ ;$$

$$\varphi \models \neg F_e(x) \iff (\exists \tau \subseteq \varphi)(\tau \Vdash \neg F_e(x))\ .$$

Lemma 2.4.5. *Let $\{B_i\}_{i\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{N}$ each of them not partially forcing definable on $\mathfrak{A}$ and $Q = \bigoplus_i B_i$. There exists a partial 1-generic enumeration φ of $\mathfrak{A}$ satisfying the following conditions:*

(1) $\varphi \leq_{\mathrm{e}} D(\mathfrak{A})' \oplus Q^+$.
(2) $\varphi^{-1}(\mathfrak{A})' \leq_{\mathrm{e}} \varphi \oplus D(\mathfrak{A})'$.
(3) $B_i \not\leq_{\mathrm{e}} \varphi^{-1}(\mathfrak{A})$ *for every* $i \in \mathbb{N}$.

Proof. We shall construct the enumeration φ by stages. On each stage s we shall define a partial finite part δ_s so that $\delta_s \subseteq \delta_{s+1}$ and take $\varphi = \bigcup_s \delta_s$, where $\varphi(x) = y \iff (\exists s)(\delta_s(x) = y \ \&\ y \neq \perp)$. We shall consider three kinds of stages. On stages $s = 3r$ we shall ensure that the mapping φ is surjective. On stages $q = 3r + 1$ we shall ensure that φ is 1-generic and on stages $3r + 2$ we shall ensure that φ omits all B_i, i.e. for every natural number i the set $B_i \not\leq_{\mathrm{e}} \varphi^{-1}(\mathfrak{A})$.

Let δ_0 be the empty finite part and suppose that δ_s is defined.

(a) Stage $s = 3r$. Let x_0 be the least natural number which does not belong to $\mathrm{dom}(\delta_s)$ and let s_0 be the least natural number which does not belong to the range of δ_s. Set $\delta_{s+1}(x_0) = s_0$ and $\delta_{s+1}(x) = \delta_s(x)$ for $x \neq x_0$.

(b) Stage $s = 3\langle e, x\rangle + 1$. Check whether there exists a partial finite part ρ such that $\rho \Vdash F_e(x)$ and $\delta_s \subseteq \rho$. If the answer is positive, then let δ_{s+1} be the partial finite part with a least extension of δ_s. If the answer is negative then let $\delta_{s+1} = \delta_s$.

(c) Stage $s = 3\langle e, i\rangle + 2$. Consider the set

$$C_e = \{x : (\exists \tau \supseteq \delta_s)(\tau \Vdash F_e(x))\} \ .$$

Clearly C_e is partially forcing definable on $\mathfrak{A}$ and hence $C_e \neq B_i$.

Let x_e be the least natural number such that

$$x_e \in C_e \ \&\ x_e \notin B_i \vee x_e \notin C_e \ \&\ x_e \in B_i \ .$$

Suppose that $x_e \in C_e$. Then there exists a τ such that

$$\delta_s \subseteq \tau \ \&\ \tau \Vdash F_e(x_e) \ . \tag{1}$$

Let δ_{s+1} be the partial finite part τ with a least code satisfying the above condition. If $x_e \notin C_e$, then set $\delta_{s+1} = \delta_s$. Notice that in this case we have that $\delta_{s+1} \Vdash \neg F_e(x_e)$.

From the construction above it follows immediately that $\varphi = \bigcup_s \delta_s$ is a partial 1-generic enumeration of $\mathfrak{A}$. The construction of φ is effective relatively $Q^+ \oplus D(\mathfrak{A})'$, i.e. φ is e-reducible to $Q^+ \oplus D(\mathfrak{A})'$ and hence it satisfies (1).

To see (2) set $\varphi^{-1}(\mathfrak{A})' = L^+$, where

$$L = \{\langle a, x\rangle \mid x \in \Gamma_a(\varphi^{-1}(\mathfrak{A}))\} \ .$$

Then

$$2\langle a, x\rangle \in \varphi^{-1}(\mathfrak{A})' \iff (\exists \tau \subseteq \varphi)(\tau \Vdash F_a(x)) \ .$$

$$2\langle a, x\rangle + 1 \in \varphi^{-1}(\mathfrak{A})' \iff (\exists \tau \subseteq \varphi)(\tau \Vdash \neg F_a(x)) \ .$$

We have also from the definition of the forcing relation:

$$\{(\tau, a, x) \mid \tau \Vdash F_a(x)\} \leq_{\mathrm{e}} D(\mathfrak{A}).$$

$$\{(\tau, a, x) \mid \tau \Vdash \neg F_a(x)\} \leq_{\mathrm{e}} D(\mathfrak{A})'.$$

Therefore $\varphi^{-1}(\mathfrak{A})' \leq_{\mathrm{e}} \varphi \oplus D(\mathfrak{A})'$.

It remains to show that $B_i \not\leq_{\mathrm{e}} \varphi^{-1}(\mathfrak{A})$ for every $i \in \mathbb{N}$. Towards a contradiction assume that $B_i \leq_{\mathrm{e}} \varphi^{-1}(\mathfrak{A})$. Then there exists an e such that

$$B_i = \{x : \varphi \models F_e(x)\} \ .$$

Consider the stage $s = 3\langle e, i\rangle + 2$. By the construction we have that there is x_e

$$x_e \notin B_i \ \& \ \delta_{s+1} \Vdash F_e(x_e) \vee x_e \in B_i \ \& \ \delta_{s+1} \Vdash \neg F_e(x_e) \ .$$

Hence by the genericity of φ

$$x_e \notin B_i \ \& \ \varphi \models F_e(x_e) \vee x_e \in B_i \ \& \ \varphi \models \neg F_e(x_e) \ .$$

A contradiction.

□

Corollary 2.4.6. *If $B \leq_{\mathrm{e}} \varphi^{-1}(\mathfrak{A})$ for all partial 1-generic enumerations φ, then B is partially forcing definable on $\mathfrak{A}$.*

Lemma 2.4.7. *There exists a partial 1-generic enumeration φ of $\mathfrak{A}$ such that $\varphi \leq_{\mathrm{e}} D(\mathfrak{A})'$ and $\varphi^{-1}(\mathfrak{A})' \leq_{\mathrm{e}} \varphi \oplus D(\mathfrak{A})'$ and hence $\varphi^{-1}(\mathfrak{A})' \leq_{\mathrm{e}} D(\mathfrak{A})'$.*

Repeat the same construction as in Lemma 2.4.5 using only stages (a) and (b).

3. The quasi-minimal degree

Another specific property of the degree spectra proved by Soskov [15] which is not valid for all upwards closed sets of degrees is the existence of quasi-minimal degree for the degree spectra. Slaman and Sorbi relativized the notion of a quasi-minimal degree for an arbitrary set of e-degrees.

Definition 3.0.4. [13] *Given any $I \subseteq \mathcal{D}_e$, we say that an e-degree $\mathbf{a}$ is I-quasi minimal if*

- $(\forall \mathbf{c} \in I)[\mathbf{c} \leq_{\mathrm{e}} \mathbf{a}]$*;*
- $(\forall$ *total* $\mathbf{c})[\mathbf{c} \leq_{\mathrm{e}} \mathbf{a} \iff (\exists \mathbf{b} \in I)(\mathbf{c} \leq_{\mathrm{e}} \mathbf{b})]$.

They proved that for every countable ideal I of e-degrees, I-quasi-minimal e-degrees exist.

Soskov generalized this notion for degree spectra of a structure.

Definition 3.0.5. [15] *Let $\mathcal{A}$ be a set of e-degrees. The degree $\mathbf{q}$ is quasi-minimal with respect to $\mathcal{A}$ if:*

- $\mathbf{q} \notin co(\mathcal{A})$.
- *If $\mathbf{a}$ is a total e-degree and $\mathbf{a} \geq_{\mathrm{e}} \mathbf{q}$, then $\mathbf{a} \in \mathcal{A}$.*
- *If $\mathbf{a}$ is a total e-degree and $\mathbf{a} \leq_{\mathrm{e}} \mathbf{q}$, then $\mathbf{a} \in co(\mathcal{A})$.*

From Selman's theorem it follows that if $\mathbf{q}$ is quasi-minimal with respect to $\mathcal{A}$, then $\mathbf{q}$ is an upper bound for $co(\mathcal{A})$.

Soskov proved the existence of a quasi-minimal degree with respect to every degree spectrum, i.e. for every structure $\mathfrak{A}$ there is an e-degree $\mathbf{q} \notin \mathrm{CS}(\mathfrak{A})$ and every total e-degree above $\mathbf{q}$ is in $\mathrm{DS}(\mathfrak{A})$ and every total e-degree below $\mathbf{q}$ is in $\mathrm{CS}(\mathfrak{A})$.

Theorem 3.0.8 (Soskov)**.** [15] *Let φ be a partial 1-generic enumeration of $\mathfrak{A}$. Then $d_{\mathrm{e}}(\varphi^{-1}(\mathfrak{A}))$ is quasi-minimal with respect to* $\mathrm{DS}(\mathfrak{A})$.

Soskov constructed a partial 1-generic enumeration φ of $\mathfrak{A}$ such that for every total function g on $\mathbb{N}$ if $g \leq_{\mathrm{e}} \varphi^{-1}(\mathfrak{A})$ then g is partially forcing definable on $\mathfrak{A}$ and hence $\mathrm{d}_e(g) \in \mathrm{CS}(\mathfrak{A})$.

As a corollary he received the theorem proved by Slaman and Sorbi [13], using the fact that every countable ideal of enumeration degrees is the co-spectrum of a structure.

Corollary 3.0.9 (Slaman and Sorbi)**.** [13] *For every countable ideal I of enumeration degrees there exists an enumeration degree $\mathbf{q}$ which is I-quasi minimal.*

We will prove three additional properties of the quasi-minimal degrees for the degree spectra.

3.1. **Properties of the quasi-minimal degrees for a degree spectrum.** The first property we want to point out is that there exist uncountably many quasi-minimal degrees with respect to the degree spectra.

Proposition 3.1.1. *For every countable structure $\mathfrak{A}$ there exist uncountably many quasi-minimal degrees with respect to* $\mathrm{DS}(\mathfrak{A})$*.*

Proof. Suppose that all quasi-minimal degrees with respect to $\mathrm{DS}(\mathfrak{A})$ are in the sequence $\mathbf{q}_0, \mathbf{q}_1, \dots, \mathbf{q}_n, \dots$ and consider sets $X_i \in \mathbf{q}_i$, for every $i \in \mathbb{N}$. Then all $\mathbf{q}_i$ are not in $\mathrm{CS}(\mathfrak{A})$ and hence every set X_i is not partially forcing definable on $\mathfrak{A}$, $i \in \mathbb{N}$.

By Lemma 2.4.5 we could construct a partial 1-generic enumeration φ of $\mathfrak{A}$ such that $X_i \not\leq_{\mathrm{e}} \varphi^{-1}(\mathfrak{A})$.

Thus $d_{\mathrm{e}}(\varphi^{-1}(\mathfrak{A}))$ is quasi-minimal with respect to $\mathrm{DS}(\mathfrak{A})$ and it is not in the sequence $\{\mathbf{q}_i\}_{i \in \mathbb{N}}$. A contradiction.

□

3.2. **Jumps of quasi-minimal degrees for a degree spectrum.** In this subsection we shall show that for every countable structure $\mathfrak{A}$ the elements of $\mathrm{DS}_1(\mathfrak{A})$ can be represented as e-jumps of quasi-minimal degrees with respect to $\mathrm{DS}(\mathfrak{A})$. This is an analogue of a jump inversion theorem of McEvoy [9] for the enumeration degrees who proved that any total e-degree above $\mathbf{0}_e{}'$ is an e-jump of a quasi-minimal degree.

We know by Proposition 2.4.2 that for every partial enumeration φ of the structure $\mathfrak{A}$ the enumeration degree of $\varphi^{-1}(\mathfrak{A})'$ belongs to $\mathrm{DS}_1(\mathfrak{A})$. Now we shall prove that every element $\mathbf{a}$ of the jump spectrum of $\mathfrak{A}$ is an e-jump of a quasi-minimal degree with respect to $\mathrm{DS}(\mathfrak{A})$, i.e. we will construct a partial 1-generic enumeration of $\mathfrak{A}$ and a set F - quasi-minimal over $\varphi^{-1}(\mathfrak{A})$ such that $F' \in \mathbf{a}$.

We will use some ideas from Ganchev [5] about a quasi-minimal set over a given set. This techniques are a modification of the regular enumerations introduced by Soskov [14] and used in [5, 16, 7] for constructing a quasi-minimal set over a given set.

Definition 3.2.1. *Let $B \subseteq \mathbb{N}$. The set $F \subseteq \mathbb{N}$ is called quasi-minimal over B if $B <_{\mathrm{e}} F$ and for every total set $X \leq_{\mathrm{e}} F$ we have that $X \leq_{\mathrm{e}} B$.*

Theorem 3.2.1. *Let $B \subseteq \mathbb{N}$ and Q be a total set such that $B' \leq_{\mathrm{e}} Q$. There exists a set F with the following properties:*

1. *$F' \equiv_{\mathrm{e}} Q$;*
2. *the set F is quasi-minimal over B.*

The set F in the above theorem is constructed as the graph of a B-generic function. We will prove this theorem in section 4.

Lemma 3.2.2. *Let* $\mathbf{q}$ *be quasi-minimal e-degree with respect to* $\mathrm{DS}(\mathfrak{A})$ *and* $d_{\mathrm{e}}(B) = \mathbf{q}$. *If the set* $F \subseteq \mathbb{N}$ *is quasi minimal over* B *then* $d_{\mathrm{e}}(F)$ *is quasi-minimal with respect to* $\mathrm{DS}(\mathfrak{A})$.

Proof. Firstly $d_{\mathrm{e}}(F) \notin \mathrm{CS}(\mathfrak{A})$. Otherwise since $B \leq_{\mathrm{e}} F$, $d_{\mathrm{e}}(B) = \mathbf{q}$ will be in $\mathrm{CS}(\mathfrak{A})$, which contradicts the fact that $\mathbf{q}$ is quasi-minimal with respect to the degree spectrum $\mathrm{DS}(\mathfrak{A})$.

Let X be a total set.

If $X \leq_{\mathrm{e}} F$, then since F is quasi-minimal over B we have that $X \leq_{\mathrm{e}} B$. Since $d_{\mathrm{e}}(B)$ is quasi-minimal with respect to the degree spectrum $\mathrm{DS}(\mathfrak{A})$ and X is total we have that $d_{\mathrm{e}}(X) \in \mathrm{CS}(\mathfrak{A})$.

If $X \geq_{\mathrm{e}} F$ then $X \geq_{\mathrm{e}} B$. Then since $d_{\mathrm{e}}(B)$ is quasi-minimal with respect to the degree spectrum of $\mathfrak{A}$ and X is a total set we know that $d_{\mathrm{e}}(X) \in \mathrm{DS}(\mathfrak{A})$.

□

Theorem 3.2.3. *For every* $\mathbf{a} \in \mathrm{DS}_1(\mathfrak{A})$ *there is a set* F*, whose e-degree is quasi-minimal with respect to* $\mathrm{DS}(\mathfrak{A})$ *and with* $F' \in \mathbf{a}$*, i.e.* $d_{\mathrm{e}}(F)' = \mathbf{a}$.

Proof. Let $\mathbf{a} \in \mathrm{DS}_1(\mathfrak{A})$. Then there is a total enumeration g of $\mathfrak{A}$ such that $d_{\mathrm{e}}(g^{-1}(\mathfrak{A}))' \leq_{\mathrm{T}} \mathbf{a}$. Consider the structure $\mathfrak{B} = (\mathbb{N}, g^{-1}(R_1), \ldots, g^{-1}(R_k))$. Note that the diagram of $\mathfrak{B}$ is $g^{-1}(\mathfrak{A})$. By Lemma 2.4.7 there is a partial 1-generic enumeration φ of $\mathfrak{B}$ such that $\varphi^{-1}(\mathfrak{B})' \leq_{\mathrm{e}} D(\mathfrak{B})' \equiv_{\mathrm{e}} g^{-1}(\mathfrak{A})'$. Then $d_{\mathrm{e}}(\varphi^{-1}(\mathfrak{B}))$ is quasi-minimal with respect to the degree spectrum $\mathrm{DS}(\mathfrak{B})$.

Let Q be a total set and $Q \in \mathbf{a}$. We have $\varphi^{-1}(\mathfrak{B})' \leq_{\mathrm{e}} g^{-1}(\mathfrak{A})' \leq_{\mathrm{e}} Q$. Then by Theorem 3.2.1 there is a set F which is quasi-minimal over $\varphi^{-1}(\mathfrak{B})$, with an e-jump equivalent to Q, i.e. $\varphi^{-1}(\mathfrak{B}) \leq_{\mathrm{e}} F$, $F' \equiv_{\mathrm{e}} Q$ and for every total set $X \leq_{\mathrm{e}} F$ we have that $X \leq_{\mathrm{e}} \varphi^{-1}(\mathfrak{B})$. Let $\mathbf{q}$ be the e-degree of F. By Lemma 3.2.2 we have that $\mathbf{q}$ is quasi-minimal with respect to $\mathrm{DS}(\mathfrak{B})$ and hence $\mathbf{q}$ is quasi-minimal with respect to $\mathrm{DS}(\mathfrak{A})$. Moreover $F' \equiv_{\mathrm{e}} Q$ and hence $d_{\mathrm{e}}(F)' = \mathbf{q}' = \mathbf{a}$.

□

Corollary 3.2.4. *The first jump spectrum of every structure* $\mathfrak{A}$ *consists exactly of the enumeration jumps of the quasi-minimal degrees for* $\mathrm{DS}(\mathfrak{A})$.

If we consider a simple computable structure $\mathfrak{A} = (\mathbb{N}, =)$, the co-spectrum of $\mathfrak{A}$ contains only $\mathbf{0}_{\mathrm{e}}$ and if $\mathbf{q}$ is a quasi-minimal for $\mathrm{DS}(\mathfrak{A})$, then $\mathbf{q}$ is a quasi-minimal e-degree. We receive the following theorem proved by McEvoy [9].

Corollary 3.2.5 (McEvoy)**.** [9] *For every total e-degree* $\mathbf{a} \geq_{\mathrm{e}} \mathbf{0}_e{}'$ *there is a quasi-minimal degree* $\mathbf{q}$ *with* $\mathbf{q}' = \mathbf{a}$.

3.3. **Splitting a total set.** Jockusch [8] showed that every nonzero Turing degree contains a semi-recursive set A, such that both A and $\overline{A}$ are not c.e. In the context of enumeration reducibility this property can be translated as follows. A nonzero enumeration degree $\mathbf{a}$ is total if and only if there is a semi-recursive set A, which is not c.e. or co-c.e. such that $\mathbf{a} = d_{\mathrm{e}}(A) \vee d_{\mathrm{e}}(\overline{A})$. Arslanov, Cooper and Kalimullin [1] showed that if A is a semi recursive set such that A and $\overline{A}$ are not c.e., then the e-degree of A is quasi-minimal.

Proposition 3.3.1. [8, 1] *For every total e-degree* $\mathbf{a}$ *there are quasi-minimal degrees* $\mathbf{p}$ *and* $\mathbf{q}$ *such that* $\mathbf{a} = \mathbf{p} \vee \mathbf{q}$.

We will prove an analogue of this theorem for the first jump spectrum of $\mathfrak{A}$.

Theorem 3.3.2. *For every element* $\mathbf{a}$ *of the first jump spectrum* $\mathrm{DS}_1(\mathfrak{A})$ *of the structure* $\mathfrak{A}$ *there exist quasi-minimal with respect to* $\mathrm{DS}(\mathfrak{A})$ *e-degrees* $\mathbf{p}$ *and* $\mathbf{q}$ *such that* $\mathbf{a} = \mathbf{p} \vee \mathbf{q}$.

Call the pair of sets (X, Y) a splitting of a set Q if $X \oplus Y \equiv_{\mathrm{e}} Q$ and $X, Y <_{\mathrm{e}} Q$. We will use a method of splitting a total set considered by Ganchev [6, 7]. This method is used for a generalization of some forcing constructions so that instead of building one set with some properties, we build two sets X and Y with the same properties but in addition (X, Y) is a splitting of Q for some given total set Q. He proved that if B is a set of natural numbers such that $B' \leq_{\mathrm{e}} Q$ and Q is total then there exist total functions f and g, such that $B \leq_{\mathrm{e}} f$, and $B \leq_{\mathrm{e}} g$ $f \oplus g \equiv_{\mathrm{e}} Q$. Ganchev proved this method of splitting a total set constructing total functions f and g. We shall prove the following theorem in the next section for partial functions.

Theorem 3.3.3. *Let* Q *be a total set and* $B' \leq_{\mathrm{e}} Q$. *There exist partial functions* ψ *and* χ *on* $\mathbb{N}$ *such that*

(1) $\psi \oplus \chi \equiv_{\mathrm{e}} Q$;
(2) *the functions* ψ *and* χ *are quasi-minimal over* B.

Using this theorem we give the proof of Theorem 3.3.2.

Proof. of Theorem 3.3.2
Let $\mathbf{a} \in \mathrm{DS}_1(\mathfrak{A})$ and g be a total enumeration of $\mathfrak{A}$ such that $d_{\mathrm{e}}(g^{-1}(\mathfrak{A}))' \leq_{\mathrm{e}} \mathbf{a}$. Let Q be a total set such that $Q \in \mathbf{a}$. Then $g^{-1}(\mathfrak{A})' \leq_{\mathrm{e}} Q$. Consider the structure $\mathfrak{B} = (\mathbb{N}, g^{-1}(R_1), \ldots, g^{-1}(R_k))$. Actually the diagram of $\mathfrak{B}$ is $g^{-1}(\mathfrak{A})$. We first construct a partial 1-generic enumeration φ of $\mathfrak{B}$, so $\varphi^{-1}(\mathfrak{B})$ is a quasi-minimal with respect to $\mathrm{DS}(\mathfrak{B})$ and by Lemma 2.4.7 we can suppose that $\varphi^{-1}(\mathfrak{B})' \leq_{\mathrm{e}} D(\mathfrak{B})'$. Denote by $B = \varphi^{-1}(\mathfrak{B})$. Notice that $B' = \varphi^{-1}(\mathfrak{B})' \leq_{\mathrm{e}} D(\mathfrak{B})' = g^{-1}(\mathfrak{A})' \leq_{\mathrm{e}} Q$. Then $B' \leq_{\mathrm{e}} Q$.

By Theorem 3.3.3 there exist partial functions ψ and χ on $\mathbb{N}$ such that the set $Q \equiv_{\mathrm{e}} \psi \oplus \chi$ and ψ and χ are quasi-minimal over B, i.e. $B \leq_{\mathrm{e}} \psi, \chi$ and for every total set X if $X \leq_{\mathrm{e}} \psi$ then $X \leq_{\mathrm{e}} B$, and similarly if $X \leq_{\mathrm{e}} \chi$ then $X \leq_{\mathrm{e}} B$.

Since the e-degree of $B = \varphi^{-1}(\mathfrak{B})$ is quasi-minimal for $\mathrm{DS}(\mathfrak{B})$ and ψ and χ are quasi-minimal over B it follows from Lemma 3.2.2 that $\mathbf{p} = d_{\mathrm{e}}(\psi)$ and $\mathbf{q} = d_{\mathrm{e}}(\chi)$ are quasi-minimal for $\mathrm{DS}(\mathfrak{B})$ and hence for $\mathrm{DS}(\mathfrak{A})$. Moreover $\mathbf{a} = \mathbf{p} \vee \mathbf{q}$.

□

4. A quasi-minimal set over a given set

In this section we will prove Theorem 3.2.1 and Theorem 3.3.3 and we will consider some properties of a relativized notion of a quasi-minimal set in the enumeration degrees. The properties of the partial generic functions and their relationship with the genericity in the Turing degrees are first considered by Case [2] and then investigated by Copestake [4]. The techniques and arguments we use are inspired by Copestake's work and are similar to those used in [15, 5, 7, 16].

4.1. **Partial B-generic functions.** Fix a set $B \subseteq \mathbb{N}$.

A *B-regular finite part* is a partial finite part $\tau : \mathbb{N} \rightharpoonup \mathbb{N} \cup \{\perp\}$ such that $\mathrm{dom}(\tau) = [0, 2q+1]$ and for all odd $z \in \mathrm{dom}(\tau)$, $\tau(z) \in B$ or $\tau(z) = \perp$.

The rank $|\tau|$ of τ we call the number of the odd elements x of $\mathrm{dom}(\tau)$, for which $\tau(x) \neq \perp$.

The partial function $\varphi : \mathbb{N} \rightharpoonup \mathbb{N}$ we call B*-regular* if $\varphi([2\mathbb{N}+1]) = B$, i.e. for every $b \in B$ there is a $2z+1 \in \mathrm{dom}(\varphi)$ such that $\varphi(2z+1) = b$. As was pointed out in subsection 2.1. we identify the partial functions with their graphs, i.e. by $B \leq_{\mathrm{e}} \varphi$ we mean $B \leq_{\mathrm{e}} \langle\varphi\rangle$, where $\langle\varphi\rangle = \{\langle x, y\rangle \mid x \in \mathrm{dom}(\varphi)\ \&\ \varphi(x) = y\}$. So $z \in B \iff (\exists x)(\langle 2x+1, z\rangle \in \langle\varphi\rangle)$. From here it follows that $B \leq_{\mathrm{e}} \varphi$ for every B-regular function φ.

Recall that for a partial finite part τ and a partial function φ on $\mathbb{N}$ we write $\tau \subseteq \varphi$ if for every $z \in \mathrm{dom}(\tau)$ we have $\tau(z) = \varphi(z)$ or $\tau(z) = \perp$ and $\varphi(z)$ is undefined.

If φ is a B-regular function then for every partial finite part $\delta \subseteq \varphi$ there exists a B-regular partial finite part $\tau \subseteq \varphi$ such that $\delta \subseteq \tau$. Moreover, there exist such B-regular partial finite parts τ of φ of arbitrary large rank.

For every partial finite part τ denote by $\langle\tau\rangle$ the set $\{\langle x, y\rangle \mid \tau(x) = y\ \&\ y \neq \perp\}$.

Let Γ_a be an enumeration operator, τ be a partial finite part and $x \in \mathbb{N}$. By $x \in \Gamma_a(\tau)$ we mean $x \in \Gamma_a(\langle\tau\rangle) \iff (\exists v)(\langle v, x\rangle \in W_a\ \&\ D_v \subseteq \langle\tau\rangle)$.

Denote by $\mathcal{R}_B$ the set of all B-regular finite parts. We will identify the finite parts with their codes since they are finite functions. It is clear that the set $\mathcal{R}_B \leq_{\mathrm{e}} B$.

Definition 4.1.1. *A partial function φ is B-*generic *if φ is B-regular and for every set S of B-regular finite parts such that $S \leq_{\mathrm{e}} B$,*

$$(\exists\tau \in \mathcal{R}_B)(\tau \subseteq \varphi\ \&\ ((\tau \in S) \vee (\forall\rho \supseteq \tau)(\rho \in \mathcal{R}_B \Rightarrow \rho \notin S))).$$

For every partial function φ denote by φ^* the total function $\varphi : \mathbb{N} \to \mathbb{N} \cup \{\perp\}$ such that if $\varphi(x)$ is defined then $\varphi^*(x) = \varphi(x)$ and $\varphi^*(x) = \perp$ otherwise.

Lemma 4.1.1. *Let φ be a B-generic function. Then $\varphi' \equiv_{\mathrm{e}} B' \oplus \varphi^*$.*

Proof. Since φ is B-regular then $B \leq_{\mathrm{e}} \varphi$ and hence $B' \leq_{\mathrm{e}} \varphi'$. On the other hand we have that $\mathrm{dom}(\varphi) \leq_e \varphi$ and then $\mathrm{dom}(\varphi)^+ \leq_e \varphi'$. But $\varphi^* \leq_{\mathrm{e}} \mathrm{dom}(\varphi)^+ \oplus \varphi \leq_{\mathrm{e}} \varphi'$.

Thus $B' \oplus \varphi^* \leq \varphi'$.

For the other direction we will use the compactness property of the enumeration operators and the fact that φ is B-generic.

We have $\varphi' = \{\langle a, x\rangle \mid x \in \Gamma_a(\varphi)\}^+$.

For any $a, x \in \mathbb{N}$ we have

(1) $X_{\langle a,x\rangle} = \{\tau \in \mathcal{R}_B \mid x \in \Gamma_a(\tau)\} \leq_{\mathrm{e}} B$;
(2) $Y_{\langle a,x\rangle} = \{\tau \in \mathcal{R}_B \mid (\exists\rho \in \mathcal{R}_B)(\tau \subseteq \rho\ \&\ x \in \Gamma_a(\rho)\} \leq_{\mathrm{e}} B$;
(3) $Z_{\langle a,x\rangle} = \{\tau \in \mathcal{R}_B \mid (\forall\rho \in \mathcal{R}_B)(\tau \subseteq \rho \Rightarrow x \notin \Gamma_a(\rho)) \leq_{\mathrm{e}} B'$.

Actually $X_{\langle a,x\rangle} \leq_{\mathrm{e}} B$ and $Y_{\langle a,x\rangle} \leq_{\mathrm{e}} X_{\langle a,x\rangle} \oplus \mathcal{R}_B$ uniformly in a, x. And $Z_{\langle a,x\rangle} = \mathcal{R}_B \setminus Y_{\langle a,x\rangle} \leq_{\mathrm{e}} B'$ uniformly in a, x.

Since φ is B-regular we have:

$$x \in \Gamma_a(\varphi) \iff (\exists\tau \in \mathcal{R}_B)(\tau \subseteq \varphi^*\ \&\ x \in \Gamma_a(\tau)) \iff (\exists\tau \subseteq \varphi^*)(\tau \in X_{\langle a,x\rangle}).$$

$$x \notin \Gamma_a(\varphi) \iff (\forall\tau \subseteq \varphi^*)(\tau \notin X_{\langle a,x\rangle}).$$

Since φ is B-generic for all a, x:

$$(\exists\tau \subseteq \varphi^*)(\tau \in X_{\langle a,x\rangle} \vee \tau \in Z_{\langle a,x\rangle}).$$

And finally:

$$x \notin \Gamma_a(\varphi) \iff (\exists\tau \subseteq \varphi^*)(\tau \in Z_{\langle a,x\rangle}).$$

Then $\{\langle a,x\rangle \mid x\in\Gamma_a(\varphi)\}^+ \leq_{\mathrm{e}} B'\oplus\varphi^*$ and hence $\varphi' \leq_{\mathrm{e}} B'\oplus\varphi^*$.

□

Lemma 4.1.2. *If φ is a B-generic partial function on $\mathbb{N}$, then $B <_{\mathrm{e}} \varphi$ and for every total set $X \leq_{\mathrm{e}} \varphi$ we have that $X \leq_{\mathrm{e}} B$.*

A function φ with the property above : $B <_{\mathrm{e}} \varphi$ and for every total set $X \leq_{\mathrm{e}} \varphi$ we have that $X \leq_{\mathrm{e}} B$ we shall call *quasi-minimal over B*.

Proof. We already know that $B \leq_{\mathrm{e}} \varphi$. Suppose that $\varphi \leq_{\mathrm{e}} B$. Then the following set of B-regular finite parts will be e-reducible to B:

$$S = \{\tau : \tau\in\mathcal{R}_B \;\&\; (\exists x\in\mathrm{dom}(\tau)\cap\mathrm{dom}(\varphi))(\varphi(x)\neq\tau(x))\}.$$

Since φ is B-generic then there is a B-regular finite part $\tau\subseteq\varphi$ such that $\tau\in S$ or $(\forall\rho\in\mathcal{R}_B)(\tau\subseteq\rho\Rightarrow\rho\notin S)$. It is clear that both cases are impossible. So φ is not e-reducible to B.

For any total set $X\subseteq\mathbb{N}$ one can construct a total function g on $\mathbb{N}$, so that $g\equiv_{\mathrm{e}} X$. To prove that φ is quasi-minimal over B, it is sufficient to show that if g is a total function on $\mathbb{N}$ and $g\leq_{\mathrm{e}}\varphi$, then $g\leq_{\mathrm{e}} B$.

Let g be a total function and $g=\Gamma_e(\varphi)$.

Consider the set

$$S = \{\tau \mid \tau\in\mathcal{R}_B \;\&\; (\exists x, y_1\neq y_2\in\mathbb{N})(\langle x,y_1\rangle\in\Gamma_e(\tau) \;\&\; \langle x,y_2\rangle\in\Gamma_e(\tau))\}.$$

Since $S\leq_{\mathrm{e}} B$, we have that there exists a B-regular partial finite part $\tau\subseteq\varphi$ such that either $\tau\in S$ or $(\forall\rho\in\mathcal{R}_B)(\rho\supseteq\tau\Rightarrow\rho\notin S)$.

Assume that $\tau\in S$. Then there exist $x, y_1\neq y_2$ such that $\langle x,y_1\rangle\in\Gamma_e(\tau)$ and $\langle x,y_2\rangle\in\Gamma_e(\tau)$. Then $g(x)=y_1$ and $g(x)=y_2$, which is impossible. So, $(\forall\rho\in\mathcal{R}_B)(\rho\supseteq\tau\Rightarrow\rho\notin S)$.

Consider now the set:

$$\begin{aligned} S_1 = \{\rho : \rho\in\mathcal{R}_B \;\&\; (\exists\mu\in\mathcal{R}_B)(\tau\subseteq\mu\subseteq\rho \;\&\; & (\exists\delta_1,\delta_2\in\mathcal{R}_B)(\delta_1\supseteq\mu \;\&\; \delta_2\supseteq\mu \;\&\\ & (\exists x, y_1\neq y_2\in\mathbb{N})(\langle x,y_1\rangle\in\Gamma_e(\delta_1) \;\&\; \langle x,y_2\rangle\in\Gamma_e(\delta_2) \;\&\\ & \mathrm{dom}(\rho)=\mathrm{dom}(\delta_1)\cup\mathrm{dom}(\delta_2) \;\&\\ & (\forall z)(z\in\mathrm{dom}(\rho)\setminus\mathrm{dom}(\mu)\Rightarrow\rho(z)=\perp))))\}. \end{aligned}$$

We have that $S_1\leq_{\mathrm{e}} B$ and hence there exists a B-regular partial finite part $\tau_1\subseteq\varphi$ such that either $\tau_1\in S_1$ or $(\forall\rho\in\mathcal{R}_B)(\rho\supseteq\tau_1\Rightarrow\rho\notin S_1)$.

Assume that $\tau_1\in S_1$. Then there exists a B-regular partial finite part μ such that $\tau\subseteq\mu\subseteq\tau_1$ and for some B-regular finite parts $\delta_1\supseteq\mu$, $\delta_2\supseteq\mu$ and for some $x, y_1\neq y_2\in\mathbb{N}$ we have

$$\begin{aligned} &\langle x,y_1\rangle\in\Gamma_e(\delta_1) \;\&\; \langle x,y_2\rangle\in\Gamma_e(\delta_2) \;\&\; \mathrm{dom}(\tau_1)=\mathrm{dom}(\delta_1)\cup\mathrm{dom}(\delta_2) \;\&\\ &(\forall z)(z\in\mathrm{dom}(\tau_1)\setminus\mathrm{dom}(\mu)\Rightarrow\tau_1(z)=\perp). \end{aligned}$$

Let $g(x)=y$. Then $\langle x,y\rangle\in\Gamma_e(\varphi)$. Hence there exists a B -regular finite part $\rho\supseteq\tau_1$ such that $\langle x,y\rangle\in\Gamma_e(\rho)$. Without loss of generality we may assume $y\neq y_1$. Define the partial finite part ρ_0 as follows:

$$\rho_0(z) = \begin{cases} \delta_1(z) & \text{if } z\in\mathrm{dom}(\delta_1),\\ \rho(z) & \text{if } z\in\mathrm{dom}(\rho)\setminus\mathrm{dom}(\delta_1). \end{cases}$$

Then ρ_0 is a B-regular finite part and $\tau\subseteq\rho_0$, $\delta_1\subseteq\rho_0$. Moreover since $\rho\supseteq\tau_1$ for all $z\in\mathrm{dom}(\rho)$ if $\rho(z)\neq\perp$, then $\rho(z)=\rho_0(z)$. Hence $\langle x,y_1\rangle\in\Gamma_e(\rho_0)$ and

$\langle x,y\rangle \in \Gamma_e(\rho_0)$. So, $\rho_0 \in S$. Since $\rho_0 \supseteq \tau$ this contradicts with the property of τ proved above.

Thus, $(\forall \rho \in \mathcal{R}_B)(\rho \supseteq \tau_1 \Rightarrow \rho \notin S_1)$.

Since $\tau \subseteq \varphi$ and $\tau_1 \subseteq \varphi$ then let $\tau_2 = \tau_1 \cup \tau$. Notice that $\tau_2 \subseteq \varphi$ and $\tau_2 \in R_B$. We shall show that for all $x, y \in \mathbb{N}$

$$g(x) = y \iff (\exists \delta \in \mathcal{R}_B)(\delta \supseteq \tau_2 \ \&\ \langle x,y\rangle \in \Gamma_e(\delta)).$$

And hence $g \leq_{\mathrm{e}} B$.

If $g(x) = y$, then $\langle x,y\rangle \in \Gamma_e(\varphi)$, and there exists $\rho \subseteq \varphi$ & $\rho \in \mathcal{R}_B$ & $\langle x,y\rangle \in \Gamma_e(\rho)$. Let $\delta = \tau_2 \cup \rho$.

For the other direction assume that $\delta_1 \supseteq \tau_2$ and $\langle x,y\rangle \in \Gamma_e(\delta_1)$. Suppose that $g(x) = y_2$ and $y \neq y_2$. Then there exists a $\delta_2 \supseteq \tau_2$ such that $\langle x, y_2\rangle \in \Gamma_e(\delta_2)$. Set

$$\rho(x) = \begin{cases} \tau_2(x) & \text{if } x \in \mathrm{dom}(\tau_2), \\ \perp & \text{if } x \in (\mathrm{dom}(\delta_1) \cup \mathrm{dom}(\delta_2)) \setminus \mathrm{dom}(\tau_2). \end{cases}$$

Clearly, $\rho \supseteq \tau_2 \supseteq \tau_1$ and $\rho \in S_1$. A contradiction.

□

For each partial finite part τ with $\mathrm{dom}(\tau) = [1, q-1]$ denote by $lh(\tau) = q$ the number of the elements in $\mathrm{dom}(\tau)$ and for every $x \in \mathbb{N}$ by $\tau * x$ we denote the extension of τ such that $\tau * x(lh(\tau)) = x$.

Recall the theorem that we have to prove.

Proof. of Theorem 3.2.1

Let $B \subseteq \mathbb{N}$ and Q be a total set such that $B' \leq_{\mathrm{e}} Q$. We have to construct a set F with the following properties:

(1) $B <_{\mathrm{e}} F$;
(2) $F' \equiv_{\mathrm{e}} Q$.
(3) for every total $X \leq_{\mathrm{e}} F$ we have that $X \leq_{\mathrm{e}} B$.

By Lemma 4.1.1 and Lemma 4.1.2 it is enough to construct a partial B-generic function φ on $\mathbb{N}$ such that $\varphi \leq_{\mathrm{e}} B' \oplus Q \leq_{\mathrm{e}} Q$ and $B' \oplus \varphi^* \equiv_{\mathrm{e}} Q$. Then the set $F = \langle\varphi\rangle$ will have all the properties (1)-(3) from the theorem since by Lemma 4.1.1 $B' \oplus \varphi^* \equiv_{\mathrm{e}} \varphi'$ thus $F' \equiv_{\mathrm{e}} Q$ and by Lemma 4.1.2 the set F will be a quasi-minimal over B.

We shall construct the function φ in stages. At each stage n we shall define a finite part δ_n so that $\delta_n \subseteq \delta_{n+1}$ and take $\varphi^* = \bigcup_n \delta_n$. Then $\varphi(x) = \varphi^*(x)$ if $\varphi(x) \neq \perp$ and $\varphi(x)$ is undefined otherwise. We shall consider three kinds of stages. At stage $n = 3e$ we shall ensure that the mapping φ is B-regular. At stages $n = 3e+1$ we shall ensure that φ is B-generic. At stages $n = 3e+2$ we shall code the elements of Q in φ.

For every partial finite part τ denote by $\nu(\tau) = \mu\rho[\tau \subseteq \rho \ \&\ \rho \in \mathcal{R}_B \ \&\ |\rho| > |\tau|]$ if any, and let $\nu(\tau)$ be undefined otherwise. The function ν gives the B-regular extension of τ with the least code if any.

Let q be a computable in Q listing of the elements of Q, i.e. q is total and the range of q is Q.

Let δ_0 be the empty finite part and suppose that δ_n is defined.

(a) Case $n = 3e$. Set $\delta_{n+1} = \nu(\delta_n)$.

(b) Case $n = 3e+1$. Check whether there exists a B-regular finite part ρ such that $\delta_n \subseteq \rho$ and $\rho \in \Gamma_e(B)$. If the answer is positive, then let δ_{n+1} be the least

B-regular extension of δ_n belonging to $\Gamma_e(B)$. If the answer is negative then let $\delta_{n+1} = \delta_n$.

(c) Case $n = 3e + 2$. Let δ_{n+1} be the least B-regular extension of $\delta_n * q(e)$.

From the construction above it follows immediately that $\varphi^* = \bigcup_n \delta_n$ is e-reducible to $B' \oplus Q$ and hence $\varphi^* \leq_{\mathrm{e}} Q$ and $\varphi \leq_{\mathrm{e}} Q$. On the other hand we can effectively in $\varphi^* \oplus B'$ list the elements of Q. So, $\varphi^* \oplus B' \equiv_{\mathrm{e}} Q$. Moreover φ is B-regular since we built it only by B-regular finite parts and at stages $3e$ we always get a B-regular extension of greater rank.

It follows from the construction also that φ is B-generic. To see this consider a set $S \leq_e B$ of B-regular finite parts and hence $S = \Gamma_e(B)$ for some e. Then at stage $3e + 1$ the finite part δ_{n+1} is B-regular and $\delta_{n+1} \in S$ or $(\forall \rho \in \mathcal{R}_B)(\rho \supseteq \delta_{n+1} \Rightarrow \rho \notin S)$. Hence φ is B-generic.

$\square$

4.2. **Splitting a total set.** We shall explain the splitting method of a total set following Ganchev [7].

Let $\kappa : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $\{y_n\}_n$ be a sequence of natural numbers. If $\tau_0 = \emptyset$, $\tau_{n+1} = \kappa(\tau_n * y_n, n)$, then we will use the notation $\kappa(\{y_n\}_n) = \bigcup_n \tau_n$.

Let P be a set of functions over the natural numbers.

Lemma 4.2.1 (Ganchev)**.** [6, 7] *If κ is computable function in the total set $Q \subseteq \mathbb{N}$ and such that for every sequence $\{y_n\}_n$ of natural numbers computable in Q, $\kappa(\{y_n\}_n) \in P$, then there exist functions $\psi, \chi \in P$ such that $Q \equiv_{\mathrm{e}} \psi \oplus \chi$.*

The idea is the following:

Let q be an enumeration of Q such that $q \leq_{\mathrm{e}} Q$. We construct two sequences of finite parts $\{\tau_n\}_n$ and $\{\sigma_n\}_n$ by the following rule:

(1) $\tau_0 = \sigma_0 = \emptyset$;
(2) $y_n = \langle lh(\sigma_n), q(2n)\rangle$;
(3) $\tau_{n+1} = \kappa(\tau_n * y_n, n)$;
(4) $z_n = \langle lh(\tau_{n+1}), q(2n + 1)\rangle$;
(5) $\sigma_{n+1} = \kappa(\sigma_n * z_n, n)$.

Define $\psi = \kappa(\{y_n\}_n)$ and $\chi = \kappa(\{z_n\}_n)$. Then $\psi, \chi \in P$ and moreover since κ is computable in Q we have $\psi \leq_{\mathrm{e}} Q$ and $\chi \leq_{\mathrm{e}} Q$. Thus $\psi \oplus \chi \leq_{\mathrm{e}} Q$.

On the other side we have

(1) $lh(\tau_0) = 0$,
(2) $lh(\sigma_i) = (\psi(lh(\tau_i)))_0$,
(3) $lh(\tau_{i+1}) = (\chi(lh(\sigma_i)))_0$.

Hence the sequence

$$lh(\tau_0), lh(\sigma_0), lh(\tau_1), lh(\sigma_1), \ldots, lh(\tau_n), lh(\sigma_n), \ldots$$

is c.e. in $\psi \oplus \chi$ and for every i we have $q(2i) = (\psi(lh(\tau_i)))_1$ and $q(2i + 1) = (\chi(lh(\sigma_i)))_1$. Thus $Q \leq_{\mathrm{e}} \psi \oplus \chi$.

We will use this method to prove Theorem 3.3.3 and so the proof of Theorem 3.3.2 will be completed. The functions ψ and χ will be constructed as partial B- generic functions, and hence they both will be quasi-minimal over B.

Proof. of Theorem 3.3.3.

Let Q be a total set and $B' \leq_{\mathrm{e}} Q$. We shall construct partial functions ψ and χ on $\mathbb{N}$ such that $\psi \oplus \chi \equiv_{\mathrm{e}} Q$ and the functions ψ and χ are quasi-minimal over B.

Let P be the class of all B-generic functions on $\mathbb{N}$. Consider a function q which is a c.e. in Q enumeration of the set Q.

We will define the function κ so that κ will assure that the partial functions ψ and χ that we are going to construct will be B-generic.

First recall that for every partial finite part τ we denoted by $\nu(\tau) = \mu\rho[\tau \subseteq \rho \;\&\; \rho \in \mathcal{R}_B \;\&\; |\rho| > |\tau|]$ if any, and $\nu(\tau)$ is undefined otherwise.

For any finite part τ and any set X of B-regular finite parts, denote by $\mu(\tau, X) = \mu\rho[\nu(\tau) \subseteq \rho \;\&\; \rho \in X]$ if any, and $\mu(\tau, X) = \nu(\tau)$, otherwise.

Denote by $X_e = \{\rho \mid \rho \in \mathcal{R}_B \;\&\; \rho \in \Gamma_e(B)\}$ for each $e \in \mathbb{N}$. Let $\mathcal{R}_B * \mathbb{N}$ be the set of all partial finite parts of the form $\tau * y$, where $\tau \in \mathcal{R}_B$ and $y \in \mathbb{N}$. Then we can define the function κ as follows: $\kappa : \mathcal{R}_B * \mathbb{N} \times \mathbb{N} \to \mathcal{R}_B$ and for every $\tau \in \mathcal{R}_B$, $n \in \mathbb{N}$, $\kappa(\tau, n) = \mu(\tau, X_n)$. It is clear that κ is computable in B' and hence in Q.

The construction of ψ and χ will be carried by stages. At each stage n we shall construct two B-regular finite parts $\tau_{n+1} \supseteq \tau_n$ and $\sigma_{n+1} \supseteq \sigma_n$ assuring that the constructed functions will be B-generic.

Let $\tau_0 = \sigma_0 = \emptyset$.

Suppose that τ_n and σ_n are defined. Denote by $y_n = \langle lh(\sigma_n), q(2n)\rangle$. Then $\kappa(\tau_n * y_n, n)$ is the least partial B-regular finite part $\rho \supseteq \tau_n * y_n$ with rank greater than τ_n such that $\rho \in \Gamma_n(B)$ if such B-regular finite part exists and $\kappa(\tau_n * y_n, n)$ is the least B-regular extension of $\tau_n * y_n$ with a rank greater than τ_n, otherwise. Let $\tau_{n+1} = \kappa(\tau_n * y_n, n)$.

Denote by $z_n = \langle lh(\tau_{n+1}), q(2n+1)\rangle$. Then $\kappa(\sigma_n * z_n, n)$ is the least partial B-regular finite part $\rho \supseteq \sigma_n * z_n$ with rank greater than σ_n such that $\rho \in \Gamma_n(B)$ if such B-regular finite part exists and $\kappa(\sigma_n * z_n, n)$ is the least B-regular extension of $\sigma_n * z_n$ with a rank greater than σ_n, otherwise. Let $\sigma_{n+1} = \kappa(\sigma_n * z_n, n)$.

Let $\psi = \bigcup_n \tau_n$ and $\chi = \bigcup_n \sigma_n$. Notice that for every n, $\psi(lh(\tau_n)) = y_n$ and $\chi(lh(\sigma_n)) = z_n$ are defined. The sequences $\{y_n\}_n$ and $\{z_n\}_n$ are computable in $B' \oplus Q$ and hence in Q. We know that the function κ is computable in Q. So by the arguments above $\psi \oplus \chi \leq_{\mathrm{e}} B' \oplus Q \leq_{\mathrm{e}} Q$. On the other hand we coded the elements of the set Q in $\psi \oplus \chi$ and hence as above $\psi \oplus \chi \equiv_{\mathrm{e}} Q$.

It remains to see that ψ and χ are B-generic. First since we chose at each stage a B-regular finite part with a greater rank the functions ψ and χ are B-regular surjective on B. To see that ψ and χ are B-generic consider a set S of B-regular finite parts and $S \leq_{\mathrm{e}} B$. Then there is an enumeration operator $\Gamma_n(B) = S$ for some $n \in \mathbb{N}$. Consider the stage n. Let $\mu = \nu(\tau_n * y_n)$. Then if there is a B-regular finite part $\rho \supseteq \mu$, then τ_{n+1} will be the least one ρ in S. Otherwise there is no B-regular extension ρ of μ such that $\rho \in S$. Thus $(\forall \rho \in \mathcal{R}_B)(\rho \supseteq \tau_{n+1} \Rightarrow \rho \notin S)$. In other words ψ is B-generic.

The B-genericity of χ is shown in the same way.

So the functions ψ and χ are B-generic and $\psi \oplus \chi \equiv_{\mathrm{e}} Q$.

□

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