Some applications of the Jump Inversion Theorem for the Degree Spectra

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Outline

- Degree spectra and jump spectra
- Every jump spectrum is spectrum
- Jump inversion theorem for the degree spectra
- Some applications

Enumeration of a Structure

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_k, =)$ be a countable abstract structure.

- An enumeration f of $\mathfrak{A}$ is a total mapping from $\mathbb{N}$ onto $\mathbb{N}$.
- For each predicate R of $\mathfrak{A}$:

$$f^{-1}(R) = \begin{array}{l} \{\langle x_1, \ldots, x_r, 0\rangle \mid R(f(x_1), \ldots, f(x_r)\} \cup \\ \{\langle x_1, \ldots, x_r, 1\rangle \mid \neg R(f(x_1), \ldots, f(x_r))\}. \end{array}$$

- $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k) \oplus f^{-1}(=)$.

Degree Spectra

Definition

The degree spectrum of $\mathfrak{A}$ is the set

$$\mathrm{DS}(\mathfrak{A}) = \{d_{\mathrm{T}}(f^{-1}(\mathfrak{A})) \mid f \text{ is an enumeration of } \mathfrak{A}\}.$$

- L. Richter [1981], J. Knight [1986].
- The degree spectra are upwards closed:

$$\mathbf{a} \in \mathrm{DS}(\mathfrak{A}), \mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{b} \in \mathrm{DS}(\mathfrak{A}).$$

- The jump spectrum of $\mathfrak{A}$ is the set $\mathrm{DS}_1(\mathfrak{A}) = \{\mathbf{a}' \mid \mathbf{a} \in \mathrm{DS}(\mathfrak{A})\}$.

Theorem

Each jump spectrum is degree spectrum of a structure, i.e. for every structure $\mathfrak{A}$ there exists a structure $\mathfrak{B}$ such that $\mathrm{DS}_1(\mathfrak{A}) = \mathrm{DS}(\mathfrak{B})$.

Moschovakis' extension

Definition

- $\bar{0} \notin \mathbb{N}$, $\mathbb{N}_0 = \mathbb{N} \cup \{\bar{0}\}$.
- A pairing function $\langle ., . \rangle$, $\mathrm{range}(\langle ., . \rangle) \cap \mathbb{N}_0 = \emptyset$.
- The least set $\mathbb{N}^* \supseteq \mathbb{N}_0$, closed under $\langle ., . \rangle$.
- Moschovakis' extension of $\mathfrak{A}$ is the structure $\mathfrak{A}^* = (\mathbb{N}^*, R_1, \ldots, R_n, =, \mathbb{N}_0, G_{\langle ., . \rangle})$.

Proposition

$\mathrm{DS}(\mathfrak{A}) = \mathrm{DS}(\mathfrak{A}^*)$.

The set $K_{\mathfrak{A}}$

- A new predicate $K_{\mathfrak{A}}$ (analogue of Kleene's set).
- For $e, x \in \mathbb{N}$ and finite part τ, let
  $\tau \Vdash F_e(x) \iff x \in W_e^{\tau^{-1}(\mathfrak{A})}$.
- $K_{\mathfrak{A}} = \{\langle \delta^*, e, x\rangle : (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}$.
- $\mathfrak{B} = (\mathfrak{A}^*, K_{\mathfrak{A}})$.

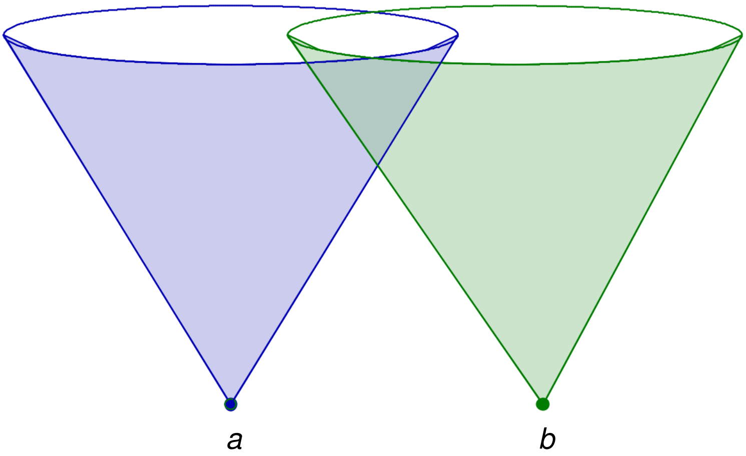
Theorem

$\mathrm{DS}_1(\mathfrak{A}) = \mathrm{DS}(\mathfrak{B})$

Inverting the jump

Given a set of enumeration degrees $\mathcal{A}$ does there exist a structure $\mathfrak{C}$ such that $\mathrm{DS}_1(\mathfrak{C}) = \mathcal{A}$?

1. Each element of $\mathcal{A}$ should be a jump of a degree.
2. $\mathcal{A}$ should be upwards closed (since each jump spectrum is a spectrum and the spectrum is upwards closed).
3. The set $\mathcal{A}$ should be a degree spectrum of a structure $\mathfrak{A}$.

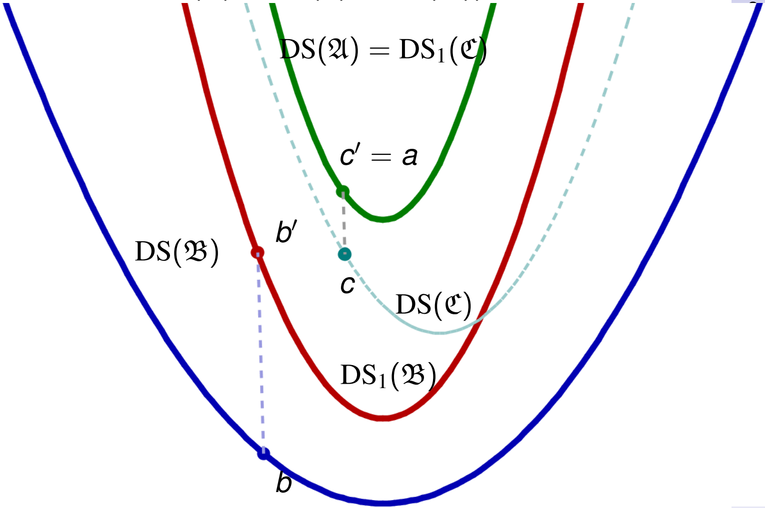

a
b

Theorem

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures such that $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_1(\mathfrak{B})$. Then there exists a structure $\mathfrak{C}$ such that $\mathrm{DS}(\mathfrak{C}) \subseteq \mathrm{DS}(\mathfrak{B})$ and $\mathrm{DS}_1(\mathfrak{C}) = \mathrm{DS}(\mathfrak{A})$.

- The structure $\mathfrak{C}$ we construct as a Marker's extension of $\mathfrak{A}$.
- We code the structure $\mathfrak{B}$ in $\mathfrak{C}$.
- In our construction we use also the relativized representation lemma for Σ^0_2 sets proved by Goncharov and Khoussainov

Theorem $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_1(\mathfrak{B}) \Longrightarrow (\exists \mathfrak{C})(\mathrm{DS}_1(\mathfrak{C}) = \mathrm{DS}(\mathfrak{A})\ \&\ \mathrm{DS}(\mathfrak{C}) \subseteq \mathrm{DS}(\mathfrak{B}))$.



Marker's Extensions

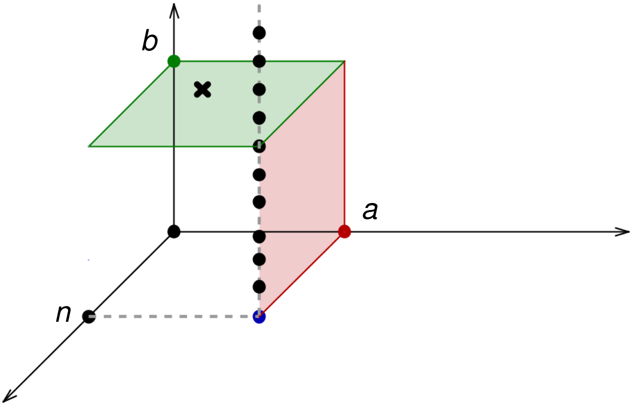
$\mathfrak{A}^{\exists\forall} = (A \cup \bigcup_{i=1}^{s} X_i \cup \bigcup_{i=1}^{s} Y_i, R_1^{\exists\forall}, \ldots, R_s^{\exists\forall}, \bar{X}_1, \ldots, \bar{X}_s, \bar{Y}_1, \ldots, \bar{Y}_s, =)$

1. $X = \{x_{\langle a_1,\ldots,a_r\rangle} \mid R(a_1,\ldots,a_r)\}$
2. $(\exists x \in X) R^{\exists}(a_1,\ldots,a_r,x) \iff R(a_1,\ldots,a_r)$.
3. $Y = \{y_{\langle a_1,\ldots,a_r,x\rangle} \mid \neg R^{\exists}(a_1,\ldots,a_r,x)\}$.
4. $(\forall y \in Y) R^{\exists\forall}(a_1,\ldots,a_r,x,y) \iff R^{\exists}(a_1,\ldots,a_r,x)$
5. $R(a_1,\ldots,a_r) \iff (\exists x \in X)(\forall y \in Y) R^{\exists\forall}(a_1,\ldots,a_r,x,y)$;
6. $(\forall y \in Y)(\exists$ a unique sequence $a_1,\ldots,a_r \in A \;\&\; x \in X)(\neg R^{\exists\forall}(a_1,\ldots,a_r,x,y))$;
7. $(\forall x \in X)(\exists$ a unique sequence $a_1,\ldots,a_r \in A)(\forall y \in Y) R^{\exists\forall}(a_1,\ldots,a_r,x,y)$.

One-to-one Representation of Σ_2^0 Sets

Goncharov and Khoussainov

1. $n \in M \Leftrightarrow (\exists \text{ a unique } a)(\forall b)Q(n,a,b)$;
2. $(\forall b)(\exists \text{ a unique pair } \langle n,a\rangle)(\neg Q(n,a,b))$;
3. $(\forall a)(\exists \text{ a unique } n)(\forall b)Q(n,a,b)$.



Theorem (Jump Inversion Theorem)

Let $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_1(\mathfrak{B})$*. Then there exists a structure* $\mathfrak{C}$ *such that* $\mathrm{DS}_1(\mathfrak{C}) = \mathrm{DS}(\mathfrak{A})$ *and* $\mathrm{DS}(\mathfrak{C}) \subseteq \mathrm{DS}(\mathfrak{B})$*.*

- The structure $\mathfrak{C}$ is constructed as

$$\mathfrak{C} = \mathfrak{B} \oplus \mathfrak{A}^{\exists\forall}.$$

Jump Inversion Theorem

Definition

n-th jump spectrum of $\mathfrak{A}$ is the set $\mathrm{DS}_n(\mathfrak{A}) = \{\mathbf{a}^{(n)} : \mathbf{a} \in \mathrm{DS}(\mathfrak{A})\}$.

By induction on *n*:

Theorem

There exists a structure $\mathfrak{A}^{(n)}$ *such that* $\mathrm{DS}_n(\mathfrak{A}) = \mathrm{DS}(\mathfrak{A}^{(n)})$.

Theorem

Let $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_n(\mathfrak{B})$. *There exists a structure* $\mathfrak{C}$ *such that* $\mathrm{DS}(\mathfrak{C}) \subseteq \mathrm{DS}(\mathfrak{B})$ *and* $\mathrm{DS}_n(\mathfrak{C}) = \mathrm{DS}(\mathfrak{A})$.

Some Applications

Definition

If **a** is the least element of $\mathrm{DS}_n(\mathfrak{A})$ then **a** is called *nth jump degree*.

- Downey and Knight by complicated construction:
- for every recursive ordinal α there exists a linear ordering $\mathfrak{A}$ such that $\mathfrak{A}$ has αth jump degree equal to $\mathbf{0}^{(\alpha)}$ but for all $\beta < \alpha$, there is no βth jump degree of $\mathfrak{A}$.
- we show a construction: for every natural number n we can find examples of structures which have $(n+1)$st jump degree but do not have kth jump degree for $k \leq n$.

The Construction

A group $\mathfrak{A}$, a subgroup of the set of rational numbers, satisfying the following conditions:

(C1) $\mathrm{DS}(\mathfrak{A}) \subseteq \{\mathbf{a} : \mathbf{0^{(n)}} \leq \mathbf{a}\}$.

(C2) $\mathrm{DS}(\mathfrak{A})$ has no degree.

(C3) $\mathfrak{A}$ has a first jump degree equal to $\mathbf{0}^{(n+1)}$.

- $\mathfrak{B} = (N; =)$
- $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_n(\mathfrak{B})$.

JIT there exists $\mathfrak{C}$,s.t. $\mathrm{DS}_n(\mathfrak{C}) = \mathrm{DS}(\mathfrak{A})$

- $\mathfrak{C}$ does not have an nth jump degree and hence it has no kth jump degree for $k \leq n$
- But $\mathrm{DS}_{n+1}(\mathfrak{C}) = \mathrm{DS}_1(\mathfrak{A})$ and hence the $(n+1)$st jump degree of $\mathfrak{C}$ is $\mathbf{0}^{(n+1)}$.

Applications

Theorem (Wehner)

There is a family of finite sets, which has no c.e. enumeration, i.e. c.e. universal set, and for each noncomputable set X there is a enumeration computable in X

Theorem (relativized)

Let $B \subseteq$ N. There is a family $\mathcal{F}$ of sets, which has no c.e. in B enumeration, and for each set $X >_T B$ there is a enumeration of the family $\mathcal{F}$, computable in X.

Applications

(Kalimullin)

$$\mathcal{F} = \{\{0\} \oplus B\} \cup \{\{1\} \oplus \overline{B}\} \cup \{\{n+2\} \oplus F \mid F \text{ fin. } F \neq W_n^B\}$$

Proposition

If a universal for $\mathcal{F}$ set U is c.e. in X then $B <_T X$.

- $B \leq_T X$;
- *If $B \equiv_T X$, then we can construct a computable in B function g, s.t. $(\forall n)(W_{g(n)}^B \neq W_n^B)$.*
- *A contradiction with the recursion theorem.*

Applications

$$\mathcal{F} = \{\{0\} \oplus B\} \cup \{\{1\} \oplus \overline{B}\} \cup \{\{n+2\} \oplus F \mid F \text{ fin.} F \neq W_n^B\}$$

Proposition

Let $B <_T X$. There exists a universal set U for the family $\mathcal{F}$, such that $U \leq_T X$.

- U is constructed in stages.
- If $F^s_{\langle n,F,i\rangle} = W^B_{n,s}$, we add a new element (from X) to $F^{s+1}_{\langle n,F,i\rangle}$.
- There is no sets which are not in the family, i.e. $F_{\langle n,F,i\rangle} \neq W_n^B$ since X is not c. e. in B.
- U is computable in X.

Applications

Theorem (Wehner, Slaman)

There exists a structure $\mathfrak{C}$, s.t. $DS(\mathfrak{C}) = \{x \mid x >_T 0\}$.

Theorem

For every n and $b \geq 0^{(n)}$ there exists $\mathfrak{C}$, s.t. $DS_n(\mathfrak{C}) = \{x \mid x >_T b\}$.

- We construct $\mathfrak{A}$, for which $DS(\mathfrak{A}) = \{x \mid x >_T b\}$, using the family $\mathcal{F}$.
- Let $\mathfrak{B} = (N; =)$. Then $b \in DS_n(\mathfrak{B})$, $b \geq 0^{(n)}$.
- $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_n(\mathfrak{B})$.

JIT There is $\mathfrak{C}$, s.t. $\mathrm{DS}_n(\mathfrak{C}) = \mathrm{DS}(\mathfrak{A})$.

Applications

Theorem

For each $n \in N$ and every Turing degree $b \geq 0^{(n)}$ there exists $\mathfrak{C}$, for which $\mathrm{DS}_n(\mathfrak{C}) = \{x \mid x >_T b\}$.

Theorem (Goncharov, Harizanov, Knight, McCoy, Miller, Solomon)

For every n there is a structure $\mathfrak{C}$, such that $\mathrm{DS}(\mathfrak{C}) = \{x \mid x^{(n)} >_T 0^{(n)}\}$, i.e. the degree spectrum contains exactly all non-low$_n$ Turing degrees.

Theorem (Harizanov, R. Miller)

There is a structure $\mathfrak{C}$, such that $\mathrm{DS}(\mathfrak{C}) = \{x \mid x' \geq_T 0''\}$.

Jump Inversion Theorem

- The Jump inversion theorem gives a method to lift some interesting results for degree spectra to the nth jump spectra.
- Questions:
  - Other interesting results for degree spectra

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