Cototal enumeration degrees and the skip operator

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joint work with Uri Andrews, Hristo Ganchev, Rutger Kuyper, Steffen Lempp, Joseph Miller and Mariya Soskova

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The enumeration degrees

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$$A = W(B) = \{x \mid \exists D(\langle x, D\rangle \in W \ \& \ D \subseteq B)\}.$$

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- The least upper bound: $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- The enumeration jump: $d_e(A)' = d_e(K_A \oplus \overline{K_A})$, where $K_A = \{\langle e, x\rangle \mid x \in W_e(A)\}$.

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Theorem (Selman)

The set A is enumeration reducible to the set B if and only if $\mathcal{E}(B) \subseteq \mathcal{E}(A)$.

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Proposition

$$A \leq_T B \Leftrightarrow A \oplus \overline{A} \leq_e B \oplus \overline{B}.$$

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A set A is *total* if its positive membership information already suffices to determine its negative membership information, i.e. if $\overline{A} \leq_e A$, or equivalently $A \equiv_e A \oplus \overline{A}$. An enumeration degree is *total* if it contains a total set.

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Within the enumeration degrees, the total degrees are an embedded copy of the Turing degrees $\mathcal{D}_T$ via $\iota : A \to A \oplus \overline{A}$. The embedding ι preserves the order, the least upper bound and the jump operation.

Motivation from symbolic dynamics by Emmanuel Jeandel

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3. The Turing degrees that compute elements of X are exactly the degrees that contain enumerations of L_X.

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Total and cototal

Definition

A set A is *cototal* if $A \leq_e \overline{A}$. A degree $\mathbf{a}$ is cototal if it contains a cotototal set.

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The cototal enumeration degrees form a proper substructure of $\mathcal{D}_e$ closed under least upper bound and the enumeration jump operator.

- The name "*total*" is coming of the following fact: given a total function f, the set $G(f) = \{\langle n, f(n)\rangle \mid n \in \omega\}$ is a total set.
- Equivalently, given a total function f, the graph-complement $\overline{G(f)}$ is cototal.
- If an enumeration degree contains a set of the form $\overline{G(f)}$, then we call it *graph-cototal*.
- So, every total enumeration degree is graph-cototal, and every graph-cototal degree is cototal.

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It is clear that every cototal degree is weakly cototal, since if $A \leq_e \overline{A}$, then $\overline{A}$ is a total set.

total $\Rightarrow$ graph-cototal $\Rightarrow$ cototal $\Rightarrow$ weakly cototal.

Σ_2^0 enumeration degrees

Proposition

Σ_2^0 enumeration degrees are cototal.

Let A be Σ_2^0. Consider the set $K_A = \bigoplus_{e<\omega} \Gamma_e(A)$. Then $A \equiv_e K_A$ and

$$\overline{K_A} = \bigoplus_{e<\omega} \overline{\Gamma_e(A)} \geq_e \overline{K} \geq_e A \equiv_e K_A.$$

Proposition

Σ_2^0 e-degrees are graph-cototal.

Corollary

Graph-cototal does not imply total.

Unique correct axiom

Theorem

An e-degree $\mathbf{a}$ *is graph-cototal if and only if* $\mathbf{a}$ *contains a cototal set* A*, such that for some enumeration operator* Γ*, we have that* $A = \Gamma(\overline{A})$ *and for every* $n \in A$ *there is a unique axiom* $\langle n, D\rangle \in \Gamma$ *such that* $D \subseteq A$*.*

Goal:
Cototal does not imply *graph-cototal*.

Maximal independent sets

Definition

Let $G = (\mathbb{N}, E)$ be a graph and $S \subseteq \mathbb{N}$.

1. S is an *independent set* for G if $i \neq j$ are in S then $(i, j) \notin E$.
2. An independent set is *maximal* if it has no proper independent superset, i.e. for every element $i \notin S$ there is a $j \in S$ such that $(i, j) \in E$.

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If S is a maximal independent set for G, then S can enumerate its complement: $i \in \overline{S}$ iff there is a $j \neq i$ such that $(i,j) \in E$ and $j \in S$.

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Theorem

Every cototal degree contains the complement of maximal independent set for $\omega^{<\omega}$.

Cototal degree not graph-cototal

Theorem

There is a cototal dregee which is not graph-cototal.

We build the complement of a maximal independent set for the graph $G = (\omega^{<\omega}; E)$.

Our other condition on the set is that it is not enumeration equivalent to a graph-cototal set.

Infinite injury $\emptyset'''$ relative to $\emptyset'$.

Maximal antichains

Proposition

If C is a maximal antichain on $\omega^{<\omega}$, then $\overline{C} \leq_e C$, i.e. $\overline{C}$ is cototal.

To determine if a string $\sigma \in \omega^{<\omega}$ is in $\overline{C}$, we wait for some element comparable but not equal to σ to enter C. Since C is an antichain, we only identify elements of $\overline{C}$ in this way. And by maximality, if $\sigma \in \overline{C}$ then something comparable but not equal to σ must eventually enter C.

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Theorem (McCarthy)

If A is cototal, then $A \equiv_e L_X$ for some minimal subshift X.

Uniformly e-pointed trees

Definition

A tree $T \subseteq 2^{<\omega}$ is *e-pointed* if it has no dead ends and every infinite path $f \in [T]$ enumerates T.

Theorem (Montalbán)

A degree spectrum is never the Turing-upward closure of an F_σ set of reals in ω^ω, unless it is an enumeration-cone.

Theorem (McCarthy)

An e-degree is cototal if and only if it contains a (uniformly) e-pointed tree.

Joins of nontrivial $\mathcal{K}$-pairs

Definition

A $\mathcal{K}$-pair is a pair of sets $\{A, B\}$ for which there is a c.e. set W such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

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Proposition (Kalimullin)

Let $\{A, B\}$ be a $\mathcal{K}$-pair. If A and B are not c.e. then:

1. $A \leq_e \overline{B}$ *and* $\overline{A} \leq_e \emptyset' \oplus B$.
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Proof: $A \oplus B \leq_e \overline{B} \oplus \overline{A} \equiv_e \overline{A \oplus B}$.

Computable metric spaces

J. Miller introduced the continuous degrees $\mathcal{D}_r$ to compare the complexity of points in computable metric spaces.

Definition

A *computable metric space* is a metric space $\mathcal{M}$ together with a countable dense sequence $Q^{\mathcal{M}} = \{q_n^{\mathcal{M}}\}_{n\in\omega}$ on which the metric is computable (as a function $\omega^2 \to \mathcal{R}$).

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Examples:

- $\mathbb{R}$ with $Q^{\mathbb{R}} = \mathbb{Q}$.
- The *Hilbert cube* $[0,1]^\omega$ with the metric

$$d(\alpha, \beta) = \sum_{n\in\omega} |\alpha(n) - \beta(n)|/2^n$$

and $Q^{[0,1]^\omega}$ the sequences of rationals in $[0,1]$ with finite support.

Continuous degrees

Definition

A *name* of a point $x \in \mathcal{M}$ is a function λ that maps positive rationals $\varepsilon > 0$ to natural numbers so that $d_{\mathcal{M}}(x, q^{\mathcal{M}}_{\lambda(\varepsilon)}) < \varepsilon$.

Definition (J. Miller)

For two points x and y in a computable metric space we say that $x \leq_r y$ if every name of y computes a name of x uniformly. This reducibility induces the *continuous degrees*.

Continuous degrees

The continuous degrees embed into $\mathcal{D}_e$. In fact, $D_T \subset \mathcal{D}_r \subset \mathcal{D}_e$.

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For $\alpha \in [0,1]^\omega$, let

$$C_\alpha = \bigoplus_{i\in\omega} \{q \in \mathbb{Q} \mid q < \alpha(i)\} \oplus \{q \in \mathbb{Q} \mid q > \alpha(i)\}.$$

Enumerating C_α is as hard as computing a name for α. So $\alpha \mapsto C_\alpha$ induces an embedding of the continuous degrees into the enumeration degrees.

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Proposition

Continuous degrees are cototal.

$\overline{C_\alpha} \equiv_e B_\alpha = \bigoplus_{i \in \omega} \{q \in \mathbb{Q} \mid q \leq \alpha(i)\} \oplus \{q \in \mathbb{Q} \mid q \geq \alpha(i)\}$.

Points in arbitrary topological spaces

Kihara and Pauly extend Miller's idea to points in arbitrary represented topological spaces.
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Definition

A represented countably based space is a pair $(\mathcal{X}, \beta)$ of a second-countable space $\mathcal{X}$ and an enumeration $\beta = \{\beta_e\}_{e \in \omega}$ of a countable open subbases of $\mathcal{X}$.

Points in arbitrary topological spaces

One can identify a point x in a represented space $(\mathcal{X}, \beta)$ with the coded neighborhood filter $Nbase_\beta(x) = \{e \in \omega \mid x \in \beta_e\}$. Any enumeration of $Nbase_\beta(x)$ is called a β-name of x.

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The $\mathcal{X}$-degrees are $\mathcal{D}_\mathcal{X} = \{deg_e(Nbase_\mathcal{X}(x)) \mid x \in \mathcal{X}\}$.

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Theorem (Kihara and Miller)

- *An e-degree is cototal if and only if it is an $\mathcal{X}$-degree of a computable G_δ space $\mathcal{X}$.*
- *There exists a decidable, computable G_δ space $\mathcal{X}$ such that the $\mathcal{X}$-degrees are exactly the cototal e-degrees.*

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The skip of A is the set $A^\Diamond = \overline{K_A}$. The skip of a degree $\mathbf{a}$ is $\mathbf{a}^\Diamond = d_e(A^\Diamond)$.

The skip operator

Recall that $\overline{K_A} = \bigoplus_e \overline{\Gamma_e(A)}$.

Proposition

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Proposition

A degree $\mathbf{a}$ is cototal if and only if $\mathbf{a} \leq \mathbf{a}^\Diamond$ if and only if $\mathbf{a}^\Diamond = \mathbf{a}'$.

$\Rightarrow A \leq_e \overline{A} \leq_e A^\Diamond$
$\Leftarrow K_A \equiv_e A \leq_e A^\Diamond = \overline{K_A}$.
Recall that $A' = K_A \oplus \overline{K_A}$.

Skip inversion

Theorem

Let $S \geq_e \emptyset'$. There is a set A such that $A^\Diamond \equiv_e S$.

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We first build a table $\hat{A}$ with one empty box in each column as a set c.e. in $\emptyset'$.

The set of empty boxes will be computable from $\emptyset'$.

Then $A = \hat{A} \cup \left\{ \langle n, s \rangle \mid n \in \overline{S} \right\}$.

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Note! $\overline{S} \leq_e A \oplus \emptyset'$. So if we start out with an S that is not total set but belongs to a total degree then the degree of A is not cototal. But $A \equiv_e K_A$ and $\overline{K_A} = A^\diamond \equiv_e S$ has a total degree and so A is in a weakly cototal degree.

Corollary

Weakly cototal does not imply cototal.

Π^0_2-sets and above

Theorem

Let $n \geq 2$. For any Π^0_n set $S \geq \emptyset'$, there is a Σ^0_n set A such that $A^{\Diamond} \equiv_e S$. Furthermore, for any Σ^0_n set $S \geq \emptyset'$, there is a Π^0_n set A such that $A^{\Diamond} \equiv_e S$.

We know that every Σ^0_2 set has a graph cototal degree.

Proposition

- *There are Π^0_2-sets that do not even have cototal enumeration degree.*
- *But every Π^0_2-set has weakly cototal degree.*

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- *There are Δ^0_3-sets that are not even weakly cototal degree.*

Skip iteration

We can define the *iterated skip operator* of an enumeration degree $\mathbf{a}$ by:

- $\mathbf{a}^{\langle 0 \rangle} = \mathbf{a}$
- $\mathbf{a}^{\langle n+1 \rangle} = (\mathbf{a}^{\langle n \rangle})^{\Diamond}$.

This iterated skip can exhibit exotic behavior:

Theorem

For all enumeration degrees, $\mathbf{a} \leq \mathbf{a}^{\Diamond\Diamond}$ *and* $\mathbf{a}^{\Diamond} \geq \mathbf{0}'$*, but not always* $\mathbf{a} \leq \mathbf{a}^{\Diamond}$.

$\overline{A} \leq_1 A^{\Diamond} \Rightarrow A \leq_1 \overline{A^{\Diamond}} \leq_1 A^{\Diamond\Diamond}$.

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$\overline{K_{\overline{K_A}}} \subseteq \overline{K_{\overline{K_B}}}$.

Any such enumeration degree lies above all total hyperarithmetic enumeration degrees.

Iterating the skip

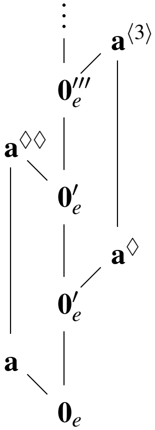

Figure: Iterated skips of a degree

Zig-zag

If $\mathbf{a}^{\langle n\rangle}$ is not cototal for every n:

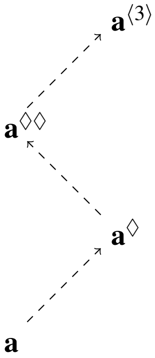

Figure: Iterated skips of a degree: the zig-zag

Generic sets

Definition

Let G and X be sets of natural numbers. G is 1- *generic relative to* $\langle X \rangle$ if and only if for every $W \subseteq 2^{<\omega}$ such that $W \leq_e X$:

$$(\exists \sigma \preceq G)[\sigma \in W \vee (\forall \tau \succeq \sigma)[\tau \notin W]].$$

Proposition

If G is 1-generic relative to $\langle X \rangle$ then:

- *$\overline{G}$ is 1-generic relative to $\langle X \rangle$.*
- $(G \oplus X)^{\Diamond} = \overline{G} \oplus X'$.

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If G is arithmetically generic, i.e. G is 1-generic relative to $\langle\emptyset^{(n)}\rangle$, for every n, then the skips of G and $\overline{G}$ form a double helix.

- If n is odd then $G^{\langle n\rangle} \equiv_e \overline{G} \oplus \emptyset^{(n)}$ and $(\overline{G})^{\langle n\rangle} \equiv_e G \oplus \emptyset^{(n)}$.
- If n is even then $G^{\langle n\rangle} \equiv_e G \oplus \emptyset^{(n)}$ and $(\overline{G})^{\langle n\rangle} \equiv_e \overline{G} \oplus \emptyset^{(n)}$.

Double zig-zag

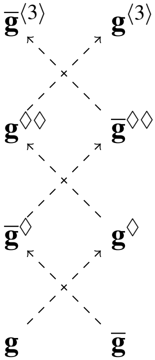

Figure: Iterated skips of a degrees of an arithmetically generic set and its complement: double zig-zag

Skips of nontrivial $\mathcal{K}$-pairs

Proposition

If $\{A,B\}$ is a non-trivial $\mathcal{K}$-pair then $A^\Diamond \equiv_e B \oplus \emptyset'$.
If $\{A,B\}$ is a non-trivial $\mathcal{K}$-pair relative to $\langle X \rangle$ then $(A \oplus X)^\Diamond \leq_e B \oplus X^\Diamond$.
The oracle X is of cototal degree iff we have equivalence above for every nontrivial $\mathcal{K}$-pair relative to $\langle X \rangle$.

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If $\{A, B\}$ is a non-trivial $\mathcal{K}$-pair relative to $\emptyset^{(n)}$ for every n then the iterate skips of A and B form a double zig-zag.

- if n is odd then $A^{\langle n \rangle} \equiv_e B \oplus \emptyset^{(n)}$ and $B^{\langle n \rangle} \equiv_e A \oplus \emptyset^{(n)}$, and
- if n is even then $A^{\langle n \rangle} \equiv_e A \oplus \emptyset^{(n)}$ and $B^{\langle n \rangle} \equiv_e B \oplus \emptyset^{(n)}$.

Skip iterations

Theorem (Ganchev, Sorbi)

For every enumeration degree $\mathbf{x} > \mathbf{0}_e$*, there is a degree* $\mathbf{a} \leq \mathbf{x}$ *such that* $\mathbf{a}$ *is half of a nontrivial* $\mathcal{K}$*-pair and such that* $\mathbf{a}' = \mathbf{x}'$*.*

$$A' \equiv_e A \oplus A^{\Diamond} \equiv_e A \oplus B \oplus \emptyset' \equiv_e B \oplus B^{\Diamond} \equiv_e B'.$$

Proposition

- *If* $\mathbf{x}$ *is high* ($\mathbf{x}' = \mathbf{0}''$)*:*

$$\mathbf{b}^{\Diamond} < \mathbf{b}' = \mathbf{b}^{\Diamond\Diamond} < \mathbf{b}'' = \mathbf{b}^{\langle 3 \rangle} < \cdots < \mathbf{b}^{(n)} = \mathbf{b}^{\langle n+1 \rangle} < \ldots$$

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- *If* $\mathbf{x}$ *is intermediate:*

$$\mathbf{b}^{\Diamond} < \mathbf{b}' < \mathbf{b}^{\Diamond\Diamond} < \mathbf{b}'' < \mathbf{b}^{\langle 3 \rangle} < \cdots < \mathbf{b}^{(n)} < \mathbf{b}^{\langle n+1 \rangle} < \ldots$$

The cototal degrees are dense

Corollary

The relation

$$SK = \left\{ (\mathbf{a}, \mathbf{a}^{\Diamond}) \mid \mathbf{a} \text{ is half of a nontrivial } \mathcal{K}\text{-pair} \right\}$$

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Question: Is the skip operator definable in $\mathcal{D}_e$?

Good e-degrees

Definition (Lachlan, Shore)

A uniformly computable sequence of finite sets $\{A_s\}_{s<\omega}$ is *a good approximation* to a set A if:

$G1 (\forall n)(\exists s)(A \upharpoonright n \subseteq A_s \subseteq A)$

$G2 (\forall n)(\exists s)(\forall t > s)(A_t \subseteq A \Rightarrow A \upharpoonright n \subseteq A_t)$.

An enumeration degree is *good* if it contains a set with a good approximation.

- Good e-degrees cannot be tops of empty intervals.
- Total enumeration degrees and enumeration degrees of n-c.e.a. sets are good.

The cototal degrees are dense

Theorem (Harris; Miller, M. Soskova)

The good enumeration degrees are exactly the cototal enumeration degrees.

If A has a good approximation then

$$A \leq_e \{\langle x, s\rangle \mid (\forall t > s)(A_t \subseteq A \Rightarrow x \in A)\} \leq_e A^{\Diamond}.$$

Every uniformly e-pointed tree has a good approximation.

Theorem (Miller, M. Soskova)

The cototal enumeration degrees are dense.

If $V <_e U$ are cototal and U has a good approximation they build Θ such that $\Theta(U)$ is the complement of a maximal independent set and

$$V <_e \Theta(U) \oplus V <_e U.$$

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- JOSEPH MILLER AND MARIYA SOSKOVA, *Density of the cototal enumeration degrees*, submitted.