Strong Jump Inversion

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Strong Jump Inversion

Definition

A structure $\mathcal{A}$ *admits strong jump inversion* provided that for all sets X, if X' computes $D(\mathcal{C})'$ for some $\mathcal{C} \cong \mathcal{A}$, then X computes $D(\mathcal{B})$ for some $\mathcal{B} \cong \mathcal{A}$.

Remark

The structure $\mathcal{A}$ admits strong jump inversion iff for all X, if $\mathcal{A}$ has a copy that is low over X, then it has a copy that is computable in X. Here when we say that $\mathcal{C}$ is low over X, we mean that $D(\mathcal{C})' \leq_T X'$.

Boolean algebras

Theorem (Downey-Jockusch)

All Boolean algebras admit strong jump inversion.

Sketch of proof.

Let $\mathcal{A}$ be a Boolean algebra that is low over X. Then X' computes the set of atoms in $\mathcal{A}$. Downey and Jockusch showed that if X' computes $(\mathcal{A}, atom(x))$, then X computes a copy of $\mathcal{A}$. The proof involves some non-uniformity. It is known that a Boolean algebra with only finitely many atoms has a computable copy. Suppose $\mathcal{A}$ has infinitely many atoms. If $\mathcal{A}$ is low over X, then there is an X-computable Boolean algebra $\mathcal{B}$ with a function f, Δ_2^0 relative to X, where $\mathcal{B}$ is generated by elements of $ran(f)$ and additional atoms, each lying below $f(a)$ for some atom a of $\mathcal{A}$. Then by isomorphism theorem, due to Vaught and Remmel $\mathcal{A} \cong \mathcal{B}$. The isomorphism between $\mathcal{A}$ and the computable copy could be Δ_4^0 by Knight and Stob, and this is the best possible by Stob unpublished. □

Equivalence structures

Example

We consider equivalence structure $\mathcal{A} = (A, \equiv)$ with infinitely many infinite classes.

Fact: An equivalence structure $\mathcal{A}$ has an X-computable copy iff $Q_{\mathcal{A}} = \{\langle n, k\rangle \mid \text{there are at least } k \text{ classes of size } n\}$ is Σ^0_2 relative to X.

Proposition

Every equivalence structure $\mathcal{A}$ with infinitely many infinite classes admits strong jump inversion.

Proof.

If $\mathcal{A}$ is low over X, then the set $Q_{\mathcal{A}}$ is Σ^0_2 relative to $\mathcal{A}$, so it is Σ^0_2 relative to X. Then $\mathcal{A}$ has an X-computable copy. □

Abelian p-groups of length ω

Example

By Ulm's Theorem: A countable Abelian p-group G is characterized up to isomorphism by the Ulm sequence $(u_n(G))_{n\in\omega}$ and the dimension of the divisible part.

- $u_n(G)$ is the number of direct summands of form $Z_{p^{n+1}}$;
- the dimension of the divisible part is the number of direct summands of form Z_{p^∞}.

Fact: If G is an Abelian p-group of length ω with a divisible part of infinite dimension, then G has an X-computable copy iff the set $\{(n,k) : u_n(G) \geq k\}$ is Σ^0_2 relative to X.

Proposition

Let G be an Abelian p-group of length ω such that the divisible part has infinite dimension. Then G admits strong jump inversion.

Not all countable structures admit strong jump inversion

Example

Jockusch and Soare showed that there are low linear orderings with no computable copy.

Example

Let T be a low completion of PA. There is a model $\mathcal{A}$ such that the atomic diagram $D(\mathcal{A})$, and even the complete diagram $D^c(\mathcal{A})$, are computable in T. Then $D(\mathcal{A})'$ is Δ^0_2. By a well-known result of Tennenbaum, since $\mathcal{A}$ is necessarily non-standard, there is no computable copy.

Further notions of jump and jump inversion

A relation R is *relatively intrinsically* Σ^0_α on a structure $\mathcal{A}$ if in all isomorphic copies $\mathcal{B}$ of $\mathcal{A}$, the image of R under the isomorphism is Σ^0_α relative to $\mathcal{B}$.

- These are the relations that are definable in $\mathcal{A}$ by computable Σ_α formulas, with parameters. [Ash, Knight, Manasse, Sleman] [Chisholm]
- For $\alpha = 1$ we can uniformly compute all of relatively intrinsically Σ^0_1 (r.i.c.e.) relations from the Turing jump of the diagram, $D(\mathcal{A})'$.
- $\mathcal{A}'$ is obtained by adding to $\mathcal{A}$ a specific named family of r.i.c.e. relations, from which all others are effectively obtained.
- A relation R is relatively intrinsically c.e. on $\mathcal{A}' \iff R$ is relatively intrinsically Σ^0_2 on $\mathcal{A}$.

Definition (Canonical jump)

[Montalbán] The *canonical jump* is a structure $\mathcal{A}' = (\mathcal{A}, (R_i)_{i\in\omega})$, where $(R_i)_{i\in\omega}$ are all r.i.c.e. relations on $\mathcal{A}$.

Strong jump inversion

Proposition (Montalbàn, Soskov, S)

If X' computes a copy of the canonical jump $\mathcal{A}'$ of $\mathcal{A}$, then there is a set Y, with $Y' \equiv_T X'$, and Y computes a copy of $\mathcal{A}$.

Proposition

For any countable structure $\mathcal{A}$, the following are equivalent:

1. $\mathcal{A}$ admits strong jump inversion.
2. For all sets X, if X' computes a copy of the canonical jump $\mathcal{A}'$ of $\mathcal{A}$, then X computes a copy of $\mathcal{A}$.
3. For all sets X and Y, if $X' \equiv_T Y'$ and Y computes a copy of $\mathcal{A}$ then so does X.

Comparing structures

Definition (Muchnik reducibility)

A structure $\mathcal{A}$ is Muchnik reducible to a structure $\mathcal{B}$, ($\mathcal{A} \leq_w \mathcal{B}$) iff every copy of $\mathcal{B}$ computes a copy of $\mathcal{A}$, or equivalently $Sp(\mathcal{B}) \subseteq Sp(\mathcal{A})$.

Definition (Effective interpretation)

An *effective interpretation* of $\mathcal{A}$ in $\mathcal{B}$ ($\mathcal{A} \leq_I \mathcal{B}$) consists of computable Σ_1 formulas, with no parameters, defining in $\mathcal{B}$ a set D and relations R, $\sim$, together with the complements of all three, where $D \subseteq \mathcal{B}^{<\omega}$, and R and $\sim$ are relations on D such that $(D, R)/_{\sim} \cong \mathcal{A}$.

Definition (Sigma reducibility)

A structure $\mathcal{A}$ is Σ reducible to a structure $\mathcal{B}$, ($\mathcal{A} \leq_\Sigma \mathcal{B}$) iff $\mathcal{A} \leq_I HF(\mathcal{B})$.

Proposition

$\mathcal{A} \leq_I \mathcal{B} \Longrightarrow \mathcal{A} \leq_\Sigma \mathcal{B}$ and $\mathcal{A} \leq_\Sigma \mathcal{B} \Longrightarrow \mathcal{A} \leq_w \mathcal{B}$.

The jump inversion of a structure

Let $\alpha < \omega_1^{CK}$ and $\mathcal{A}$ be a countable structure such that $\mathcal{A} \geq_w 0^{(\alpha)}$. Does there exist a structure $\mathcal{C}$ such that $\mathcal{C}^{(\alpha)} \equiv_w \mathcal{A}$?

- The jump inversion theorem holds for successor ordinals [Goncharov-Harizanov-Knight-McCoy-Miller-Solomon, Vatev], not in these terms.
- If $\mathcal{A} \geq_w \mathcal{B}'$ then there exists a structure $\mathcal{C} \geq_w \mathcal{B}$ and $\mathcal{C}' \equiv_w \mathcal{A}$ [Soskov, S].
- The jump inversion theorem does not hold for $\alpha = \omega$ [Soskov].
- Jump inversion theorem for $\equiv_\Sigma$ [Stukachev].

Structural jump

Definition (Structural jump)

(Montalbàn) A *structural jump* of $\mathcal{A}$ is an expansion $\mathcal{A}' = (\mathcal{A}, (R_i)_{i\in\omega})$ such that each R_i has a Σ_1^c defining formula that we can compute from i, and every relation that is relatively intrinsically Σ_2^0 on $\mathcal{A}$ is r.i.c.e. on $\mathcal{A}' \oplus \emptyset'$.

Here the structure $\mathcal{A}' \oplus \emptyset'$ is the expansion of $\mathcal{A}'$ by a family of relations that encode the set $\emptyset'$.

For certain classes of structures, there is a structural jump formed by adding a finite set of such relations.

- for Boolean algebras $(\mathcal{A}, atom(x))$ is the structural jump
- for linear orders $(\mathcal{A}, succ(x, y))$ is the structural jump
- for Q-vector spaces $(\mathcal{A}, \vec{LD})$ is the structural jump
- for equivalence structures which have one equivalence class of each finite size there is no finite complete set of relations for the structural jump

B_n-types

Definition

Let S be a countable family of sets. An *enumeration* of S is a set R of pairs (i, k) such that S is the family of sets $R_i = \{k : (i, k) \in R\}$.

Definition

1. A *B_n-formula* is a finite Boolean combination of ordinary finite elementary Σ_n-formulas.
2. A *B_n-type* is the set of B_n-formulas in the complete type of some tuple in some structure for the language.

Effective type completion

Definition

Let S be a set of B_1-types including all those realized in $\mathcal{A}$. Let R be an enumeration of S. An *R-labeling of $\mathcal{A}$* is a function taking each tuple $\bar{a}$ in $\mathcal{A}$ to an R-index for the B_1-type of $\bar{a}$.

Definition (Effective type completion)

The structure $\mathcal{A}$ satisfies *effective type completion* if there is a uniform effective procedure that, given a B_1-type $p(\bar{u})$ realized in $\mathcal{A}$ and an existential formula $\varphi(\bar{u}, x)$ such that $(\exists x)\varphi(\bar{u}, x) \in p(\bar{u})$, yields a B_1-type $q(\bar{u}, x)$ with $\varphi(\bar{u}, x) \in q(\bar{u}, x)$, such that if $\bar{a}$ in $\mathcal{A}$ realizes $p(\bar{u})$, then some b in $\mathcal{A}$ realizes $q(\bar{a}, x)$.

The general result

Theorem

A structure $\mathcal{A}$ admits strong jump inversion if it satisfies the following conditions:

1. There is a computable enumeration R of a set of B_1-types including all those realized by tuples in $\mathcal{A}$.
2. $\mathcal{A}$ satisfies effective type completion.
3. For all sets X, if X' computes the jump of some copy of $\mathcal{A}$, then X' computes a copy of $\mathcal{A}$ with an R-labeling.

Moreover, if $\mathcal{C}$ is a copy of $\mathcal{A}$ with an X'-computable R-labeling, then we get an X-computable copy $\mathcal{B}$ of $\mathcal{A}$ with an X'-computable isomorphism from $\mathcal{B}$ to $\mathcal{C}$.

Weekly 1-saturation

Definition

A B_1-type $q(\bar{u}, x)$ is *generated by the formulas of B_1-type $p(\bar{u})$ and existential formulas* if $q(\bar{u}, x) \supseteq p(\bar{u})$, and for any universal formula $\psi(\bar{u}, x)$, we have $\psi(\bar{u}, x) \in q(\bar{u}, x)$ iff there is a finite conjunction $\chi(\bar{u}, x)$ of existential formulas in $q(\bar{u}, x)$ s.t. $(\exists x)[\chi(\bar{u}, x) \,\&\, \neg(\psi(\bar{u}, x))] \notin p(\bar{u})$.

Definition

$\mathcal{A}$ is *weakly* 1-*saturated* provided that any B_1-type $q(\bar{u}, x)$ generated by formulas of the B_1-type $p(\bar{u})$ and existential formulas, where $p(\bar{u})$ realized on a tuple $\bar{a}$, $q(\bar{a}, x)$ is realized in $\mathcal{A}$.

If $q(\bar{u}, x)$ is a such B_1-type over $p(\bar{u})$ then $q(\bar{u}, x)$ is consistent with all extensions of $p(\bar{u})$ to a complete type in variables $\bar{u}$.

Proposition

Weekly 1-saturation $\Longrightarrow$ Effective type completion

Linear orderings I

Every B_1-type $p(\bar{u})$ is determined uniquely by the sizes of the intervals of $\bar{u}$. There is a computable enumeration R of all B_1-types realized in linear orderings s.t. from the index i of the B_1-type R_i (eff.)$\Longrightarrow$ the sizes.

Let $\mathcal{A}$ be a linear ordering such that every infinite interval can be split into two infinite parts. Then $\mathcal{A}$ is weakly 1-saturated.

Frolov proved strong jump inversion for two special classes of linear orderings. We present here that our conditions hold for them.

Theorem

Let $\mathcal{A}$ be a linear ordering such that each element lies on a maximal discrete set that is finite. Suppose there is a finite bound on the sizes of these sets. Then $\mathcal{A}$ admits strong jump inversion. Moreover, if $\mathcal{A}$ is low over X, then there is an X-computable copy with an isomorphism that is Δ^0_2 relative to X.

It is Δ^0_2 relative to $\mathcal{A}$ to say that the interval (a, b) has size n for some fixed n. It is Σ^0_1 relative to $\mathcal{A}$ to say that the interval is infinite.

Linear orderings II

The *block equivalence relation* $\sim$ on a linear ordering $\mathcal{A}$, is $a \sim b$ iff $[a, b]$ is finite. For any linear ordering $\mathcal{A}$, each equivalence class under this relation is an interval that is either finite or of order type ω, ω^*, or $\zeta = \omega^* + \omega$.

Theorem

Let $\mathcal{A}$ be a linear ordering for which the quotient $\mathcal{A}/_\sim$ has order type η. Suppose also that in $\mathcal{A}$, every infinite interval has arbitrarily large finite successor chains. Then $\mathcal{A}$ admits strong jump inversion. Moreover, if $\mathcal{A}$ is low over X, then there is an X-computable copy $\mathcal{B}$ with an isomorphism that is Δ^0_3 over X from $\mathcal{A}$ to $\mathcal{B}$.

If $\mathcal{A}$ is low over X, then there is a copy $\mathcal{B}$ of $\mathcal{A}$ with an R-labeling that is Δ^0_2 over X and there is an isom. $f : \mathcal{B} \cong \mathcal{A}$ such that f is $\Delta^0_3(X)$. At stage s, we construct (using the Δ^0_2 oracle) an approximation $\mathcal{A}_{n,s}$ of the linear ordering of the first n elements in which the intervals are either correctly labeled with a finite number at most s, or else carry the label ∞. We have a finite sub-ordering $\mathcal{B}_s$ of $\mathcal{B}$ in which the intervals are labeled by size.

Boolean algebras

We say that *a has size n* in a Boolean algebra $\mathcal{B}$, if it is the join of n atoms of $\mathcal{B}$, otherwise we call it infinite. Here we consider Boolean algebras with no 1-atoms, so every infinite element splits into two infinite elements. For a tuple $\bar{a}$ in $\mathcal{B}$, the B_1-type of $\bar{a}$ is uniquely determined by the sizes in $\mathcal{B}$ of the atoms in the finite sub-algebra generated by $\bar{a}$. We can define a computable enumeration R of all B_1-types realized in $\mathcal{B}$.

Lemma

If $\mathcal{A}$ is a Boolean algebra with no 1-atoms, then $\mathcal{A}$ is weakly 1-saturated.

Proposition

Let $\mathcal{A}$ be an infinite Boolean algebra with no 1-atoms. Then $\mathcal{A}$ admits strong jump inversion. Moreover, if $\mathcal{A}$ is low over X, there is an X-computable copy $\mathcal{B}$ with an isomorphism that is Δ^0_3 relative to X.

On each step construct $\bar{b} \in \mathcal{B}$ with R-labels and $f_s : \bar{d} \to \bar{c}$: (1) the finite algebras $\bar{d}$ and $\bar{c}$ agree; (2) the approx. R-labels agree; (3) the number of atoms below of each element agrees.

Special Trees

Proposition

Suppose $\mathcal{A}$ is a tree, subtree of $\omega^{<\omega}$ such that the top node (the root) is infinite (i.e., it has infinitely many successors), and each infinite node has only finitely many successors that are terminal, with the rest all infinite. Then $\mathcal{A}$ admits strong jump inversion.

- Computable enumeration R of the B_1-types: The B_1-type of a tuple $\bar{a}$ is determined by the subtree generated by $\bar{a}$ and labels "infinite" or "terminal" on the nodes. We have a computable enumeration of all possible labeled finite such subtrees.
- Weak 1-saturation: Consider a B_1-type $p(\bar{a}, x)$, generated by formulas true of $\bar{a}$ and existential formulas. The type may locate x in the subtree generated by $\bar{a}$. Then the type is realized. The type may locate x properly below some infinite a_i, or at some level not below any a_i. Again the type is realized by a new infinite element.
- Let $\mathcal{A}$ be low. Then there is a Δ^0_2 R-labeling of $\mathcal{A}$.

Models of a theory with few B_1-types

Lerman and Schmerl gave conditions under which an $\aleph_0$-categorical theory T has a computable model. They assumed that the theory is arithmetical and $T \cap \Sigma_{n+1}$ is Σ^0_n for each n. Knight showed that the assumption that T is arithmetical could be dropped, and, instead, $T \cap \Sigma_{n+1}$ is Σ^0_n uniformly in n.

Theorem (Lerman-Schmerl)

Let T be an $\aleph_0$-categorical theory that is Δ^0_N and suppose that for all $1 \leq n < N$, $T \cap \Sigma_{n+1}$ is Σ^0_n. Then T has a computable model.

Lemma

For any $n < N$, if $\mathcal{A}$ is a model whose B_{n+1}-diagram is computable in X', and $T \cap \Sigma_{n+2}$ is Σ^0_1 in X, then there is a model $\mathcal{B}$ whose B_n-diagram is computable in X.

Note: There are non-$\aleph_0$-categorical theories satisfying the conditions of the Theorem above: $T = [\eta + \{2\} + \eta] \cdot \mathcal{A}$, where $\mathcal{A}$ is non-$\aleph_0$-categorical LO.

Models of a theory with few B_1-types

Theorem

Let T be an elementary first order theory, in a computable language, such that $T \cap \Sigma_2$ is Σ^0_1. Suppose that for each tuple of variables $\bar{x}$, there are only finitely many B_1-types in variables $\bar{x}$ consistent with T. Then every model $\mathcal{A}$ admits strong jump inversion. Moreover, if $\mathcal{A}$ is low over X, then there is an X-computable copy $\mathcal{B}$ with an isomorphism that is Δ^0_2 relative to X.

- Uniformly in each tuple of variables $\bar{x}$, we build a c.e. tree $P_{\bar{x}}$ whose paths represent the B_1-types in $\bar{x}$. Using the fact that $T \cap \Sigma_2$ is c.e. and $P_{\bar{x}}$ has only finitely many paths we construct a computable enumeration of the B_1-types.
- All of the B_1-types are principal, in the sense that each B_1-type contains some formula not contained in any of the other.
- Let $\mathcal{A}$ be low. For a tuple of variables $\bar{x}$, Δ^0_2 can find generating formulas for all of the B_1-types. Then there is a Δ^0_2 R-labeling of $\mathcal{A}$.

Differentially closed fields

A *differential field* is a field with one or more derivations satisfying the following familiar rules:

1. $\delta(u+v) = \delta(u) + \delta(v)$, and
2. $\delta(u \cdot v) = u \cdot \delta(v) + \delta(u) \cdot v$.

We consider differential fields of characteristic 0, and with a single derivation δ.

Roughly speaking, a *differentially closed field DCF* is a differential field in which differential polynomials have roots, where a differential polynomial is a polynomial $p(x)$ in x and its various derivatives.

- A. Robinson showed that the theory DCF_0 admits elimination of quantifiers.
- L. Blum gave a computable set of axioms, showing that the theory is decidable.
- Thus, the elimination of quantifiers is effective.

Differential polynomials

A differential polynomial $p(x)$, over a differential field K, may be thought of as an algebraic polynomial in $K[x, \delta(x), \delta^{(2)}(x), \dots, \delta^{(n)}(x)]$, for some n. We write $K\langle x\rangle$ for the set of differential polynomials over K.

For $p(x) \in K\langle x\rangle$:

- The *order* is the greatest n such that $\delta^{(n)}(x)$ appears non-trivially in $p(x)$. The order of 0 polynomial is ∞.
- The *degree* is the highest power k of $\delta^{(n)}(x)$ that appears, if $p(x)$ is of finite order n.

Blum's axioms say that a differentially closed field (of characteristic 0 and with a single derivation), is a differential field K such that

1. for any pair of differential polynomials $p(x), q(x) \in K\langle x\rangle$ such that the order of $q(x)$ is less than that of $p(x)$, there is some x satisfying $p(x) = 0$ and $q(x) \neq 0$,
2. if $p(x)$ has order 0, then $p(x)$ has a root. (K is algebraically closed)

Types

We want to understand the types, in any number of variables, realized in models of DCF_0.

Definition

A differential polynomial $p(x) \in K\langle x \rangle$ of order n is said to be *irreducible* if it is irreducible when considered as an algebraic polynomial in $K[x, \delta(x), \ldots, \delta^{(n)}(x)]$ We count the 0 polynomial as irreducible.

If $p(x) \in K\langle x \rangle$ is irreducible, then the corresponding type $\lambda_{K,p}$ is a complete type over K, and all types over K (in "x") have this form.

Each type over K is determined by an irreducible differential polynomial $p(x) \in K\langle x \rangle$. If $p(x)$ is irreducible of order n, the corresponding type $\lambda_{K,p}$ consists of formulas provable from the axioms of DCF_0, the atomic diagram of K, the formula $p(x) = 0$, and further formulas $q(x) \neq 0$, for $q(x)$ of order less than n. The formulas $q(x) \neq 0$, taken together, say that $x, \delta(x), \ldots, \delta^{(n-1)}(x)$ are algebraically independent over K.

Toward strong jump inversion

Marker and R. Miller showed that all models of DCF_0 admit strong jump inversion.
Marker and Miller gave a method for coding an arbitrary countable graph in a model of DCF_0.

Our goal is to obtain this result using our general result. Among the countable models of DCF_0, only the saturated one is weakly 1-saturated.
We will need to show effective type-completion.
The hard problem is to receive a computable enumeration of the B_1 types.

By quantifier elimination, we can pass effectively from a quantifier-free type $\lambda(\bar{x})$ to the complete type generated by $DCF_0 \cup \lambda(\bar{x})$.

Computable enumeration of types

We will enumerate effectively all quantifier-free types in K.

Lemma

There is a uniform effective procedure that, given a differential field K and a type $\lambda(x)$ over K, yields a differential field $K' \supseteq K$ that is generated over K by an element x realizing λ.

Lemma

There is a uniform effective procedure that, given a differential field K and $p(x) \in K\langle x \rangle$, enumerates a type $\lambda(x)$ for x over K. Moreover, if $p(x)$ is irreducible, then $\lambda(x) = \lambda_{K,p}$.

Proposition

There is a computable enumeration R of all complete types realized in models of DCF_0.

Every countable model of DCF_0 admits strong jump inversion

- *There is a computable enumeration R of the complete types realized in models of DCF_0, and thus, of the B_1 types.*
- *There is a uniform effective procedure for computing, from a type $p(\bar{u})$ and a formula $\varphi(\bar{u}, x)$, consistent with $p(\bar{u})$, a type $q(\bar{u}, x)$ such that if $\bar{c}$ satisfies $p(\bar{u})$, then some a satisfies $q(\bar{c}, x)$. The type $q(\bar{c}, x)$ will be realized in the differential closure of $\bar{c}$. (Marker and Miller)*
- *If $\mathcal{A}$ is a model of DCF_0 that is low over X. Then X' computes an R-labeling of $\mathcal{A}$, We have an R-labeling that is computable in $D(\mathcal{A})'$, and hence, in X', since $\mathcal{A}$ is low over X.*

A countable complete theory T has a decidable saturated model iff there is a computable enumeration of all types realized in models of T (Morley).

Corollary

The saturated model of DCF_0 has a decidable copy.

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