Generalization of the notion of jump sequence of sets for sequences of structures

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Abstract

We study the notion of relatively intrinsically c.e. sets with respect to a sequence of structures. We propose a generalization of the notion of jump sequence of sets to jump sequence of structures and study the relatively intrinsically c.e. sets in this notion.

1 Introduction

Let $\mathfrak{A} = (A; R_1, \dots, R_k)$ be a countable abstract structure. An enumeration f of $\mathfrak{A}$ is a bijection from $\mathbb{N}$ onto A. For an arbitrary set $X \subseteq A^a$ the pullback of X under the enumeration f is denoted by $f^{-1}(X)$ and defined as $\{\langle x_1 \dots x_a \rangle : (f(x_1), \dots, f(x_a)) \in X\}$. The pullback of the structure $\mathfrak{A}$ under f is $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \dots \oplus f^{-1}(R_k)$. We will consider only structures $\mathfrak{A} = (A; R_1, \bar{R}_1 \dots, R_k, \bar{R}_k)$ where equality is among the predicates $R_1, \dots, R_k$.

Definition 1.1. *A set $R \subseteq A$ is relatively intrinsically c.e. in $\mathfrak{A}$ if and only if $f^{-1}(R)$ is c.e. in $f^{-1}(\mathfrak{A})$ for every enumeration f of $\mathfrak{A}$.*

Ash, Knight, Manasse, Slaman[1] and independently Chisholm[2] show that the relatively intrinsically c.e. sets in a structure $\mathfrak{A}$ and the sets that are definable in $\mathfrak{A}$ by means of computable infinitary Σ^0_1 formulas coincide.

We will generalize the notion of jump sequence of a sequence of sets which is the main tool in many results and proofs of Soskov such as the jump inversion theorem for the enumeration jump, the regular enumerations, Ash's theorem for abstract structures and ω-enumeration degrees.

Definition 1.2. *(Soskov) Let $\mathcal{X} = \{X_n\}_{n<\omega}$ and $(\forall n)(X_n \subseteq \mathbb{N})$. The jump sequence $\mathcal{P}(\mathcal{X}) = \{\mathcal{P}_n(\mathcal{X})\}_{n<\omega}$ of $\mathcal{X}$ is defined inductively:*

(i) $\mathcal{P}_0(\mathcal{X}) = X_0$;

(ii) $\mathcal{P}_{n+1}(\mathcal{X}) = \mathcal{P}_n(\mathcal{X})'_e \oplus X_{n+1}$. *Here $\mathcal{P}_n(\mathcal{X})'_e$ is the enumeration jump of $\mathcal{P}_n(\mathcal{X})$.*

We generalize the above notion to a sequence of structures in the following way:

Definition 1.3. *Given a sequence of structures $\vec{\mathfrak{A}} = \{\mathfrak{A}_i\}_{i<\omega}$ the n-th polynomial of $\vec{\mathfrak{A}}$ is a structure $\mathcal{P}_n(\vec{\mathfrak{A}})$ defined inductively:*

(i) $\mathcal{P}_0(\vec{\mathfrak{A}}) = \mathfrak{A}_0$;

(ii) $\mathcal{P}_{n+1}(\vec{\mathfrak{A}}) = \mathcal{P}_n(\vec{\mathfrak{A}})' \oplus \mathfrak{A}_{n+1}$. *Here the jump of a structure and the join of two structures are appropriately defined.*

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We denote by $\mathfrak{A}^{(n)}$ the n-th jump of structure $\mathfrak{A}$ defined inductively:

$$\mathfrak{A}^{(0)} = \mathfrak{A}; \qquad \mathfrak{A}^{(n+1)} = (\mathfrak{A}^{n})'.$$

Definition 1.4. *We call two structures $\mathfrak{A}$ and $\mathfrak{B}$ equivalent: $\mathfrak{A} \equiv \mathfrak{B}$ if they have the same relatively intrinsically c.e. subsets of the common part of the domains of $\mathfrak{A}$ and $\mathfrak{B}$.*

Our main result is the following:

Theorem 1.5. *For every sequence of structures $\vec{\mathfrak{A}}$, there exists a structure $\mathfrak{M}$ such that for every n we have $\mathcal{P}_n(\vec{\mathfrak{A}}) \equiv \mathfrak{M}^{(n)}$.*

2 Preliminaries

2.1 Enumeration and $\leq_n$ Reducibility

We shall assume a fixed Gödel enumeration $W_0, \ldots, W_a, \ldots$ of the computably enumerable sets. By D_v we shall denote the finite set with canonical code v. Each c.e. set W_a determines an enumeration operator $W_a : \mathcal{P}(\mathbb{N}) \longrightarrow \mathcal{P}(\mathbb{N})$, so that for any sets of natural numbers A and B

$$A = W_a(B) \iff (\forall x)(x \in A \iff (\exists v)(\langle x, v\rangle \in W_a \wedge D_v \subseteq B)).$$

The set A *is enumeration reducible to* B ($A \leq_e B$) if there exists a c.e. set W such that $A = W(B)$. Let $A \equiv_e B \iff A \leq_e B \;\&\; B \leq_e A$. The relation $\equiv_e$ is an equivalence relation and the respective equivalence classes are called enumeration degrees.

For every set A of natural numbers let $A^+ = A \oplus (\mathbb{N} \setminus A)$. Clearly a set B is c.e. in A if and only if $B \leq_e A^+$. A set A is *total* if $A \equiv_e A^+$.

Given a set A of natural numbers, set $L_A = \{\langle a, x\rangle : x \in W_a(A)\}$ and let the *enumeration jump* of A be the set L_A^+. We will denote it by A'_e. One property of the enumeration jump is $(A^+)'_e \equiv_{\mathrm{e}} (A'_T)^+$ uniformly in A. It is obvious that if A is total then $A'_e \equiv_{\mathrm{T}} A'_T$.

Enumeration reducibility is further generalized to a notion of enumeration reducibility of sets to sequences of sets and to a notion of enumeration reducibility of sequences of sets to sequences of sets. The starting point of these generalizations is Selman's Theorem which states that the set X is enumeration reducible to the set Y if for all sets B, Y is c.e. in B implies X is c.e. in B. The following definition in a different notation is given by Ash:

Definition 2.1. *Given a set X of natural numbers and a sequence $\mathcal{Y} = \{Y_k\}_{k\in\omega}$ of sets of natural numbers, let $X \leq_n \mathcal{Y}$ if for all sets $Z \subseteq \mathbb{N}$, $\mathcal{Y}$ is c.e. in Z implies X is Σ^0_{n+1} in Z. Here $\mathcal{Y}$ is c.e. in Z means that $(\forall k)(Y_k$ is c.e. in $Z_T^{(k)}$ uniformly in $k)$.*

Ash presents a characterization of "$\leq_n$" using computable infinitary propositional sentences. Another characterization in terms of enumeration reducibility is obtained by Soskov and Kovachev:

Theorem 2.2. *(Soskov) $X \leq_n \mathcal{Y}$ if and only if $X \leq_e \mathcal{P}_n(\mathcal{Y})$.*

Soskov further generalized the notion to a sequence of structures:

Definition 2.3. *Let $\vec{\mathfrak{A}}$ is a sequence of structures and the union of their domains is A. For $R \subseteq A$ we say that $R \leq_n \vec{\mathfrak{A}}$ if $f^{-1}(R) \leq_n f^{-1}(\vec{\mathfrak{A}})$ for every enumeration f of $\vec{\mathfrak{A}}$.*

Theorem 2.4 (Soskov[4]). *For every sequence of structures $\vec{\mathfrak{A}}$, there exists a structure $\mathfrak{M}$, such that for each n, the relatively intrinsically Σ_{n+1} sets in $\mathfrak{M}$ sets coincide with the sets $R \leq_n \vec{\mathfrak{A}}$.*

The structure $\mathfrak{M}$ is the Marker's extension of the sequence of structures $\vec{\mathfrak{A}}$ as defined below.

First we will define the n-th Marker's extension $\mathfrak{M}_n(R)$ of $R \subseteq A^m$, where A is the union of the domains. Let $X_0, X_1, \dots X_n$ be new infinite disjoint countable sets - companions to $\mathfrak{M}_n(R)$. Fix bijections:

$h_0 : R \to X_0$

$h_1 : (A^m \times X_0) \setminus G_{h_0} \to X_1$

$\dots$

$h_n : (A^m \times X_0 \times X_1 \dots \times X_{n-1}) \setminus G_{h_{n-1}} \to X_n$

Let $\mathfrak{M}_n(R) = (A \cup X_0 \cup \dots \cup X_n; X_0, X_1, \dots X_n, G_{h_n})$.

Now for every n construct the n-th Marker's extension $\mathfrak{M}_n(\mathfrak{A}_n)$ of $\mathfrak{A}_n$ by constructing the n-th Marker's extension for all of its predicates $A_n, R_1^n, \dots R_{m_n}^n$ with disjoint companions and let $\mathfrak{M}_n(\mathfrak{A}_n) = \mathfrak{M}_n(A_n) \cup \mathfrak{M}_n(R_1^n) \cup \dots \cup \mathfrak{M}_n(R_{m_n}^n)$. Finally for the whole sequence of structures set $\mathfrak{M}$ to be $\bigcup_n \mathfrak{M}_n(\mathfrak{A}_n)$ with one additional predicate for A. For further details refer to [4].

2.2 Moschkovakis' Extension and the Jump Structure

Let $\mathfrak{A} = (A; R_1, \dots, R_s)$ be a countable structure and let equality be among the predicates $R_1, \dots, R_s$. Following Moschkovakis[3] the least acceptable extension of the structure $\mathfrak{A}$ is defined as follows.

Let 0 be an object which does not belong to A and Π be a pairing operation chosen so that neither 0 nor any element of A is an ordered pair. Let A^* be the least set containing all elements of $A_0 = A \cup \{0\}$ and closed under operation Π.

Let L and R be the decoding functions on A^* satisfying the following conditions:
$L(0) = R(0) = 0$; $(\forall t \in A)(L(t) = R(t) = 1^*)$; $(\forall s, t \in A^*)(L(\Pi(s,t)) = s \;\&\; R(\Pi(s,t)) = t)$.

We associate an element n^* of A^* with each natural number n by induction:

$$0^* = 0; \qquad (n+1)^* = \Pi(0, n^*).$$

The set of all elements n^* defined above will be denoted by N^*.

The pairing function allows us to code finite sequences of elements: let $\Pi_1(t_1) = t_1$, $\Pi_{n+1}(t_1, t_2, \dots, t_{n+1}) = \Pi(t_1, \Pi_n(t_2, \dots, t_{n+1}))$ for every $t_1, t_2, \dots, t_{n+1} \in A^*$.

For each predicate R_i of the structure $\mathfrak{A}$ define the respective predicate R_i^* on A^* by

$$R_i^*(t) \iff (\exists a_1 \in A) \dots (\exists a_{r_i} \in A)(t = \Pi_{r_i}(a_1, \dots, a_{r_i}) \;\&\; R_i(a_1, \dots, a_{r_i})).$$

Definition 2.5. *Moschovakis' extension of* $\mathfrak{A}$ *is the structure*

$$\mathfrak{A}^* = (A^*; A_0, R_1^*, \dots, R_s^*, G_\Pi, G_L, G_R, =),$$

where G_Π, G_L *and* G_R *are the graphs of* Π, L *and* R *respectively.*

We will now define the jump of a structure [5]. We define a forcing with conditions all finite mappings of $\mathbb{N}$ into A. For any $e, x \in \mathbb{N}$ and for every finite mapping δ of $\mathbb{N}$ into A, define *the forcing relations* $\delta \Vdash F_e(x)$ and $\delta \Vdash \neg F_e(x)$ as follows:

$$\delta \Vdash F_e(x) \iff x \in W_e^{\delta^{-1}(\mathfrak{A})}; \qquad \delta \Vdash \neg F_e(x) \iff (\forall \tau \supseteq \delta)(\tau \not\Vdash F_e(x)).$$

Where $\delta^{-1}(\mathfrak{A})$ is a finite function that is an initial part of the characteristic function of $f^{-1}(\mathfrak{A})$ for an enumeration $f \supseteq \delta$ of $\mathfrak{A}$. We also assume that if the oracle is called with an argument outside the domain of δ then the computation $\{e\}^{\delta^{-1}(\mathfrak{A})}(x)$ halts unsuccessfully.

With each finite mapping $\tau \neq \emptyset$ such that $\mathrm{dom}(\tau) = \{x_1 < \dots < x_n\}$ and $\tau(x_i) = s_i, 1 \leq i \leq n$, we associate an element $\tau^* = \Pi_n(\Pi(x_1^*, s_1), \dots, \Pi(x_n^*, s_n))$ of A^*. Let $\tau^* = 0$ if $\tau = \emptyset$.

Define $K_\mathfrak{A} = \{\Pi_3(\delta^*, e^*, x^*) \mid (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x)) \;\&\; e^*, x^* \in N^*\}$. The set $K_\mathfrak{A}$ is an analogue of the Kleene set K.

Definition 2.6. *We define the jump of structure* $\mathfrak{A}$ *to be*

$$\mathfrak{A}' = (A^*, A_0, R_1^*, \ldots, R_s^*, G_\Pi, G_L, G_R, =, K_{\mathfrak{A}}).$$

The main property of the jump structure, obtained in [6], is that for all $X \subseteq A$:

Theorem 2.7. X *is relatively intrinsically* Σ_{n+1} *in* $\mathfrak{A} \iff X$ *is relatively intrinsically* Σ_1 *in* $\mathfrak{A}^{(n)}$.

Let A be a set, $X \subseteq A$ and f, g are enumerations of A. We will denote by $E_X^{f,g}$ the set:

$$E_X^{f,g} = \{\langle x, y\rangle \mid f(x) = g(y) \in X\}.$$

The following two lemmas give the connection between enumerations of a structure and enumerations of its jump structure. They can be proved following [5](Propositions 13 and 15).

Lemma 2.8. *Let* $\mathfrak{A}$ *be a countable structure with domain* A. *For every enumeration* f *of* $\mathfrak{A}$ *there exists an enumeration* g *of* $\mathfrak{A}'$ *such that* $g^{-1}(\mathfrak{A}') \leq_{\mathrm{T}} (f^{-1}(\mathfrak{A}))'_T$ *and* $E_A^{f,g}$ *is c.e. in* $(f^{-1}(\mathfrak{A}))'_T$.

Lemma 2.9. *Let* $\mathfrak{A}$ *be a countable structure with domain* A. *For every enumeration* f *of* $\mathfrak{A}'$ *there exists an enumeration* g *of* $\mathfrak{A}$ *such that* $(g^{-1}(\mathfrak{A}))'_T \leq_{\mathrm{T}} f^{-1}(\mathfrak{A}')$ *and* $E_A^{f,g}$ *is c.e. in* $f^{-1}(\mathfrak{A}')$.

We now define the join of two structures:

Definition 2.10. *Let* $\mathfrak{A} = (A; R_1, \ldots, R_s, =)$ *and* $\mathfrak{B} = (B; P_1, \ldots, P_t, =)$ *be countable structures in the languages* $\mathfrak{L}_1$ *and* $\mathfrak{L}_2$. *Suppose that* $\mathfrak{L}_1 \cap \mathfrak{L}_2 = \{=\}$ *and* $A \cap B = \emptyset$. *Let* $\mathfrak{L} = \mathfrak{L}_1 \cup \mathfrak{L}_2 \cup \{A, B\}$ *where* A *and* B *are unary predicates. Define* $\mathfrak{A} \oplus \mathfrak{B} = (A \cup B; R_1, \ldots, R_s, P_1, \ldots, P_t, A, B, =)$ *in language* $\mathfrak{L}$ *where predicates* A *and* B *are true only on the elements of the domain of* $\mathfrak{A}$ *and* $\mathfrak{B}$ *respectively.*

In order to satisfy this definition we will only consider sequences of structures $\vec{\mathfrak{A}} = \{\mathfrak{A}_i\}_{i\in\omega}$ where the domains of $\mathfrak{A}_i$ and $\mathfrak{A}_j$ don't have common elements for all $i \neq j$.

3 Proof of main result

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_i\}_{i\in\omega}$ be a sequence of structures and the domain of $\mathfrak{A}_i$ is A_i. Denote by $A^{\leq n} = \bigcup_{i=0}^n A_i$. Let f be an enumeration of $A = \bigcup_{i\in\omega} A_i$ then we denote by $\mathcal{P}_n^f$ the following:

$$\mathcal{P}_0^f = f^{-1}(\mathfrak{A}_0); \qquad \mathcal{P}_{n+1}^f = (\mathcal{P}_n^f)'_e \oplus f^{-1}(\mathfrak{A}_{n+1}).$$

Note that for the structures we are considering the set $\mathcal{P}_n^f$ is total for all n and f.

First we prove two lemmas which follow from Lemma 2.8 and Lemma 2.9 by induction:

Lemma 3.1. *For every enumeration* f *of* $\vec{\mathfrak{A}}$ *and every* $n \in \mathbb{N}$ *there exists an enumeration* g_n *of* $\mathcal{P}_n(\vec{\mathfrak{A}})$, *such that* $g_n^{-1}(\mathcal{P}_n(\vec{\mathfrak{A}})) \leq_{\mathrm{T}} \mathcal{P}_n^f$ *and* $E_{A^{\leq n}}^{f,g_n}$ *is c.e. in* $\mathcal{P}_n^f$.

Lemma 3.2. *Let* $n \in \mathbb{N}$. *For every enumeration* g *of* $\mathcal{P}_n(\vec{\mathfrak{A}})$ *there exist an enumeration* f_n *of* $\vec{\mathfrak{A}}$, *such that* $\mathcal{P}_n^{f_n} \leq_{\mathrm{T}} g^{-1}(\mathcal{P}_n(\vec{\mathfrak{A}}))$ *and* $E_{A^{\leq n}}^{g,f_n}$ *is c.e. in* $g^{-1}(\mathcal{P}_n(\vec{\mathfrak{A}}))$.

Now using these two lemmas above we can prove the following:

Proposition 3.3. *Let* $n \in \mathbb{N}$ *and* $X \subseteq A^{\leq n}$. *We have the following equivalence:*

$$X \text{ is relatively intrinsically c.e. in } \mathcal{P}_n(\vec{\mathfrak{A}}) \iff X \leq_{\mathrm{n}} \vec{\mathfrak{A}}.$$

PROOF: ($\Rightarrow$) Suppose that X is relatively intrinsically c.e. in $\mathcal{P}_n(\vec{\mathfrak{A}})$ and let f be an enumeration of A. By Theorem 2.2 we should only prove that $f^{-1}(X) \leq_{\mathrm{e}} \mathcal{P}_n^f$. According to Lemma 3.1 for the enumeration f we can find enumeration g_n of $\mathcal{P}_n(\vec{\mathfrak{A}})$ such that $g_n^{-1}(\mathcal{P}_n(\vec{\mathfrak{A}})) \leq_{\mathrm{T}} \mathcal{P}_n^f$ and $E_{A^{\leq n}}^{f,g_n}$ is c.e. in $\mathcal{P}_n^f$. Now note the equivalence:

$$x \in f^{-1}(X) \iff (\exists y)(\langle x, y\rangle \in E_{A^{\leq n}}^{f,g_n} \ \&\ y \in g_n^{-1}(X)).$$

Because X is relatively intrinsically c.e. in $\mathcal{P}_n(\vec{\mathfrak{A}})$, the set $g_n^{-1}(X)$ is c.e. in $g_n^{-1}(\mathcal{P}_n(\vec{\mathfrak{A}}))$. Also $g_n^{-1}(\mathcal{P}_n(\vec{\mathfrak{A}})) \leq_{\mathrm{T}} \mathcal{P}_n^f$ from the properties of g_n. Then we have that $g_n^{-1}(X)$ is c.e. in P_n^f. We also know that $E_{A^{\leq n}}^{f,g_n}$ is c.e. in $\mathcal{P}_n^f$, so we can conclude that $f^{-1}(X)$ is c.e. in $\mathcal{P}_n^f$. Since $\mathcal{P}_n^f$ is a total set we have that $f^{-1}(X) \leq_{\mathrm{e}} P_n^f$.

($\Leftarrow$) Suppose that $X \leq_{\mathrm{n}} \vec{\mathfrak{A}}$. Let g be an enumeration of $\mathcal{P}_n(\vec{\mathfrak{A}})$. According to Lemma 3.2 for enumeration g there is an enumeration f_n of $\vec{\mathfrak{A}}$, such that $\mathcal{P}_n^{f_n} \leq_{\mathrm{T}} g^{-1}(\mathcal{P}_n(\vec{\mathfrak{A}}))$ and $E_{A^{\leq n}}^{g,f_n}$ is c.e. in $g^{-1}(\mathcal{P}_n(\vec{\mathfrak{A}}))$.

Because $X \leq_{\mathrm{n}} \vec{\mathfrak{A}}$ we have that $f_n^{-1}(X) \leq_{\mathrm{e}} \mathcal{P}_n^{f_n}$. The set $\mathcal{P}_n^f$ is total so we also have that $f_n^{-1}(X)$ is c.e. in $\mathcal{P}_n^{f_n}$. We also know that $\mathcal{P}_n^{f_n} \leq_{\mathrm{T}} g^{-1}(\mathcal{P}_n(\vec{\mathfrak{A}}))$ by the properties of f_n and so the set $f_n^{-1}(X)$ is c.e. in $g^{-1}(\mathcal{P}_n(\vec{\mathfrak{A}}))$. Now note the equivalence:

$$x \in g^{-1}(X) \iff (\exists y)(\langle x, y\rangle \in E_{A^{\leq n}}^{g,f_n} \ \&\ y \in f_n^{-1}(X)).$$

It is obvious that $g^{-1}(X)$ is c.e. in $g^{-1}(\mathcal{P}_n(\vec{\mathfrak{A}}))$. □

We can now prove our main result Theorem 1.5. Note that the structure $\mathfrak{M}$ is the Marker's extension of the sequence $\vec{\mathfrak{A}}$ and the domains of $\mathfrak{M}$ and $\mathcal{P}_n(\vec{\mathfrak{A}})$ depend on the Moskovakis' extension. We shall assume that the common part of the domains is exactly $A^{\leq n}$.

PROOF: [of Theorem 1.5] Let $n \in \mathbb{N}$. By Theorem 2.7 X is relatively intrinsically c.e. in $\mathfrak{M}^{(n)}$ if and only if when X is relatively intrinsically Σ_{n+1} in $\mathfrak{M}$.

Now by Theorem 2.4 we have that this is equivalent to $X \leq_{\mathrm{n}} \vec{\mathfrak{A}}$.

Lastly using the previous Proposition 3.3 we conclude that X is relatively intrinsically c.e. in $\mathfrak{M}^{(n)}$ if and only if X is relatively intrinsically c.e. in $\mathcal{P}_n(\vec{\mathfrak{A}})$. Which can be written as:

$$\mathfrak{M}^{(n)} \equiv \mathcal{P}_n(\vec{\mathfrak{A}}).$$

□

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