Effective coding and decoding structures.

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Borel embedding

Definition (Friedman-Stanley, 1989)

We say that a class $\mathcal{K}$ of structures is *Borel embeddable* in a class of structures $\mathcal{K}'$, and we write $\mathcal{K} \leq_B \mathcal{K}'$, if there is a Borel function $\Phi : \mathcal{K} \to \mathcal{K}'$ such that for $\mathcal{A}, \mathcal{B} \in \mathcal{K}$, $\mathcal{A} \cong \mathcal{B}$ iff $\Phi(\mathcal{A}) \cong \Phi(\mathcal{B})$.

Theorem

The following classes lie on top under $\leq_B$.

1. undirected graphs (Lavrov,1963; Nies, 1996; Marker, 2002)
2. fields of any fixed characteristic (Friedman-Stanley; R. Miller-Poonen-Schoutens-Shlapentokh, 2018)
3. 2-step nilpotent groups (Mal'tsev, 1949; Mekler, 1981)
4. linear orderings (Friedman-Stanley)

Turing computable embeddings

Definition (Calvert-Cummins-Knight-S. Miller, 2004)

We say that a class $\mathcal{K}$ is *Turing computably embedded* in a class $\mathcal{K}'$, and we write $\mathcal{K} \leq_{tc} \mathcal{K}'$, if there is a Turing operator $\Phi : \mathcal{K} \to \mathcal{K}'$ such that for all $\mathcal{A}, \mathcal{B} \in \mathcal{K}$, $\mathcal{A} \cong \mathcal{B}$ iff $\Phi(\mathcal{A}) \cong \Phi(\mathcal{B})$.

A Turing computable embedding represents an effective coding procedure.

Theorem

The following classes lie on top under $\leq_{tc}$.

1. undirected graphs
2. fields of any fixed characteristic
3. 2-step nilpotent groups
4. linear orderings

Medvedev reducibility

A *problem* is a subset of 2^ω or ω^ω.
Problem P is Medvedev reducible to problem Q if there is a Turing operator Φ that takes elements of Q to elements of P.

Definition

We say that $\mathcal{A}$ is *Medvedev reducible* to $\mathcal{B}$, and we write $\mathcal{A} \leq_s \mathcal{B}$, if there is a Turing operator that takes copies of $\mathcal{B}$ to copies of $\mathcal{A}$.

Supposing that $\mathcal{A}$ is coded in $\mathcal{B}$, a Medvedev reduction of $\mathcal{A}$ to $\mathcal{B}$ represents an effective decoding procedure.

Effective interpretability

Definition (Montlbán)

A structure $\mathcal{A} = (A, R_i)$ is *effectively interpreted* in a structure $\mathcal{B}$ if there is a set $D \subseteq \mathcal{B}^{<\omega}$, computable Σ_1-definable over $\emptyset$, and there are relations $\sim$ and R_i^* on D, computable Δ_1-definable over $\emptyset$, such that $(D, R_i^*)/_\sim \cong \mathcal{A}$.

Definition (R. Miller)

A *computable functor* from $\mathcal{B}$ to $\mathcal{A}$ is a pair of Turing operators Φ, Ψ such that Φ takes copies of $\mathcal{B}$ to copies of $\mathcal{A}$ and Ψ takes isomorphisms between copies of $\mathcal{B}$ to isomorphisms between the corresponding copies of $\mathcal{A}$, so as to preserve identity and composition.

Equivalence

The main result gives the equivalence of the two definitions.

Theorem (Harrison-Trainor, Melnikov, R. Miller and Montalbán)

For structures $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}$ is effectively interpreted in $\mathcal{B}$ iff there is a computable functor Φ, Ψ from $\mathcal{B}$ to $\mathcal{A}$.

Corollary

If $\mathcal{A}$ is effectively interpreted in $\mathcal{B}$, then $\mathcal{A} \leq_s \mathcal{B}$.

Coding and Decoding

Proposition (Kalimullin, 2010)

There exist $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \leq_s \mathcal{B}$ but $\mathcal{A}$ is not effectively interpreted in $\mathcal{B}$.

Proposition

If $\mathcal{A}$ is computable, then it is effectively interpreted in all structures $\mathcal{B}$.

Proof.

Let $D = \mathcal{B}^{<\omega}$. Let $\bar{b} \sim \bar{c}$ if $\bar{b}, \bar{c}$ are tuples of the same length. For simplicity, suppose $\mathcal{A} = (\omega, R)$, where R is binary. If $\mathcal{A} \models R(m, n)$, then $R^*(\bar{b}, \bar{c})$ for all $\bar{b}$ of length m and $\bar{c}$ of length n. Thus, $(D, R^*)/_\sim \cong \mathcal{A}$. □

Borel interpretability

Harrison-Trainor, Miller and Montlbán, 2018, defined Borel versions of the notion of effective interpretation and computable functor.

Definition

1. For a Borel interpretation of $\mathcal{A} = (A, R_i)$ in $\mathcal{B}$ the set $D \subseteq \mathcal{B}^{<\omega}$ the relations $\sim$ and R_i^* on D, are definable by formulas of $L_{\omega_1 \omega}$.
2. For a Borel functor from $\mathcal{B}$ to $\mathcal{A}$, the operators Φ and Ψ are Borel.

Their main result gives the equivalence of the two definitions.

Theorem (Harrison-Trainor, Miller and Montlbán)

A structure $\mathcal{A}$ is interpreted in $\mathcal{B}$ using $L_{\omega_1 \omega}$-formulas iff there is a Borel functor Φ, Ψ from $\mathcal{B}$ to $\mathcal{A}$.

Graphs and linear orderings

Graphs and linear orderings both lie on top under Turing computable embeddings.

Graphs also lie on top under effective interpretation.

Question: What about linear orderings under effective interpretation?

And under using $L_{\omega_1\omega}$-formulas?

Interpreting graphs in linear orderings

Proposition

There is a graph G such that for all linear orderings L, $G \not\leq_s L$.

Proof.

Let S be a non-computable set. Let G be a graph such that every copy computes S.
We may take G to be a "daisy" graph", consisting of a center node with a "petal" of length $2n+3$ if $n \in S$ and $2n+4$ if $n \notin S$.
Now, apply:

Proposition (Richter)

For a linear ordering L, the only sets computable in all copies of L are the computable sets.

□

Interpreting a graph in the jump of linear ordering

We are identifying a structure $\mathcal{A}$ with its atomic diagram. We may consider an interpretation of $\mathcal{A}$ in the jump $\mathcal{B}'$ of $\mathcal{B}$. Note that the relations definable in $\mathcal{B}'$ by computable Σ_1 relations are the ones definable in $\mathcal{B}$ by computable Σ_2 relations.

Proposition

There is a graph G such that for all linear orderings L, $G \not\leq_s L'$.

Proof.

Let S be a non-Δ^0_2 set. Let G be a graph such that every copy computes S. Then apply:

Proposition (Knight, 1986)

For a linear ordering L, the only sets computable in all copies of L' (or in the jumps of all copies of L), are the Δ^0_2 sets.

□

Interpreting a graph in the second jump of linear ordering

Proposition

For any set S, there is a linear ordering L such that for all copies of L, the second jump of L computes S.

Proof.

We may take L to be a "shuffle sum" of $n+1$ for $n \in S \oplus S^c$ and ω. □

Proposition

For any graph G, there is a linear ordering L such that $G \leq_s L''$. In fact, G is interpreted in L using computable Σ_3 formulas.

Proof.

Let S be the diagram of a specific copy G_0 of G and let L be a linear order such that $S \leq_s L''$. We have computable functor that takes the second jump of any copy of L to G_0, and takes all isomorphisms between copies of L to the identity isomorphism on G_0. □

Friedman-Stanley embedding of graphs in orderings

Friedman and Stanley determined a Turing computable embedding $L : G \to L(G)$, where $L(G)$ is a sub-ordering of $\mathbb{Q}^{<\omega}$ under the lexicographic ordering.

1. Let $(A_n)_{n\in\omega}$ be an effective partition of $\mathbb{Q}$ into disjoint dense sets.
2. Let $(t_n)_{1\leq n}$ be a list of the atomic types in the language of directed graphs.

Definition

For a graph G, the elements of $L(G)$ are the finite sequences $r_0 q_1 r_1 \dots r_{n-1} q_n r_n k \in \mathbb{Q}^{<\omega}$ such that for $i < n$, $r_i \in A_0$, $r_n \in A_1$, and for some $a_1, \dots, a_n \in G$, satisfying t_m, $q_i \in A_{a_i}$ and $k < m$.

No uniform interpretation of G in $L(G)$

Theorem

There are not $L_{\omega_1\omega}$ formulas that, for all graphs G, interpret G in $L(G)$.

The idea of Proof: We may think of an ordering as a directed graph. It is enough to show the following.

Proposition

1. ω_1^{CK} is not interpreted in $L(\omega_1^{CK})$ using computable infinitary formulas.
2. For all X, ω_1^X is not interpreted in $L(\omega_1^X)$ using X-computable infinitary formulas.

Proof of (1)

The Harrison ordering H has order type $\omega_1^{CK}(1+\eta)$. It has a computable copy.

Let I be the initial segment of H of order type ω_1^{CK}. Thinking of H as a directed graph, we can form the linear ordering $L(H)$. We consider $L(I) \subseteq L(H)$.

Lemma

$L(I)$ is a computable infinitary elementary substructure of $L(H)$.

Proposition (Main)

There do not exist computable infinitary formulas that define an interpretation of H in $L(H)$ and an interpretation of I in $L(I)$.

To prove (1), we suppose that there are computable infinitary formulas interpreting ω_1^{CK} in $L(\omega_1^{CK})$. Using Barwise Compactness theorem, we get essentially H and I with these formulas interpreting H in $L(H)$ and I in $L(I)$.

Proof of the Proposition(Main)

Lemma

1. For any $\bar{b} \in L(I)$, and $c \in L(I)$ there is an automorphism of $L(I)$ taking $\bar{b}$ to a tuple $\bar{b}'$ entirely to the right of c.
2. For any $\bar{b} \in L(I)$, and $c \in L(I)$ there is also an automorphism taking $\bar{b}$ to a tuple $\bar{b}''$ entirely to the left of c.

Lemma

Suppose that we have computable Σ_γ formulas D, $\circledless$ and $\sim$, defining an interpretation of H in $L(H)$ and I in $L(I)$. Then in $D^{L(I)}$ there is a fixed n, and there are n-tuples, all satisfying the same Σ_γ formulas, and representing arbitrarily large ordinals $\alpha < \omega_1^{CK}$.

We arrive at a contradiction by producing tuples $\bar{b}, \bar{b}', \bar{c}$ in $D^{L(I)}$, $\bar{b}$ and $\bar{b}'$ are automorphic, $\bar{b}, \bar{c}$ and $\bar{c}, \bar{b}'$ satisfy the same Σ_γ formulas, and the ordinal represented by $\bar{b}$ and $\bar{b}'$ is smaller than that represented by $\bar{c}$. Then $\bar{b}, \bar{c}$ should satisfy $\circledless$, while $\bar{c}, \bar{b}'$ should not.

Conjecture

We believe that Friedman and Stanley did the best that could be done.

Conjecture. For any Turing computable embedding Θ of graphs in orderings, there do not exist $L_{\omega_1\omega}$ formulas that, for all graphs G, define an interpretation of G in $\Theta(G)$.

M. Harrison-Trainor and A. Montlbán came to a similar result very recently by a totally different construction. Their result is that there exist structures which cannot be computably recovered from their tree of tuples. They proved :

1. There is a structure $\mathcal{A}$ with no computable copy such that $T(\mathcal{A})$ has a computable copy.
2. For each computable ordinal α there is a structure $\mathcal{A}$ such that the Friedman and Stanley Borel interpretation $L(\mathcal{A})$ is computable but $\mathcal{A}$ has no Δ^0_α copy.

Mal'tsev embedding of fields in groups

If F is a field, we denote by $H(F)$ the multiplicative group of matrices of kind

$$h(a,b,c) = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

where $a, b, c \in F$. Note that $h(0,0,0) = 1$.
Groups of kind $H(F)$ are known as *Heisenberg groups*.

Theorem (Mal'tsev)

There is a copy of F defined in $H(F)$ with parameters.

Natural isomorphisms

For a non-commuting pair (u, v), where $u = h(u_1, u_2, u_3)$ and $v = h(v_1, v_2, v_3)$, let

$$\Delta_{(u,v)} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

Theorem

The function f that takes $x \in F$ to $h(0, 0, \Delta_{(u,v)} \cdot_F x)$ is an isomorphism.

Morozov's isomorphism

Lemma (Morozov)

Let (u, v) and (u', v') be non-commuting pairs in $G = H(F)$. Let $F_{(u,v)}$ and $F_{(u',v')}$ be the copies of F defined in G with these pairs of parameters. There is an isomorphism g from $F_{(u,v)}$ onto $F_{(u',v')}$ defined in G by an existential formula with parameters u, v, u', v'.

Note that $\Delta_{(u,v)}$ is the multiplicative identity in $F_{(u,v)}$.
Let $g(x) = y \iff x = \Delta_{(u,v)} \cdot_{(u',v')} y$.

Computable functor

Theorem

There is a computable functor Φ, Ψ from $H(F)$ to F.

- For $G \cong H(F)$, $\Phi(G)$ is the copy of F obtained by taking the first non-commuting pair (u, v) in G and forming $(D; +; \cdot_{(u,v)})$.
- Take (G_1, f, G_2), where $G_i = H(F)$, and $G_1 \cong_f G_2$. Let (u, v), (u', v') be the first non-commuting pairs in G_1, G_2, respectively.
  - Let h be the isomorphism from $F_{(f(u),f(v))}$ onto $F_{(u',v')}$ defined in G_2 with parameters $f(u), f(v), u', v'$.
  - Let f' be the restriction of f to the center of G_1.
  - Then $\Psi(G_1, f, G_2) = h \circ f'$.

Finitely existential interpretation and generalizing

Corollary (Alvir,Calvert,Harizanov,Knight,Miller,Morozov,S,Weisshaar)

F is effectively interpreted in $H(F)$.

$(u, v, x) \sim (u', v', x')$ holds if Morozov's isomorphism from $F_{(u,v)}$ to $F_{(u',v')}$ takes x to x'.

Proposition

Suppose $\mathcal{A}$ has a copy $\mathcal{A}_{\bar{b}}$ defined in $(\mathcal{B}, \bar{b})$, using computable Σ_1 formulas, where the orbit of $\bar{b}$ is defined by a computable Σ_1 formula $\varphi(\bar{x})$. Suppose also that there is a computable Σ_1 formula $\psi(\bar{b}, \bar{b}', u, v)$ that, for any tuples $\bar{b}$, $\bar{b}'$ satisfying $\varphi(\bar{x})$, defines a specific isomorphism $f_{\bar{b},\bar{b}'}$ from $\mathcal{A}_{\bar{b}}$ onto $\mathcal{A}_{\bar{b}'}$. We suppose that for each $\bar{b}$ satisfying φ, $f_{\bar{b},\bar{b}}$ is the identity isomorphism, and for any $\bar{b}$, $\bar{b}'$, and $\bar{b}''$ satisfying φ, $f_{\bar{b}',\bar{b}''} \circ f_{\bar{b},\bar{b}'} = f_{\bar{b},\bar{b}''}$. Then there is an effective interpretation of $\mathcal{A}$ in $\mathcal{B}$.

W. Calvert, D. Cummins, J. F. Knight, and S. Miller
Comparing classes of finite structures
Algebra and Logic, vo. 43(2004), pp. 374-392.

H. Friedman and L. Stanley
A Borel reducibility theory for classes of countable structures
JSL, vol. 54(1989), pp. 894-914.

J. Knight, A. Soskova, and S. Vatev
Coding in graphs and linear orderings
https://arxiv.org/abs/1903.06948

M. Harrison-Trainor, A. Melnikov, R. Miller, and A. Montalbán
Computable functors and effective interpretability,
JSL, vol. 82(2017), pp. 77-97.

M. Harrison-Trainor, R. Miller, and A. Montalbán
Borel functors and infinitary interpretations,
JSL, vol. 83(2018), no. 4, pp. 1434-1456.

THANK YOU