Cototal enumeration degrees

Alexandra A. Soskova[1]

Sofia University

joint work with Andrews, Ganchev, Kuyper, Lempp, Miller and M. Soskova

Siena University Logic Seminar

[1]Supported by Bulgarian National Science Fund DN 02/16 /19.12.2016 and Sofia University Science Fund, 80-10-147/20.04.2017

The enumeration degrees

Definition

$A \leq_e B$ if there is a c.e. set W, such that

$$A = W(B) = \{x \mid \exists D(\langle x, D\rangle \in W \;\&\; D \subseteq B)\}.$$

The enumeration degrees

Definition

$A \leq_e B$ if there is a c.e. set W, such that

$$A = W(B) = \{x \mid \exists D(\langle x, D\rangle \in W \,\&\, D \subseteq B)\}\,.$$

- $A \equiv_e B$ if $A \leq_e B$ and $B \leq_e A$.

The enumeration degrees

Definition

$A \leq_e B$ if there is a c.e. set W, such that

$$A = W(B) = \{x \mid \exists D(\langle x, D \rangle \in W \;\&\; D \subseteq B)\}.$$

- $A \equiv_e B$ if $A \leq_e B$ and $B \leq_e A$.
- The enumeration degree of a set A is $d_e(A) = \{B \mid A \equiv_e B\}$.

The enumeration degrees

Definition

$A \leq_e B$ if there is a c.e. set W, such that

$$A = W(B) = \{x \mid \exists D(\langle x, D\rangle \in W \;\&\; D \subseteq B)\}.$$

- $A \equiv_e B$ if $A \leq_e B$ and $B \leq_e A$.
- The enumeration degree of a set A is $d_e(A) = \{B \mid A \equiv_e B\}$.
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.

The enumeration degrees

Definition

$A \leq_e B$ if there is a c.e. set W, such that

$$A = W(B) = \{x \mid \exists D(\langle x, D\rangle \in W \;\&\; D \subseteq B)\}\,.$$

- $A \equiv_e B$ if $A \leq_e B$ and $B \leq_e A$.
- The enumeration degree of a set A is $d_e(A) = \{B \mid A \equiv_e B\}$.
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- The least element: $\mathbf{0_e} = d_e(\emptyset)$, the set of all c.e. sets.

The enumeration degrees

Definition

$A \leq_e B$ if there is a c.e. set W, such that

$$A = W(B) = \{x \mid \exists D(\langle x, D\rangle \in W \ \&\ D \subseteq B)\}.$$

- $A \equiv_e B$ if $A \leq_e B$ and $B \leq_e A$.
- The enumeration degree of a set A is $d_e(A) = \{B \mid A \equiv_e B\}$.
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- The least element: $\mathbf{0_e} = d_e(\emptyset)$, the set of all c.e. sets.
- The least upper bound: $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.

The enumeration degrees

Definition

$A \leq_e B$ if there is a c.e. set W, such that

$$A = W(B) = \{x \mid \exists D(\langle x, D\rangle \in W \;\&\; D \subseteq B)\}\,.$$

- $A \equiv_e B$ if $A \leq_e B$ and $B \leq_e A$.
- The enumeration degree of a set A is $d_e(A) = \{B \mid A \equiv_e B\}$.
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- The least element: $\mathbf{0_e} = d_e(\emptyset)$, the set of all c.e. sets.
- The least upper bound: $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- The enumeration jump: $d_e(A)' = d_e(K_A \oplus \overline{K_A})$, where $K_A = \{\langle e, x\rangle \mid x \in W_e(A)\}$.

Selman's theorem

Equivalently, $A \leq_e B$ if there is a single Turing functional which uniformly, given any enumeration of B, outputs an enumeration of A.

Selman's theorem

Equivalently, $A \leq_e B$ if there is a single Turing functional which uniformly, given any enumeration of B, outputs an enumeration of A.

Definition

Given a set A, let $\mathcal{E}(A)$ denote the collection of all Turing degrees computing enumerations of A, called the enumeration cone of A.

Theorem (Selman)

A is enumeration reducible to B if and only if $\mathcal{E}(B) \subseteq \mathcal{E}(A)$.

What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

Proposition

$$A \leq_T B \Leftrightarrow A \oplus \overline{A} \leq_e B \oplus \overline{B}.$$

What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

Proposition

$$A \leq_T B \Leftrightarrow A \oplus \overline{A} \leq_e B \oplus \overline{B}.$$

A set A is *total* if its positive membership information already suffices to determine its negative membership information, i.e. if $\overline{A} \leq_e A$. An enumeration degree is *total* if it contains a total set.

What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

Proposition

$$A \leq_T B \Leftrightarrow A \oplus \overline{A} \leq_e B \oplus \overline{B}.$$

A set A is *total* if its positive membership information already suffices to determine its negative membership information, i.e. if $\overline{A} \leq_e A$. An enumeration degree is *total* if it contains a total set.

Within the enumeration degrees, the total degrees are an embedded copy of the Turing degrees $\mathcal{D}_T$ via $\iota : A \to A \oplus \overline{A}$. The embedding ι preserves the order, the least upper bound and the jump operation.

Total and cototal

Definition

A set A is *cototal* if $A \leq_e \overline{A}$. A degree $\mathbf{a}$ is cototal if it contains a cototal set.

For every set A the set $A \oplus \overline{A}$ is cototal.
So, every total e-dergree is cototal.

Total and cototal

Definition

A set A is *cototal* if $A \leq_e \overline{A}$. A degree $\mathbf{a}$ is cototal if it contains a cototal set.

For every set A the set $A \oplus \overline{A}$ is cototal.
So, every total e-dergree is cototal.
The cototal enumeration degrees form a proper substructure of $\mathcal{D}_e$ closed under least upper bound and the enumeration jump operator.

- The name "*total*" is coming of the following fact: given a total function f, the set $G(f) = \{\langle n, f(n)\rangle \mid n \in \omega\}$ is a total set.
- Equivalently, given a total function f, the graph-complement $\overline{G(f)}$ is cototal.
- If an enumeration degree contains a set of the form $\overline{G(f)}$, then we call it *graph-cototal*.
- So every total enumeration degree is graph-cototal, and every graph-cototal is cototal.

Motivation from symbolic dynamics by Emmanuel Jeandel

Definition

- A subshift is a closed subset $X \subseteq 2^\omega$ such that if $a\alpha \in X$ then $\alpha \in X$.

Motivation from symbolic dynamics by Emmanuel Jeandel

Definition

- A subshift is a closed subset $X \subseteq 2^\omega$ such that if $a\alpha \in X$ then $\alpha \in X$.
- X is minimal if there is no $Y \subset X$, such that Y is a subshift.

Motivation from symbolic dynamics by Emmanuel Jeandel

Definition

- A subshift is a closed subset $X \subseteq 2^\omega$ such that if $a\alpha \in X$ then $\alpha \in X$.
- X is minimal if there is no $Y \subset X$, such that Y is a subshift.
- The language of X is the set $L_X = \{\sigma \in 2^{<\omega} \mid \exists\alpha \in X(\sigma \text{ is a subword of } \alpha)\}$.

Motivation from symbolic dynamics by Emmanuel Jeandel

Definition

- A subshift is a closed subset $X \subseteq 2^\omega$ such that if $a\alpha \in X$ then $\alpha \in X$.
- X is minimal if there is no $Y \subset X$, such that Y is a subshift.
- The language of X is the set $L_X = \{\sigma \in 2^{<\omega} \mid \exists \alpha \in X(\sigma \text{ is a subword of } \alpha)\}$.
- $\overline{L_X}$ is the set of forbidden words.

Motivation from symbolic dynamics by Emmanuel Jeandel

Definition

- A subshift is a closed subset $X \subseteq 2^\omega$ such that if $a\alpha \in X$ then $\alpha \in X$.
- X is minimal if there is no $Y \subset X$, such that Y is a subshift.
- The language of X is the set
$L_X = \{\sigma \in 2^{<\omega} \mid \exists \alpha \in X(\sigma \text{ is a subword of } \alpha)\}$.
- $\overline{L_X}$ is the set of forbidden words.

1. If X is minimal and $\sigma \in L_X$ then for every $\alpha \in X$, σ is a subword of α. So every element of X can enumerate the set L_X.

Motivation from symbolic dynamics by Emmanuel Jeandel

Definition

- A subshift is a closed subset $X \subseteq 2^\omega$ such that if $a\alpha \in X$ then $\alpha \in X$.
- X is minimal if there is no $Y \subset X$, such that Y is a subshift.
- The language of X is the set
  $L_X = \{\sigma \in 2^{<\omega} \mid \exists \alpha \in X(\sigma \text{ is a subword of } \alpha)\}$.
- $\overline{L_X}$ is the set of forbidden words.

1. If X is minimal and $\sigma \in L_X$ then for every $\alpha \in X$, σ is a subword of α. So every element of X can enumerate the set L_X.
2. If we can enumerate L_X then we can compute a member of X.
3. The Turing degrees that compute elements of X are exactly the degrees that contain enumerations of L_X.

Motivation from symbolic dynamics by Emmanuel Jeandel

Definition

- A subshift is a closed subset $X \subseteq 2^\omega$ such that if $a\alpha \in X$ then $\alpha \in X$.
- X is minimal if there is no $Y \subset X$, such that Y is a subshift.
- The language of X is the set
  $L_X = \{\sigma \in 2^{<\omega} \mid \exists \alpha \in X(\sigma \text{ is a subword of } \alpha)\}$.
- $\overline{L_X}$ is the set of forbidden words.

1. If X is minimal and $\sigma \in L_X$ then for every $\alpha \in X$, σ is a subword of α. So every element of X can enumerate the set L_X.
2. If we can enumerate L_X then we can compute a member of X.
3. The Turing degrees that compute elements of X are exactly the degrees that contain enumerations of L_X.
4. (Jaendel) If we can enumerate the set of forbidden words $\overline{L_X}$ then we can enumerate L_X. So, $L_X \leq_e \overline{L_X}$.
5. (McCarthy) If A is cototal, then $A \equiv_e L_X$ for some minimal subshift X.

Three definitions

Definition

A set A is *cototal* if $A \leq_e \overline{A}$. A degree $\mathbf{a}$ is cototal if it contains a cototal set.

Three definitions

Definition

A set A is *cototal* if $A \leq_e \overline{A}$. A degree $\mathbf{a}$ is cototal if it contains a cototal set.

Definition

A set A is *graph-cototal* if $A \equiv_e \overline{G_f}$ for some total function f. A degree $\mathbf{a}$ is graph-cototal if it contains a graph-cototal set.

Three definitions

Definition

A set A is *cototal* if $A \leq_e \overline{A}$. A degree $\mathbf{a}$ is cototal if it contains a cototal set.

Definition

A set A is *graph-cototal* if $A \equiv_e \overline{G_f}$ for some total function f. A degree $\mathbf{a}$ is graph-cototal if it contains a graph-cototal set.

Definition

A set A is *weakly cototal* if there is a set $A \equiv_e B$, such that $\overline{B}$ is total.

Three definitions

Definition

A set A is *cototal* if $A \leq_e \overline{A}$. A degree $\mathbf{a}$ is cototal if it contains a cototal set.

Definition

A set A is *graph-cototal* if $A \equiv_e \overline{G_f}$ for some total function f. A degree $\mathbf{a}$ is graph-cototal if it contains a graph-cototal set.

Definition

A set A is *weakly cototal* if there is a set $A \equiv_e B$, such that $\overline{B}$ is total.

It is clear that every cototal degree is weakly cototal, since if $A \leq_e \overline{A}$, then $\overline{A}$ is a total set.

total $\Rightarrow$ graph-cototal $\Rightarrow$ cototal $\Rightarrow$ weakly cototal.

Σ^0_2 e-degrees

Proposition

Σ^0_2 e-degrees are (graph-)cototal.

Let A be Σ^0_2. Consider the set $K_A = \bigoplus_{e<\omega} \Gamma_e(A)$. Then $A \equiv_e K_A$ and

$$\overline{K_A} = \bigoplus_{e<\omega} \overline{\Gamma_e(A)} \geq_e \overline{K} \geq_e A \equiv_e K_A.$$

Corollary

Graph-cototal does not imply *total*.

Unique correct axiom

Proposition

- There are Π^0_2-sets that do not even have cototal enumeration degree.
- But every Π^0_2-set has weakly cototal degree.
  $A \equiv_e A \oplus K \Rightarrow \overline{A \oplus K} \equiv_e \overline{A} \oplus \overline{K} \equiv_e \overline{K} \in \mathbf{0}'_e$
- There are Δ^0_3-sets that are not even weakly cototal.

Theorem

An e-degree $\mathbf{a}$ is graph-cototal if and only if $\mathbf{a}$ contains a cototal set A, such that for some enumeration operator Γ, we have that $A = \Gamma(\overline{A})$ and for every $n \in A$ there is a unique axiom $\langle n, D\rangle \in \Gamma$ such that $D \subseteq A$.

Goal:
Cototal does not imply *graph-cototal*.

Maximal independent sets

Definition

Let $G = (\mathbb{N}, E)$ be a graph and $S \subseteq \mathbb{N}$.

1. S is an *independent set* for G if $i \neq j$ are in S then $(i, j) \notin E$.
2. An independent set is *maximal* if it has no proper independent superset, i.e. for every element $i \notin S$ there is a $j \in S$ such that $(i, j) \in E$.

Maximal independent sets

Definition

Let $G = (\mathbb{N}, E)$ be a graph and $S \subseteq \mathbb{N}$.

1. S is an *independent set* for G if $i \neq j$ are in S then $(i, j) \notin E$.
2. An independent set is *maximal* if it has no proper independent superset, i.e. for every element $i \notin S$ there is a $j \in S$ such that $(i, j) \in E$.

If S is a maximal independent set for G, then S can enumerate its complement: $i \in \overline{S}$ iff there is a $j \neq i$ such that $(i, j) \in E$ and $j \in S$.

Maximal independent sets

Definition

Let $G = (\mathbb{N}, E)$ be a graph and $S \subseteq \mathbb{N}$.

1. S is an *independent set* for G if $i \neq j$ are in S then $(i, j) \notin E$.
2. An independent set is *maximal* if it has no proper independent superset, i.e. for every element $i \notin S$ there is a $j \in S$ such that $(i, j) \in E$.

If S is a maximal independent set for G, then S can enumerate its complement: $i \in \overline{S}$ iff there is a $j \neq i$ such that $(i, j) \in E$ and $j \in S$.

Theorem

Every cototal degree contains the complement of maximal independent set for $\omega^{<\omega}$.

Theorem

There is a cototal dregee which is not graph-cototal.

Maximal antichains

Proposition

If C is a maximal antichain on $\omega^{<\omega}$, then $\overline{C} \leq_e C$, i.e. $\overline{C}$ is cototal.

To determine if a string $\sigma \in \omega^{<\omega}$ is in $\overline{C}$, we wait for some element comparable but not equal to σ to enter C. Since C is an antichain, we only identify elements of $\overline{C}$ in this way. And by maximality, if $\sigma \in \overline{C}$ then something comparable but not equal to σ must eventually enter C.

Theorem (McCarthy)

Every cototal degree contains the complement of a maximal antichain in $\omega^{<\omega}$.

Maximal antichains

Proposition

If C is a maximal antichain on $\omega^{<\omega}$, then $\overline{C} \leq_e C$, i.e. $\overline{C}$ is cototal.

To determine if a string $\sigma \in \omega^{<\omega}$ is in $\overline{C}$, we wait for some element comparable but not equal to σ to enter C. Since C is an antichain, we only identify elements of $\overline{C}$ in this way. And by maximality, if $\sigma \in \overline{C}$ then something comparable but not equal to σ must eventually enter C.

Theorem (McCarthy)

Every cototal degree contains the complement of a maximal antichain in $\omega^{<\omega}$.

Definition (Montalban)

A tree $T \subseteq 2^{<\omega}$ is *e-pointed* if it has no dead ends and every infinite path $f \in [T]$ enumerates T.

Theorem (McCarthy)

An e-degree is cototal if and only if it contains a (uniformly) e-pointed tree.

Joins of nontrivial $\mathcal{K}$-pairs

Definition

A $\mathcal{K}$-pair is a pair of sets $\{A, B\}$ for which there is a c.e. set W such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

Joins of nontrivial $\mathcal{K}$-pairs

Definition

A $\mathcal{K}$-pair is a pair of sets $\{A, B\}$ for which there is a c.e. set W such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

Proposition (Kalimullin)

Let $\{A, B\}$ be a $\mathcal{K}$-pair. If A and B are not c.e. then:

1. $A \leq_e \overline{B}$ and $\overline{A} \leq_e \emptyset' \oplus B$.
2. $B \leq_e \overline{A}$ and $\overline{B} \leq_e \emptyset' \oplus A$.

Joins of nontrivial $\mathcal{K}$-pairs

Definition

A $\mathcal{K}$-pair is a pair of sets $\{A, B\}$ for which there is a c.e. set W such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

Proposition (Kalimullin)

Let $\{A, B\}$ be a $\mathcal{K}$-pair. If A and B are not c.e. then:

1. $A \leq_e \overline{B}$ and $\overline{A} \leq_e \emptyset' \oplus B$.
2. $B \leq_e \overline{A}$ and $\overline{B} \leq_e \emptyset' \oplus A$.

Proposition

If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair then $A \oplus B$ is cototal.

Joins of nontrivial $\mathcal{K}$-pairs

Definition

A $\mathcal{K}$-pair is a pair of sets $\{A, B\}$ for which there is a c.e. set W such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

Proposition (Kalimullin)

Let $\{A, B\}$ be a $\mathcal{K}$-pair. If A and B are not c.e. then:

1. $A \leq_e \overline{B}$ and $\overline{A} \leq_e \emptyset' \oplus B$.
2. $B \leq_e \overline{A}$ and $\overline{B} \leq_e \emptyset' \oplus A$.

Proposition

If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair then $A \oplus B$ is cototal.

Proof: $A \oplus B \leq_e \overline{B} \oplus \overline{A} \equiv_e \overline{A \oplus B}$.

Continuous degrees

J. Miller introduced the continuous degrees $\mathcal{D}_r$ to compare the complexity of points in computable metric spaces. A point x in a computable metric space can be described by a sequence of "rational" points that limit to it. For two points $x; y$ we say that $x \leq_r y$ if every description of y computes a description of x. The continuous degrees embed into $\mathcal{D}_e$. In fact, $D_T \subset \mathcal{D}_r \subset \mathcal{D}_e$.

Definition (J. Miller)

An e-degree is continuous if it contains a set of the form $A = \bigoplus_{i<\omega}(\{q \mid q < \alpha_i\} \oplus \{q \mid q > \alpha_i\})$, where $\{\alpha_i\}_{i<\omega}$ is a sequence of real numbers.

Continuous degrees

J. Miller introduced the continuous degrees $\mathcal{D}_r$ to compare the complexity of points in computable metric spaces. A point x in a computable metric space can be described by a sequence of "rational" points that limit to it. For two points $x; y$ we say that $x \leq_r y$ if every description of y computes a description of x. The continuous degrees embed into $\mathcal{D}_e$. In fact, $D_T \subset \mathcal{D}_r \subset \mathcal{D}_e$.

Definition (J. Miller)

An e-degree is continuous if it contains a set of the form $A = \bigoplus_{i<\omega}(\{q \mid q < \alpha_i\} \oplus \{q \mid q > \alpha_i\})$, where $\{\alpha_i\}_{i<\omega}$ is a sequence of real numbers.

Proposition

Continuous degrees are cototal.

$\overline{A} \equiv_e B = \bigoplus_{i<\omega}(\{q \mid q \leq \alpha_i\} \oplus \{q \mid q \geq \alpha_i\})$.

Kihara and Pauly extend Miller's idea to points in arbitrary represented topological spaces.

The skip operator

Recall that $\overline{K_A} = \bigoplus_e \overline{\Gamma_e(A)}$.

The skip operator

Recall that $\overline{K_A} = \bigoplus_e \overline{\Gamma_e(A)}$.

Proposition

If $A \leq_e B$ then $\overline{K_A} \leq_1 \overline{K_B}$.

The skip operator

Recall that $\overline{K_A} = \bigoplus_e \overline{\Gamma_e(A)}$.

Proposition

If $A \leq_e B$ then $\overline{K_A} \leq_1 \overline{K_B}$.

Definition

The skip of A is the set $A^\diamond = \overline{K_A}$. The skip of a degree $\mathbf{a}$ is $\mathbf{a}^\diamond = d_e(A^\diamond)$.

The skip operator

Recall that $\overline{K_A} = \bigoplus_e \overline{\Gamma_e(A)}$.

Proposition

If $A \leq_e B$ then $\overline{K_A} \leq_1 \overline{K_B}$.

Definition

The skip of A is the set $A^\Diamond = \overline{K_A}$. The skip of a degree $\mathbf{a}$ is $\mathbf{a}^\Diamond = d_e(A^\Diamond)$.

Proposition

A degree $\mathbf{a}$ is cototal if and only if $\mathbf{a} \leq \mathbf{a}^\Diamond$ if and only if $\mathbf{a}^\Diamond = \mathbf{a}'$.

$\Rightarrow A \leq_e \overline{A} \leq_e A^\Diamond$

$\Leftarrow K_A \equiv_e A \leq_e A^\Diamond = \overline{K_A}$.

Recall that $A' = K_A \oplus \overline{K_A}$.

Skip inversion

Theorem

Let $S \geq_e \emptyset'$. There is a set A such that $A^\Diamond \equiv_e S$.

Skip inversion

Theorem

Let $S \geq_e \emptyset'$. There is a set A such that $A^\Diamond \equiv_e S$.

We build A so that:

1. $S \leq_e \overline{A}$.
2. $\overline{K_A} \leq_e S$.

Skip inversion

Theorem

Let $S \geq_e \emptyset'$. There is a set A such that $A^\Diamond \equiv_e S$.

We build A so that:

1. $S \leq_e \overline{A}$.
2. $\overline{K_A} \leq_e S$.

We first build a table $\hat{A}$ with one empty box in each column as a set c.e. in $\emptyset'$.

The set of empty boxes will be computable from $\emptyset'$.

Then $A = \hat{A} \cup \{\langle n, s\rangle \mid n \in \overline{S}\}$.

Skip inversion

Theorem

Let $S \geq_e \emptyset'$. There is a set A such that $A^{\Diamond} \equiv_e S$.

We build A so that:

1. $S \leq_e \overline{A}$.
2. $\overline{K_A} \leq_e S$.

We first build a table $\hat{A}$ with one empty box in each column as a set c.e. in $\emptyset'$.

The set of empty boxes will be computable from $\emptyset'$.

Then $A = \hat{A} \cup \left\{ \langle n, s \rangle \mid n \in \overline{S} \right\}$.

Note! $\overline{S} \leq_e A \oplus \emptyset'$. So if we start out with an S that is not total set but belongs to a total degree then A is not cototal. But $A \equiv_e K_A$ and $\overline{K_A} = A^{\Diamond} \equiv_e S$ has a total degree and so A is weakly cototal.

Corollary

Weakly cototal does not imply *cototal*.

Skip iteration

We can define the *iterated skip operator* of an enumeration degree $\mathbf{a}$ by:

- $\mathbf{a}^{\langle 0\rangle} = \mathbf{a}$
- $\mathbf{a}^{\langle n+1\rangle} = (\mathbf{a}^{\langle n\rangle})^{\Diamond}$.

This iterated skip can exhibit exotic behavior:

Theorem

For all enumeration degrees, $\mathbf{a} \leq \mathbf{a}^{\Diamond\Diamond}$ and $\mathbf{a}^{\Diamond} \geq \mathbf{0}'$, but not always $\mathbf{a} \leq \mathbf{a}^{\Diamond}$.

$\overline{A} \leq_1 A^{\Diamond} \Rightarrow A \leq_1 \overline{A^{\Diamond}} \leq_1 A^{\Diamond\Diamond}$.

Skip iteration

We can define the *iterated skip operator* of an enumeration degree $\mathbf{a}$ by:

- $\mathbf{a}^{\langle 0 \rangle} = \mathbf{a}$
- $\mathbf{a}^{\langle n+1 \rangle} = (\mathbf{a}^{\langle n \rangle})^{\Diamond}$.

This iterated skip can exhibit exotic behavior:

Theorem

For all enumeration degrees, $\mathbf{a} \leq \mathbf{a}^{\Diamond\Diamond}$ and $\mathbf{a}^{\Diamond} \geq \mathbf{0}'$, but not always $\mathbf{a} \leq \mathbf{a}^{\Diamond}$.

$\overline{A} \leq_1 A^{\Diamond} \Rightarrow A \leq_1 \overline{A^{\Diamond}} \leq_1 A^{\Diamond\Diamond}$.

By Knaster-Tarski's fixed point theorem:

Theorem

There is an enumeration degree $\mathbf{a}$ such that $\mathbf{a} = \mathbf{a}^{\Diamond\Diamond}$.

Skip iteration

We can define the *iterated skip operator* of an enumeration degree $\mathbf{a}$ by:

- $\mathbf{a}^{\langle 0\rangle} = \mathbf{a}$
- $\mathbf{a}^{\langle n+1\rangle} = (\mathbf{a}^{\langle n\rangle})^{\Diamond}$.

This iterated skip can exhibit exotic behavior:

Theorem

For all enumeration degrees, $\mathbf{a} \leq \mathbf{a}^{\Diamond\Diamond}$ and $\mathbf{a}^{\Diamond} \geq \mathbf{0}'$, but not always $\mathbf{a} \leq \mathbf{a}^{\Diamond}$.

$\overline{A} \leq_1 A^{\Diamond} \Rightarrow A \leq_1 \overline{A^{\Diamond}} \leq_1 A^{\Diamond\Diamond}$.

By Knaster-Tarski's fixed point theorem:

Theorem

There is an enumeration degree $\mathbf{a}$ such that $\mathbf{a} = \mathbf{a}^{\Diamond\Diamond}$.

$A \subseteq B \Rightarrow K_A \subseteq K_B$

Skip iteration

We can define the *iterated skip operator* of an enumeration degree $\mathbf{a}$ by:

- $\mathbf{a}^{\langle 0 \rangle} = \mathbf{a}$
- $\mathbf{a}^{\langle n+1 \rangle} = (\mathbf{a}^{\langle n \rangle})^{\Diamond}$.

This iterated skip can exhibit exotic behavior:

Theorem

For all enumeration degrees, $\mathbf{a} \leq \mathbf{a}^{\Diamond\Diamond}$ and $\mathbf{a}^{\Diamond} \geq \mathbf{0}'$, but not always $\mathbf{a} \leq \mathbf{a}^{\Diamond}$.

$\overline{A} \leq_1 A^{\Diamond} \Rightarrow A \leq_1 \overline{A^{\Diamond}} \leq_1 A^{\Diamond\Diamond}$.

By Knaster-Tarski's fixed point theorem:

Theorem

There is an enumeration degree $\mathbf{a}$ such that $\mathbf{a} = \mathbf{a}^{\Diamond\Diamond}$.

$$A \subseteq B \Rightarrow K_A \subseteq K_B \Rightarrow \overline{K_A} \supseteq \overline{K_B}$$

Skip iteration

We can define the *iterated skip operator* of an enumeration degree $\mathbf{a}$ by:

- $\mathbf{a}^{\langle 0 \rangle} = \mathbf{a}$
- $\mathbf{a}^{\langle n+1 \rangle} = (\mathbf{a}^{\langle n \rangle})^{\Diamond}$.

This iterated skip can exhibit exotic behavior:

Theorem

For all enumeration degrees, $\mathbf{a} \leq \mathbf{a}^{\Diamond\Diamond}$ and $\mathbf{a}^{\Diamond} \geq \mathbf{0}'$, but not always $\mathbf{a} \leq \mathbf{a}^{\Diamond}$.

$\overline{A} \leq_1 A^{\Diamond} \Rightarrow A \leq_1 \overline{A^{\Diamond}} \leq_1 A^{\Diamond\Diamond}$.

By Knaster-Tarski's fixed point theorem:

Theorem

There is an enumeration degree $\mathbf{a}$ such that $\mathbf{a} = \mathbf{a}^{\Diamond\Diamond}$.

$$A \subseteq B \Rightarrow K_A \subseteq K_B \Rightarrow \overline{K_A} \supseteq \overline{K_B} \Rightarrow K_{\overline{K_A}} \supseteq K_{\overline{K_B}}$$

Skip iteration

We can define the *iterated skip operator* of an enumeration degree $\mathbf{a}$ by:

- $\mathbf{a}^{\langle 0 \rangle} = \mathbf{a}$
- $\mathbf{a}^{\langle n+1 \rangle} = (\mathbf{a}^{\langle n \rangle})^{\Diamond}$.

This iterated skip can exhibit exotic behavior:

Theorem

For all enumeration degrees, $\mathbf{a} \leq \mathbf{a}^{\Diamond\Diamond}$ and $\mathbf{a}^{\Diamond} \geq \mathbf{0}'$, but not always $\mathbf{a} \leq \mathbf{a}^{\Diamond}$.

$\overline{A} \leq_1 A^{\Diamond} \Rightarrow A \leq_1 \overline{A^{\Diamond}} \leq_1 A^{\Diamond\Diamond}$.

By Knaster-Tarski's fixed point theorem:

Theorem

There is an enumeration degree $\mathbf{a}$ such that $\mathbf{a} = \mathbf{a}^{\Diamond\Diamond}$.

$A \subseteq B \Rightarrow K_A \subseteq K_B \Rightarrow \overline{K_A} \supseteq \overline{K_B} \Rightarrow K_{\overline{K_A}} \supseteq K_{\overline{K_B}} \Rightarrow$
$\overline{K_{\overline{K_A}}} \subseteq \overline{K_{\overline{K_B}}}$.

Any such enumeration degree lies above all total hyperarithmetic enumeration degrees.

Iterating the skip

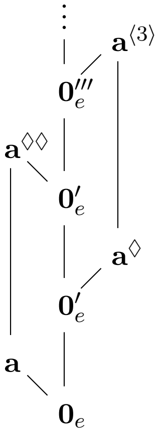

Figure: Iterated skips of a degree

Zig-zag

If $\mathbf{a}^{\langle n \rangle}$ is not cototal for every n:

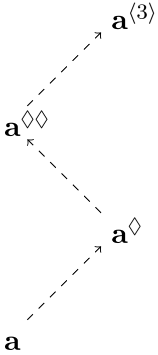

Figure: Iterated skips of a degree: the zig-zag

Generic sets

Definition

Let G and X be sets of natural numbers. G is 1- *generic relative to* $\langle X \rangle$ if and only if for every $W \subseteq 2^{<\omega}$ such that $W \leq_e X$:

$$(\exists \sigma \preceq G)[\sigma \in W \vee (\forall \tau \succeq \sigma)[\tau \notin W]].$$

Proposition

If G is 1-generic relative to $\langle X \rangle$ then:

- $\overline{G}$ is 1-generic relative to $\langle X \rangle$.
- $(G \oplus X)^{\Diamond} = \overline{G} \oplus X'$.

Generic sets

Definition

Let G and X be sets of natural numbers. G is 1- *generic relative to* $\langle X \rangle$ if and only if for every $W \subseteq 2^{<\omega}$ such that $W \leq_e X$:

$$(\exists \sigma \preceq G)[\sigma \in W \vee (\forall \tau \succeq \sigma)[\tau \notin W]].$$

Proposition

If G is 1-generic relative to $\langle X \rangle$ then:

- $\overline{G}$ is 1-generic relative to $\langle X \rangle$.
- $(G \oplus X)^{\Diamond} = \overline{G} \oplus X'$.

If G is arithmetically generic, i.e. G is 1-generic relative to $\langle \emptyset^{(n)} \rangle$, for every n, then the skips of G and $\overline{G}$ form a double helix.

- If n is odd then $G^{\langle n \rangle} \equiv_e \overline{G} \oplus \emptyset^{(n)}$ and $(\overline{G})^{\langle n \rangle} \equiv_e G \oplus \emptyset^{(n)}$.
- If n is even then $G^{\langle n \rangle} \equiv_e G \oplus \emptyset^{(n)}$ and $(\overline{G})^{\langle n \rangle} \equiv_e \overline{G} \oplus \emptyset^{(n)}$.

Double zig-zag

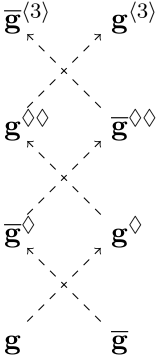

Figure: Iterated skips of a degrees of an arithmetically generic set and its complement: double zig-zag

Skips of nontrivial $\mathcal{K}$-pairs

Proposition

If $\{A, B\}$ is a non-trivial $\mathcal{K}$-pair then $A^\Diamond \equiv_e B \oplus \emptyset'$.
If $\{A, B\}$ is a non-trivial $\mathcal{K}$-pair relative to $\langle X \rangle$ then $(A \oplus X)^\Diamond \leq_e B \oplus X^\Diamond$.
The oracle X is of cototal degree iff we have equivalence above for every nontrivial $\mathcal{K}$-pair relative to $\langle X \rangle$.

Skips of nontrivial $\mathcal{K}$-pairs

Proposition

If $\{A, B\}$ is a non-trivial $\mathcal{K}$-pair then $A^{\Diamond} \equiv_e B \oplus \emptyset'$.
If $\{A, B\}$ is a non-trivial $\mathcal{K}$-pair relative to $\langle X \rangle$ then $(A \oplus X)^{\Diamond} \leq_e B \oplus X^{\Diamond}$.
The oracle X is of cototal degree iff we have equivalence above for every nontrivial $\mathcal{K}$-pair relative to $\langle X \rangle$.

If $\{A, B\}$ is a non-trivial $\mathcal{K}$-pair relative to $\emptyset^{(n)}$ for every n then the iterate skips of A and B form a double zig-zag.

- if n is odd then $A^{\langle n \rangle} \equiv_e B \oplus \emptyset^{(n)}$ and $B^{\langle n \rangle} \equiv_e A \oplus \emptyset^{(n)}$, and
- if n is even then $A^{\langle n \rangle} \equiv_e A \oplus \emptyset^{(n)}$ and $B^{\langle n \rangle} \equiv_e B \oplus \emptyset^{(n)}$.

Skip iterations

Theorem (Ganchev, Sorbi)

For every enumeration degree $\mathbf{x} > \mathbf{0}_e$, there is a degree $\mathbf{a} \leq \mathbf{x}$ such that $\mathbf{a}$ is half of a nontrivial $\mathcal{K}$-pair and such that $\mathbf{a}' = \mathbf{x}'$.

$$A' \equiv_e A \oplus A^{\Diamond} \equiv_e A \oplus B \oplus \emptyset' \equiv_e B \oplus B^{\Diamond} \equiv_e B'.$$

Proposition

- If $\mathbf{x}$ is high ($\mathbf{x}' = \mathbf{0}''$):

$$\mathbf{b}^{\Diamond} < \mathbf{b}' = \mathbf{b}^{\Diamond\Diamond} < \mathbf{b}'' = \mathbf{b}^{\langle 3 \rangle} < \cdots < \mathbf{b}^{(n)} = \mathbf{b}^{\langle n+1 \rangle} < \ldots$$

Skip iterations

Theorem (Ganchev, Sorbi)

For every enumeration degree $\mathbf{x} > \mathbf{0}_e$, there is a degree $\mathbf{a} \leq \mathbf{x}$ such that $\mathbf{a}$ is half of a nontrivial $\mathcal{K}$-pair and such that $\mathbf{a}' = \mathbf{x}'$.

$$A' \equiv_e A \oplus A^{\Diamond} \equiv_e A \oplus B \oplus \emptyset' \equiv_e B \oplus B^{\Diamond} \equiv_e B'.$$

Proposition

- If $\mathbf{x}$ is high ($\mathbf{x}' = \mathbf{0}''$):

$$\mathbf{b}^{\Diamond} < \mathbf{b}' = \mathbf{b}^{\Diamond\Diamond} < \mathbf{b}'' = \mathbf{b}^{\langle 3 \rangle} < \cdots < \mathbf{b}^{(n)} = \mathbf{b}^{\langle n+1 \rangle} < \dots$$

- If $\mathbf{x}$ is intermediate:

$$\mathbf{b}^{\Diamond} < \mathbf{b}' < \mathbf{b}^{\Diamond\Diamond} < \mathbf{b}'' < \mathbf{b}^{\langle 3 \rangle} < \cdots < \mathbf{b}^{(n)} < \mathbf{b}^{\langle n+1 \rangle} < \dots$$

The cotatal degrees are dense

Corollary

The relation

$$SK = \left\{ (\mathbf{a}, \mathbf{a}^{\Diamond}) \mid \mathbf{a} \text{ is half of a nontrivial } \mathcal{K}\text{-pair} \right\}$$

is first-order definable in $\mathcal{D}_e$.

The cototal degrees are dense

Corollary

The relation

$$SK = \left\{ (\mathbf{a}, \mathbf{a}^{\Diamond}) \mid \mathbf{a} \text{ is half of a nontrivial } \mathcal{K}\text{-pair} \right\}$$

is first-order definable in $\mathcal{D}_e$.

Question: Is the skip operator definable in $\mathcal{D}_e$?

Good e-degrees

Definition (Lachlan, Shore)

A uniformly computable sequence of finite sets $\{A_s\}_{s<\omega}$ is *a good approximation* to a set A if:
$G1 (\forall n)(\exists s)(A \upharpoonright n \subseteq A_s \subseteq A)$
$G2 (\forall n)(\exists s)(\forall t > s)(A_t \subseteq A \Rightarrow A \upharpoonright n \subseteq A_t)$.

An enumeration degree is *good* if it contains a set with a good approximation.

- Good e-degrees cannot be tops of empty intervals.
- Total enumeration degrees and enumeration degrees of n-c.e.a. sets are good.

The cototal degrees are dense

Theorem (Harris; Miller, M. Soskova)

The good enumeration degrees are exactly the cototal enumeration degrees.

If A has a good approximation then

$$A \leq_e \{\langle x, s\rangle \mid (\forall t > s)(A_t \subseteq A \Rightarrow x \in A)\} \leq_e A^{\Diamond}.$$

Every uniformly e-pointed tree has a good approximation.

Theorem (Miller, M. Soskova)

The cototal enumeration degrees are dense.

If $V <_e U$ are cototal and U has a good approximation they build Θ such that $\Theta(U)$ is the complement of a maximal independent set and

$$V <_e \Theta(U) \oplus V <_e U.$$