# A note on $\omega$-jump inversion of degree spectra of structures

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# The problem

*In this note I. Soskov provides a negative solution to the $\omega$-jump inversion problem for degree spectra of structures.*

# Degree spectra

**Definition.** Let $\mathfrak{A}$ be a countable structure. The *spectrum* of $\mathfrak{A}$ is the set of Turing degrees

$Sp(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a}$ computes the diagram of an isomorphic copy of $\mathfrak{A}\}$.

*For $\alpha < \omega_1^{CK}$ the $\alpha$-th jump spectrum of $\mathfrak{A}$ is the set* $Sp_\alpha(\mathfrak{A}) = \{\mathbf{a}^{(\alpha)} \mid \mathbf{a} \in Sp(\mathfrak{A})\}$.

# The jump inversion theorem

*Let $\alpha < \omega_1^{CK}$ and $\mathfrak{A}$ be a countable structure such that all elements of $Sp(\mathfrak{A})$ are above $\mathbf{0}^{(\alpha)}$.*

*Does there exist a structure $\mathfrak{M}$ such that $Sp_\alpha(\mathfrak{M}) = Sp(\mathfrak{A})$?*

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# The jump inversion theorem - the positive solutions

2005 ***S. Goncharov, V. Harizanov, J. Knight, C. McCoy, R. Miller, R. Solomon***, *Enumerations in computable structure theory, Annals of Pure and Applied Logic, **136**, 219-246.*

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# The jump inversion theorem - a negative solution

**Theorem.** *[Soskov] There is a structure* $\mathfrak{A}$ *with* $Sp(\mathfrak{A}) \subseteq \{\mathbf{b} \mid \mathbf{0}^{(\omega)} \leq \mathbf{b}\}$ *for which there is no structure* $\mathfrak{M}$ *with* $Sp_\omega(\mathfrak{M}) = Sp(\mathfrak{A})$.

# Enumeration reducibility

**Definition.** Given two sets of natural numbers $X$ and $Y$, say that $X$ is enumeration reducible to $Y$ ($X \leq_e Y$) if for some $e$, $X = W_e(Y)$, i.e.

$$(\forall x)(x \in X \iff (\exists v)(\langle x, v\rangle \in W_e \wedge D_v \subseteq Y)).$$

**Definition.** Let $X \equiv_e Y$ if $X \leq_e Y$ and $Y \leq_e X$.
The enumeration degree of $X$ is $d_e(X) = \{Y \subseteq \mathbb{N} \mid X \equiv_e Y\}$.
By $D_e$ we shall denote the set of all enumeration degrees.

# The enumeration reducibility

**Definition.** Given a set $X \subseteq \mathbb{N}$, denote by $X^+ = X \oplus (\mathbb{N} \setminus X)$. A set $X$ is called *total* iff $X \equiv_e X^+$.

**Theorem.** *For any sets $X$ and $Y$:*

(i) *$X$ is c.e. in $Y$ iff $X \leq_e Y^+$.*

(ii) *$X \leq_T Y$ iff $X^+ \leq_e Y^+$.*

**Theorem.** *[Selman] $X \leq_e Y$ iff for all total $Z$*

$$(Y \leq_e Z \Rightarrow X \leq_e Z).$$

# The enumeration jump

**Definition.** For any $X \subseteq \mathbb{N}$ set $J_e(X) = \{\langle e, x\rangle \mid x \in W_e(X)\}$. The *enumeration jump* $X'$ *of* $X$ is the set $J_e(X)^+$.

- $J_T(X)^+ \equiv_e (X^+)'$.
- $X' \leq_T (X^+)' \leq_T J_T(X)$.
- *for total* $X$, $X' \equiv_T J_T(X)$.
- *The enumeration jump of an e-degree is always a total degree and agrees with the Turing jump under the standard embedding* $\iota : D_T \to D_e$ *by* $\iota(d_T(X)) = d_e(X^+)$.

# Enumeration reducibility of sequences of sets

**Definition.** Let $\mathcal{X} = \{X_n\}_{n<\omega}$ and $\mathcal{Y} = \{Y_n\}_{n<\omega}$ be sequences of sets of natural numbers. Then $\mathcal{X}$ *is enumeration reducible to* $\mathcal{Y}$ ($\mathcal{X} \leq_e \mathcal{Y}$) if for all $n$, $X_n \leq_e Y_n$ uniformly in $n$. In other words, if there exists a computable function $\mu$ such that for all $n$, $X_n = W_{\mu(n)}(Y_n)$.

**Definition.** Let $\mathcal{X} = \{X_n\}_{n<\omega}$ be a sequence of sets of natural numbers. The *jump sequence* $\mathcal{P}(\mathcal{X}) = \{\mathcal{P}_n(\mathcal{X})\}_{n<\omega}$ of $\mathcal{X}$ is defined by induction:

(i) $\mathcal{P}_0(\mathcal{X}) = X_0$;

(ii) $\mathcal{P}_{n+1}(\mathcal{X}) = \mathcal{P}_n(\mathcal{X})' \oplus X_{n+1}$.

# Enumeration reducibility of sequences of sets

*By $\mathcal{P}_\omega(\mathcal{X})$ we shall denote the set $\bigoplus_n \mathcal{P}_n(\mathcal{X})$.*
*Clearly $\mathcal{X} \leq_e \mathcal{P}(\mathcal{X})$ and hence $\bigoplus_n X_n \leq_e \mathcal{P}_\omega(\mathcal{X})$.*

**Proposition.** *For all sequences $\mathcal{X}$ of sets of natural numbers the set $\mathcal{P}_\omega(\mathcal{X})$ is total.*

**Proposition.** *Let $\mathcal{X} = \{X_n\}_{n<\omega}$ be a sequence of sets of natural numbers, $M \subseteq \mathbb{N}$ and $\mathcal{X} \leq_e \{M^{(n)}\}_{n<\omega}$. Then $\mathcal{P}(\mathcal{X}) \leq_e \{M^{(n)}\}_{n<\omega}$.*

# Co-spectra of structures

**Definition.** Let $\mathfrak{M}$ be a countable structure and $\alpha < \omega_1^{CK}$. The $\alpha$-th co-spectrum of $\mathfrak{M}$ is the set

$$CoSp_\alpha(\mathfrak{M}) = \{\mathbf{a} \mid \mathbf{a} \in D_e \wedge (\forall \mathbf{b} \in Sp_\alpha(\mathfrak{M}))(\mathbf{a} \leq_e \mathbf{b})\}.$$

**Definition.** Let $\alpha < \omega_1^{CK}$. A subset $R$ of $\mathbb{N}$ is $\Sigma^c_\alpha$ definable in $\mathfrak{M}$ if there exist a computable function $\gamma$ taking as values codes of computable $\Sigma_\alpha$ infinitary formulas $F_{\gamma(x)}$ and finitely many parameters $t_1, \dots, t_m$ of $|\mathfrak{M}|$ such that

$$x \in R \iff \mathfrak{M} \models F_{\gamma(x)}(t_1, \dots, t_m).$$

**Theorem.**[Ash,Knight,Mannase,Slaman] *Let $\alpha < \omega_1^{CK}$. Then*

1. *If $\alpha < \omega$ then* **a** $\in CoSp_\alpha(\mathfrak{M})$ *if and only if all elements of* **a** *are $\Sigma^c_{\alpha+1}$ definable in $\mathfrak{M}$.*
2. *If $\omega \leq \alpha$ then* **a** $\in CoSp_\alpha(\mathfrak{M})$ *if and only if all elements of* **a** *are $\Sigma^c_\alpha$ definable in $\mathfrak{M}$.*

# A property of the $\omega$ co-spectra

**Theorem.** *Let* $\mathfrak{M}$ *be a countable structure and* $\mathbf{a} \in CoSp_\omega(\mathfrak{M})$*. Then there exists a total enumeration degree* $\mathbf{b}$ *such that* $\mathbf{a} \leq_e \mathbf{b}$ *and* $\mathbf{b} \in CoSp_\omega(\mathfrak{M})$*,*

**Proof.**

Fix an element $R$ of $\mathbf{a} \in CoSp_\omega(\mathfrak{M})$.
$R$ is $\Sigma^c_\omega$ definable in $\mathfrak{M}$ and hence there exists a computable function $\gamma$ and parameters $t_1, \dots, t_m$ of $|\mathfrak{M}|$ such that

$$x \in R \iff \mathfrak{M} \models F_{\gamma(x)}(t_1, \dots, t_m).$$

$F_{\gamma(x)}$ is a c.e. disjunction of computable $\Sigma_{n+1}$ infinitary formulae. Hence there exists a computable function $\delta(n, x)$ such that for all $n$ and $x$, $\delta(n, x)$ yields a code of some computable $\Sigma_{n+1}$ infinitary formula $F_{\delta(n,x)}$ and

$$x \in R \iff (\exists n)(\mathfrak{M} \models F_{\delta(n,x)}(t_1, \dots, t_m)).$$

□

# A property of the $\omega$ co-spectra

**Proof.**

For each $n \in \mathbb{N}$ denote by

$$R_n = \{x \mid x \in \mathbb{N} \wedge \mathfrak{M} \models F_{\delta(n,x)}(t_1, \dots, t_m)\}.$$

Let $B$ be the diagram of some isomorphic copy $\mathfrak{B}$ of $\mathfrak{M}$ on the natural numbers and let $\kappa$ be an isomorphism from $\mathfrak{M}$ to $\mathfrak{B}$ and $x_1 = \kappa(t_1), \dots, x_m = \kappa(t_m)$. Then

$$x \in R_n \iff \mathfrak{B} \models F_{\delta(n,x)}(x_1, \dots, x_m).$$

Hence

$$\mathcal{P}(\{R_n\}_{n<\omega}) \leq_e \{B^{(n)}\}_{n<\omega} \text{ uniformly in } n.$$

Thus

$$\mathcal{P}_\omega(\{R_n\}_{n<\omega}) \leq_e B^{(\omega)}.$$

□

# A property of the $\omega$ co-spectra

**Proof.**

Set $\mathbf{b} = d_e(\mathcal{P}_\omega(\{R_n\}_{n<\omega}))$.

- $\mathbf{b} \in CoSp_\omega(\mathfrak{M})$;
- $\mathbf{b}$ is a total degree;
- $\mathbf{a} \leq_e \mathbf{b}$.

□

# A negative solution for the $\omega$-jump inversion problem

- *Let $Y$ be a set which is quasi-minimal above $\emptyset^{(\omega)}$, i.e. $\emptyset^{(\omega)} <_e Y$ and if $X$ is a total set and $X \leq_e Y$ then $X \leq_e \emptyset^{(\omega)}$, e.g. $Y = \emptyset^{(\omega)} \oplus G$, where $G$ is one-generic relatively $\emptyset^{(\omega)}$.*
- *$d_e(Y)$ does not contain any total set.*
- *Let $CoSp(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \leq_e d_e(Y)\}$. Then $Sp(\mathfrak{A}) \subseteq \{\mathbf{b} \mid \mathbf{0}^{(\omega)} \leq_T \mathbf{b}\}$.*
- *Assume that there exists a countable structure $\mathfrak{M}$ such that $Sp_\omega(\mathfrak{M}) = Sp(\mathfrak{A})$. Then $CoSp_\omega(\mathfrak{M}) = CoSp(\mathfrak{A})$.*
- *Hence there exists a total degree $\mathbf{b}$ in $CoSp(\mathfrak{A})$ such that $d_e(Y) \leq \mathbf{b} \leq d_e(Y)$.*
  *A contradiction.*

**Theorem.** *If $Y$ is quasi-minimal above $\emptyset^{(\omega)}$ and $\mathfrak{A}$ is a structure with $CoSp(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \leq_e d_e(Y)\}$ then there is no structure $\mathfrak{M}$ with $Sp_\omega(\mathfrak{M}) = Sp(\mathfrak{A})$.*

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**Theorem.** *If $Y$ is quasi-minimal above $\emptyset^{(\omega)}$ and $\mathfrak{A}$ is a structure with $CoSp(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \leq_e d_e(Y)\}$ then there is no structure $\mathfrak{M}$ with $Sp_\omega(\mathfrak{M}) = Sp(\mathfrak{A})$.*

# A structure $\mathfrak{A}$ with $CoSp(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \leq_e d_e(Y)\}$

*Consider a non-trivial group $G \subseteq Q$.*
*For every $a \neq 0$ element of $G$ and every prime number $p$ set*

$$h_p(a) = \begin{cases} k & \text{if } k \text{ is the greatest number such that } p^k | a \text{ in } G, \\ \infty & \text{if } p^k | a \text{ in } G \text{ for all } k. \end{cases}$$

*Let $p_0, p_1, \dots$ be the standard enumeration of the prime numbers and set*

$$S_a(G) = \{\langle i, j\rangle : j \leq h_{p_i}(a)\}.$$

*If $a$ and $b$ are non-zero elements of $G$, then $S_a(G) \equiv_e S_b(G)$.*
*Denote by $\mathbf{d}_G = d_e(S_a(G))$, for some non-zero element $a$ of $G$.*

## A structure $\mathfrak{A}$ with $CoSp(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \leq_e d_e(Y)\}$

**Proposition.** *[Coles, Downey and Slaman]*
$Sp(G) = \{\mathbf{b} \mid \mathbf{b}$ *is total* & $\mathbf{d}_G \leq_e \mathbf{b}\}$.

**Corollary.** $CoSp(G) = \{\mathbf{a} \mid \mathbf{a} \leq_e \mathbf{d}_G\}$.

**Proof.**

Clearly $\mathbf{a} \in CoSp(G)$ if and only if for all total $\mathbf{b}$,
$\mathbf{d}_G \leq_e \mathbf{b} \Rightarrow \mathbf{a} \leq_e \mathbf{b}$.
According Selman's Theorem the last is equivalent to $\mathbf{a} \leq_e \mathbf{d}_G$. □

# A structure $\mathfrak{A}$ with $CoSp(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \leq_e d_e(Y)\}$

*Consider the set*

$$S = \{\langle i, j\rangle : (j = 0) \vee (j = 1 \;\&\; i \in Y)\}.$$

*Clearly* $S \equiv_e Y$.
*Let* $G$ *be the least subgroup of* $Q$ *containing the set*

$$\{1/p_i^j : \langle i, j\rangle \in S\}.$$

*Then* $1 \in G$ *and* $S_1(G) = S$. *So,* $\mathbf{d}_G = d_e(Y)$.

**Theorem.** $CoSp(G) = \{\mathbf{a} \mid \mathbf{a} \leq_e d_e(Y)\}$.

Ash, C., Knight, J., Manasse, M., Slaman, T.:
Generic copies of countable structures.
Ann. Pure Appl. Logic **42** (1989) 195–205

Goncharov, S., Harizanov, V., Knight, J., McCoy, C., Miller, R., Solomon, R.:
Enumerations in computable structure theory.
Annals of Pure and Applied Logic **136** (2005) 219–246

Soskov I. N.,
Degree spectra and co-spectra of structures.
Ann. Univ. Sofia **96** (2003) 45–68

Soskova, A., Soskov, I.:
A jump inversion theorem for the degree spectra.
Journal of Logic and Computation **19** (2009) 199–215

Thank you!