Structural properties of spectra and ω-spectra

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Enumeration reducibility

Definition. We say that $\Gamma : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is an *enumeration operator* iff for some c.e. set W_e for each $B \subseteq \mathbb{N}$

$$\Gamma(B) = \{x | (\exists D)[\langle x, D\rangle \in W_e \,\& D \subseteq B]\}.$$

Definition. The set A is *enumeration reducible to* the set B ($A \leq_e B$), if $A = \Gamma(B)$ for some e-operator Γ.
The enumeration degree of A is $d_e(A) = \{B \subseteq \mathbb{N} | A \equiv_e B\}$.
The set of all enumeration degrees is denoted by $\mathcal{D}_e$.

The enumeration jump

Definition. Given a set A, denote by $A^+ = A \oplus (\mathbb{N} \setminus A)$.

Theorem. *For any sets A and B:*

1. *A is c.e. in B iff $A \leq_e B^+$.*
2. *$A \leq_T B$ iff $A^+ \leq_e B^+$.*

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Definition. For any set A let $K_A = \{\langle i, x\rangle | x \in \Gamma_i(A)\}$. Set $A' = K_A^+$.

Definition. A set A is called *total* iff $A \equiv_e A^+$.

Let $d_e(A)' = d_e(A')$. The enumeration jump is always a total degree and agrees with the Turing jump under the standard embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$ by $\iota(d_T(A)) = d_e(A^+)$.

Enumeration degree spectra

Let $\mathfrak{A} = (A; R_1, \dots, R_k)$ be a countable structure. An enumeration of $\mathfrak{A}$ is every total surjective mapping of $\mathbb{N}$ onto A.

Given an enumeration f of $\mathfrak{A}$ and a subset of B of A^n, let

$$f^{-1}(B) = \{\langle x_1, \dots, x_n\rangle \mid (f(x_1), \dots, f(x_n)) \in B\}.$$

$f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq)$.

Definition. *The enumeration degree spectrum of $\mathfrak{A}$* is the set

$$DS(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) \mid f \text{ is an enumeration of } \mathfrak{A}\}.$$

If **a** is the least element of $DS(\mathfrak{A})$, then **a** is called the *e-degree of $\mathfrak{A}$*.

Enumeration degree spectra

Proposition. *The enumeration degree spectrum is closed upwards with respect to total e-degrees, i.e. if* $\mathbf{a} \in DS(\mathfrak{A})$, $\mathbf{b}$ *is a total e-degree* $\mathbf{a} \leq_e \mathbf{b}$ *then* $\mathbf{b} \in DS(\mathfrak{A})$.

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Let $\mathfrak{A}^+ = (A, R_1, \ldots, R_k, R_1^c, \ldots, R_k^c)$.

Proposition.
$\iota(DS_T(\mathfrak{A})) = DS(\mathfrak{A}^+)$.

Co-spectra

Definition. Let $\mathcal{A}$ be a nonempty set of enumeration degrees. The *co-set of* $\mathcal{A}$ is the set $co(\mathcal{A})$ of all lower bounds of $\mathcal{A}$. Namely

$$co(\mathcal{A}) = \{\mathbf{b} : \mathbf{b} \in \mathcal{D}_e \ \& \ (\forall \mathbf{a} \in \mathcal{A})(\mathbf{b} \leq_e \mathbf{a})\}.$$

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Definition. Given a structure $\mathfrak{A}$, set $CS(\mathfrak{A}) = co(DS(\mathfrak{A}))$.
If $\mathbf{a}$ is the greatest element of $CS(\mathfrak{A})$ then we call $\mathbf{a}$ the *co-degree* of $\mathfrak{A}$.

The formally definable sets on $\mathfrak{A}$

Definition. A Σ_1^+ formula with free variables among $X_1, \ldots, X_r$ is a c.e. disjunction of existential formulae of the form $\exists Y_1 \ldots \exists Y_k \theta(\bar{Y}, \bar{X})$, where θ is a finite conjunction of atomic formulae.

Definition. A set $B \subseteq \mathbb{N}$ is *formally definable* on $\mathfrak{A}$ if there exists a recursive function $\gamma(x)$, such that $\bigvee_{x \in \mathbb{N}} \Phi_{\gamma(x)}$ is a Σ_1^+ formula with free variables among $X_1, \ldots, X_r$ and elements $t_1, \ldots, t_r$ of A such that the following equivalence holds:

$$x \in B \iff \mathfrak{A} \models \Phi_{\gamma(x)}(X_1/t_1, \ldots, X_r/t_r) \ .$$

Theorem. *Let $B \subseteq \mathbb{N}$. Then*

1. *$d_e(B) \in CS(\mathfrak{A})$ iff*
2. *B is formally definable on $\mathfrak{A}$.*

Some examples

- For every linear ordering $\mathrm{DS}(\mathfrak{A})$ contains a minimal pair of degrees [Richter] and hence $\mathbf{0}_e$ is the co-degree of $\mathfrak{A}$. So, if $\mathfrak{A}$ has a degree $\mathbf{a}$, then $\mathbf{a} = \mathbf{0}_e$.
- For a linear ordering $\mathfrak{A}$, $\mathrm{CS}_1(\mathfrak{A})$ consists of all e-degrees of Σ^0_2 sets [Knight]. The first co-degree of $\mathfrak{A}$ is $\mathbf{0}'_e$.
- There exists a structure $\mathfrak{A}$ [Slaman,Whener]

$$\mathrm{DS}(\mathfrak{A}) = \{\mathbf{a} : \mathbf{a} \text{ is total and } \mathbf{0}_e < \mathbf{a}\}.$$

Clearly the structure $\mathfrak{A}$ has co-degree $\mathbf{0}_e$ but has no degree.

Representing the principle countable ideals as co-spectra

Example. *Let G be a torsion free abelian group of rank 1. [Coles, Downey, Slaman; Soskov] There exists an enumeration degree* $\mathbf{s}_G$ *such that*

- $DS(G) = \{\mathbf{b} : \mathbf{b}$ *is total and* $\mathbf{s}_G \leq_e \mathbf{b}\}$.
- *The co-degree of G is* $\mathbf{s}_G$.
- *G has a degree iff* $\mathbf{s}_G$ *is a total e-degree.*

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- $DS(G) = \{\mathbf{b} : \mathbf{b}$ *is total and* $\mathbf{s}_G \leq_e \mathbf{b}\}$.
- *The co-degree of G is* $\mathbf{s}_G$.
- *G has a degree iff $\mathbf{s}_G$ is a total e-degree.*

For every $\mathbf{d} \in \mathcal{D}_e$ there exists a G, s.t. $\mathbf{s}_G = \mathbf{d}$.

Corollary. *Every principle ideal of enumeration degrees is $CS(G)$ for some G.*

Representing non-principle countable ideals as co-spectra

Theorem.[Soskov] *Every countable ideal is the co-spectrum of a structure.*

Proof.

Let $B_0, \ldots, B_n, \ldots$ be a sequence of sets of natural numbers. Set $\mathfrak{A} = (\mathbb{N}; G_f; \sigma)$,

$$f(\langle i, n \rangle) = \langle i+1, n \rangle;$$
$$\sigma = \{\langle i, n \rangle : n = 2k+1 \vee n = 2k \ \& \ i \in B_k\}.$$

Then $CS(\mathfrak{A}) = I(d_e(B_0), \ldots, d_e(B_n), \ldots)$

□

Spectra with a countable base

Definition. Let $\mathcal{B} \subseteq \mathcal{A}$ be sets of degrees. Then $\mathcal{B}$ is a base of $\mathcal{A}$ if

$$(\forall \mathbf{a} \in \mathcal{A})(\exists \mathbf{b} \in \mathcal{B})(\mathbf{b} \leq \mathbf{a}).$$

Spectra with a countable base

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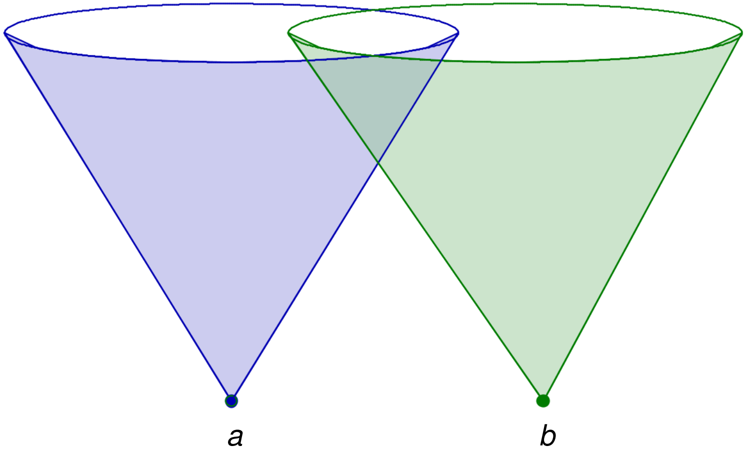
Theorem. *A structure $\mathfrak{A}$ has e-degree if and only if DS$(\mathfrak{A})$ has a countable base.*

An upwards closed set of degrees which is not a degree spectra of a structure



The minimal pair theorem

Theorem. *Let* $\mathbf{c} \in DS_2(\mathfrak{A})$. *There exist* $\mathbf{f},\mathbf{g} \in DS(\mathfrak{A})$ *such that* $\mathbf{f},\mathbf{g}$ *are total,* $\mathbf{f}'' = \mathbf{g}'' = \mathbf{c}$ *and* $CS(\mathfrak{A}) = co(\{\mathbf{f},\mathbf{g}\})$.

Notice that for every enumeration degree $\mathbf{b}$ *there exists a structure* $\mathfrak{A}_{\mathbf{b}}$ *such that* $DS(\mathfrak{A}_{\mathbf{b}}) = \{\mathbf{x} \in \mathcal{D}_T | \mathbf{b} <_e \mathbf{x}\}$. *Hence*

Corollary.*[Rozinas] For every* $\mathbf{b} \in \mathcal{D}_e$ *there exist total* $\mathbf{f}, \mathbf{g}$ *below* $\mathbf{b}''$ *which are a minimal pair over* $\mathbf{b}$.

The quasi-minimal degree

Definition. Let $\mathcal{A}$ be a set of enumeration degrees. The degree $\mathbf{q}$ is quasi-minimal with respect to $\mathcal{A}$ if:

- $\mathbf{q} \notin co(\mathcal{A})$.
- If $\mathbf{a}$ is total and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathcal{A}$.
- If $\mathbf{a}$ is total and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in co(\mathcal{A})$.

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Theorem. *For every structure $\mathfrak{A}$ there exists a quasi-minimal with respect to $DS(\mathfrak{A})$ degree.*

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Theorem. *For every structure $\mathfrak{A}$ there exists a quasi-minimal with respect to $DS(\mathfrak{A})$ degree.*

Corollary.[*Slaman and Sorbi*] *Let I be a countable ideal of enumeration degrees. There exists an enumeration degree $\mathbf{q}$ s.t.*

1. *If $\mathbf{a} \in I$ then $\mathbf{a} <_e \mathbf{q}$.*
2. *If $\mathbf{a}$ is total and $\mathbf{a} \leq_e \mathbf{q}$ then $\mathbf{a} \in I$.*

Jumps of quasi-minimal degrees

Proposition. *For every countable structure* $\mathfrak{A}$ *there exist uncountably many quasi-minimal degrees with respect to* $DS(\mathfrak{A})$.

Proposition. *The first jump spectrum of every structure* $\mathfrak{A}$ *consists exactly of the enumeration jumps of the quasi-minimal degrees.*

Corollary.[McEvoy] *For every total e-degree* $\mathbf{a} \geq_e \mathbf{0}'_e$ *there is a quasi-minimal degree* $\mathbf{q}$ *with* $\mathbf{q}' = \mathbf{a}$.

Splitting a total set

Proposition.*[Jockusch] For every total e-degree* $\mathbf{a}$ *there are quasi-minimal degrees* $\mathbf{p}$ *and* $\mathbf{q}$ *such that* $\mathbf{a} = \mathbf{p} \vee \mathbf{q}$.

Proposition. *For every element* $\mathbf{a}$ *of the jump spectrum of a structure* $\mathfrak{A}$ *there exists quasi-minimal with respect to* $\mathfrak{A}$ *degrees* $\mathbf{p}$ *and* $\mathbf{q}$ *such that* $\mathbf{a} = \mathbf{p} \vee \mathbf{q}$.

Every jump spectrum is the spectrum of a structure

Let $\mathfrak{A} = (A; R_1, \dots, R_n)$.
Let $\bar{0} \notin A$. *Set* $A_0 = A \cup \{\bar{0}\}$. *Let* $\langle ., . \rangle$ *be a pairing function s.t. none of the elements of* A_0 *is a pair and* A^* *be the least set containing* A_0 *and closed under* $\langle ., . \rangle$. *Let L and R be the decoding functions.*

Definition. *Moschovakis' extension* of $\mathfrak{A}$ is the structure

$$\mathfrak{A}^* = (A^*, R_1, \dots, R_n, A_0, G_{\langle .,. \rangle}, G_L, G_R).$$

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Let $K_{\mathfrak{A}} = \{\langle \delta, e, x \rangle : (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}$.
Set $\mathfrak{A}' = (\mathfrak{A}^*, K_{\mathfrak{A}}, A^* \setminus K_{\mathfrak{A}})$.

Theorem. $DS_1(\mathfrak{A}) = DS(\mathfrak{A}')$.

The jump inversion theorem

Let $\alpha < \omega_1^{CK}$ and $\mathfrak{A}$ be a countable structure such that all elements of $DS(\mathfrak{A})$ are above $\mathbf{0}^{(\alpha)}$.
Does there exist a structure $\mathfrak{M}$ such that $DS_\alpha(\mathfrak{M}) = DS(\mathfrak{A})$?

Theorem. *[Soskov, AS] $\alpha = 1$. If $DS(\mathfrak{A}) \subseteq DS_1(\mathfrak{B})$ then there exists a structure $\mathfrak{C}$ such that $DS(\mathfrak{C}) \subseteq DS(\mathfrak{B})$ and $DS_1(\mathfrak{C}) = DS(\mathfrak{A})$.*

Method: Marker's extensions.

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Method: Marker's extensions.

Remark.

2009 *Montalban Notes on the jump of a structure, Mathematical Theory and Computational Practice, 372–378.*

2009 *Stukachev A jump inversion theorem for the semilattices of Sigma-degrees, Siberian Electronic Mathematical Reports, v. 6, 182 – 190*

Jump inversion theorem

- The jump inversion theorem holds for successor ordinals [Goncharov-Harizanov-Knight-McCoy-Miller-Solomon, 2006; Vatev,2013]
- The jump inversion theorem does not hold for ω. [Soskov 2013]

ω-Enumeration Degrees

- Uniform reducibility on sequences of sets.
- For the sequence of sets of natural numbers $\mathcal{B} = \{B_n\}_{n<\omega}$ call *the jump class of* $\mathcal{B}$ the set

$$J_{\mathcal{B}} = \{d_{\mathrm{T}}(X) \mid (\forall n)(B_n \text{ is c.e. in } X^{(n)} \text{ uniformly in } n)\} \ .$$

Definition. $\mathcal{A} \leq_\omega \mathcal{B}$ ($\mathcal{A}$ is *ω-enumeration reducible to* $\mathcal{B}$) if $J_{\mathcal{B}} \subseteq J_{\mathcal{A}}$

- $\mathcal{A} \equiv_\omega \mathcal{B}$ if $J_{\mathcal{A}} = J_{\mathcal{B}}$.

ω-Enumeration Degrees

- The relation $\leq_\omega$ induces a partial ordering of $\mathcal{D}_\omega$ with least element $\mathbf{0}_\omega = d_\omega(\emptyset_\omega)$, where $\emptyset_\omega$ is the sequence with all members equal to $\emptyset$.
- $\mathcal{D}_\omega$ is further an upper semi-lattice, with least upper bound induced by $\mathcal{A} \oplus \mathcal{B} = \{X_n \oplus Y_n\}_{n<\omega}$.
- If $A \subseteq \mathbb{N}$ denote by $A \uparrow \omega = \{A, \emptyset, \emptyset, \dots\}$.
- The mapping $\kappa(d_e(A)) = d_\omega(A \uparrow \omega)$ gives an isomorphic embedding of $\mathcal{D}_e$ to $\mathcal{D}_\omega$, where $A \uparrow \omega = \{A, \emptyset, \emptyset, \dots\}$.

ω-Enumeration Degrees

Let $\mathcal{B} = \{B_n\}_{n<\omega}$.
The jump sequence $\mathcal{P}(\mathcal{B}) = \{\mathcal{P}_n(\mathcal{B})\}_{n<\omega}$:

1. $\mathcal{P}_0(\mathcal{B}) = B_0$
2. $\mathcal{P}_{n+1}(\mathcal{B}) = (\mathcal{P}_n(\mathcal{B}))' \oplus B_{n+1}$

Definition. *$\mathcal{A}$ is enumeration reducible $\mathcal{B}$ ($\mathcal{A} \leq_{\mathrm{e}} \mathcal{B}$) iff $A_n \leq_{\mathrm{e}} B_n$ uniformly in n.*

Theorem.*[Soskov, Kovachev] $\mathcal{A} \leq_\omega \mathcal{B} \iff \mathcal{A} \leq_{\mathrm{e}} \mathcal{P}(\mathcal{B})$.*

ω-Enumeration Jump

Definition. The ω-enumeration jump of $\mathcal{A}$ is $\mathcal{A}' = \{\mathcal{P}_{n+1}(\mathcal{A})\}_{n<\omega}$

- $J'_{\mathcal{A}} = \{\mathbf{a}' \mid \mathbf{a} \in J_{\mathcal{A}}\}$.
- The jump is monotone and agrees with the enumeration jump.
- Soskov and Ganchev: Strong jump inversion theorem: for $\mathbf{a}^{(n)} \leq \mathbf{b}$ there exists a *least* $\mathbf{x} \geq \mathbf{a}$ such that $\mathbf{x}^{(n)} = \mathbf{b}$. So, every degree $\mathbf{x}$ in the range of the jump operator has a least jump invert.
- Soskov and Ganchev: if we add a predicate for the jump operator to the language of partial orders then the natural copy of the enumeration degrees in the omega enumeration degrees becomes first order definable.
- The two structures have the same automorphism group.
- Ganchev and Sariev: The jump operator in the upper semi-lattice of the ω-enumeration degrees is first order definable.

ω- Degree Spectra

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_k, =, \neq)$ be an abstract structure and $\mathcal{B} = \{B_n\}_{n<\omega}$ be a fixed sequence of subsets of $\mathbb{N}$.
The enumeration f of the structure $\mathfrak{A}$ is *acceptable with respect to* $\mathcal{B}$, if for every n,

$$f^{-1}(B_n) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(n)} \text{ uniformly in } n.$$

Denote by $\mathcal{E}(\mathfrak{A}, \mathcal{B})$ - the class of all acceptable enumerations.

Definition. *The ω- degree spectrum of $\mathfrak{A}$ with respect to $\mathcal{B} = \{B_n\}_{n<\omega}$* is the set

$$\mathrm{DS}(\mathfrak{A}, \mathcal{B}) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) \mid f \in \mathcal{E}(\mathfrak{A}, \mathcal{B})\}$$

Proposition. $\mathrm{DS}(\mathfrak{A}, \mathcal{B})$ *is upwards closed with respect to total e-degrees.*

ω-Co-Spectra

For every $\mathcal{A} \subseteq \mathcal{D}_\omega$ let $co(\mathcal{A}) = \{\mathbf{b} \mid \mathbf{b} \in \mathcal{D}_\omega \;\&\; (\forall \mathbf{a} \in \mathcal{A})(\mathbf{b} \leq_\omega \mathbf{a})\}$.

Definition. *The ω-co-spectrum of $\mathfrak{A}$ with respect to $\mathcal{B}$* is the set

$$\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = co(\mathrm{DS}(\mathfrak{A}, \mathcal{B})).$$

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Definition. *The ω-co-spectrum of $\mathfrak{A}$ with respect to $\mathcal{B}$* is the set

$$\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = co(\mathrm{DS}(\mathfrak{A}, \mathcal{B})).$$

Proposition. $co(\mathcal{A}) = co(\{\mathbf{a} : \mathbf{a} \in \mathcal{A} \ \& \ \mathbf{a} \text{ is total}\})$.

Corollary. $\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = co(\{\mathbf{a} \mid \mathbf{a} \in \mathrm{DS}(\mathfrak{A}, \mathcal{B}) \ \& \ \mathbf{a} \text{ is a total e-degree}\})$.

Minimal pair theorem

Theorem. *For every structure* $\mathfrak{A}$ *and every sequence* $\mathcal{B}$ *there exist total enumeration degrees* $\mathbf{f}$ *and* $\mathbf{g}$ *in* $\mathrm{DS}(\mathfrak{A}, \mathcal{B})$ *such that for every* ω*-enumeration degree* $\mathbf{a}$ *and* $k \in \mathbb{N}$*:*

$$\mathbf{a} \leq_\omega \mathbf{f}^{(k)} \ \& \ \mathbf{a} \leq_\omega \mathbf{g}^{(k)} \Rightarrow \mathbf{a} \in \mathrm{CS}_k(\mathfrak{A}, \mathcal{B}) \ .$$

Quasi-Minimal Degree

Theorem. *For every structure $\mathfrak{A}$ and every sequence $\mathcal{B}$, there exists $F \subseteq \mathbb{N}$, such that* $\mathbf{q} = d_\omega(F \uparrow \omega)$ *and:*

1. $\mathbf{q} \notin \mathrm{CS}(\mathfrak{A}, \mathcal{B})$;
2. *If* $\mathbf{a}$ *is a total e-degree and* $\mathbf{a} \geq_\omega \mathbf{q}$ *then* $\mathbf{a} \in \mathrm{DS}(\mathfrak{A}, \mathcal{B})$
3. *If* $\mathbf{a}$ *is a total e-degree and* $\mathbf{a} \leq_\omega \mathbf{q}$ *then* $\mathbf{a} \in \mathrm{CS}(\mathfrak{A}, \mathcal{B})$.

Countable ideals of ω-enumeration degrees

- $I = \mathrm{CS}(\mathfrak{A}, \mathcal{B})$ is a countable ideal.
- $\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = I(\mathbf{f}_\omega) \cap I(\mathbf{g}_\omega)$ where $I(\mathbf{f}_\omega)$ and $I(\mathbf{g}_\omega)$ are the principal ideals of ω-enumeration degrees with greatest elements $\mathbf{f}_\omega = \kappa(\mathbf{f})$ and $\mathbf{g}_\omega = \kappa(\mathbf{g})$, the images of **f** and **g** under the embedding κ of $\mathcal{D}_e$ in $\mathcal{D}_\omega$.
- Denote by $I^{(k)}$ - the least ideal, containing all kth ω-jumps of the elements of I.

Proposition. *[Ganchev]* $I = I(\mathbf{f}_\omega) \cap I(\mathbf{g}_\omega) \Longrightarrow I^{(k)} = I(\mathbf{f}_\omega^{(k)}) \cap I(\mathbf{g}_\omega^{(k)})$ *for every* k.

- $I(\mathbf{f}_\omega^{(k)}) \cap I(\mathbf{g}_\omega^{(k)}) = \mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$ for each k.
- Thus $I^{(k)} = \mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$.

Corollary. $\mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$ *is the least ideal containing all kth ω-jumps of the elements of* $\mathrm{CS}(\mathfrak{A}, \mathcal{B})$.

Countable ideals of ω-enumeration degrees

There is a countable ideal I of ω-enumeration degrees for which there is no structure $\mathfrak{A}$ and sequence $\mathcal{B}$ such that $I = \mathrm{CS}(\mathfrak{A}, \mathcal{B})$.

- Consider $\mathcal{A} = \{\mathbf{0}_\omega, \mathbf{0}_\omega{}', \mathbf{0}''_\omega, \dots, \mathbf{0}^{(n)}_\omega, \dots\}$;
- $I = I(d_\omega(\mathcal{A})) = \{\mathbf{a} \mid \mathbf{a} \in \mathcal{D}_\omega \ \& \ (\exists n)(\mathbf{a} \leq_\omega \mathbf{0}^{(n)})\}$
- Assume that there is a structure $\mathfrak{A}$ and a sequence $\mathcal{B}$ such that $I = \mathrm{CS}(\mathfrak{A}, \mathcal{B})$
- Then there is a minimal pair $\mathbf{f}$ and $\mathbf{g}$ for $\mathrm{DS}(\mathfrak{A}, \mathcal{B})$, so $I^{(n)} = I(\mathbf{f}^{(n)}_\omega) \cap I(\mathbf{g}^{(n)}_\omega)$ for each n.
- But $\mathbf{f}_\omega \geq \mathbf{0}^{(n)}_\omega$ and $\mathbf{g}_\omega \geq \mathbf{0}^{(n)}_\omega$ for each n.
- If $F \in \mathbf{f}$ and $G \in \mathbf{g}$ then $F \geq_T \emptyset^{(n)}$ and $G \geq_T \emptyset^{(n)}$ for every n .
- Then by Enderton and Putnam [1970], Sacks [1971] $F'' \geq_T \emptyset^{(\omega)}$ and $G'' \geq \emptyset^{(\omega)}$ and hence $\mathbf{f}'' \geq_T \mathbf{0}^{(\omega)}_T$ and $\mathbf{g}'' \geq_T \mathbf{0}^{(\omega)}_T$.
- Then $\kappa(\iota(\mathbf{0}^{(\omega)}_T)) \in I(\mathbf{f}''_\omega) \cap I(\mathbf{g}''_\omega)$.
- But $\kappa(\iota(\mathbf{0}^{(\omega)}_T)) \notin I''$ since all elements of I'' are bounded by $\mathbf{0}^{(k+2)}_\omega$ for some k.
- Hence $I'' \neq I(\mathbf{f}_{\omega''}) \cap I(\mathbf{g}_{\omega''})$. A contradiction.

Degree spectra

- Questions:
  - Describe the sets of enumeration degrees which are equal to $DS(\mathfrak{A})$ for some structure $\mathfrak{A}$.
  - For a countable ideal $I \subseteq \mathcal{D}_\omega$ if there is an exact pair then are there a structure $\mathfrak{A}$ and a sequence $\mathcal{B}$ so that $\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = I$?
  - Is it true that for every structure $\mathfrak{A}$ and every sequence $\mathcal{B}$ there exists a structure $\mathfrak{B}$ such that $\mathrm{CS}_\omega(\mathfrak{B}) = \mathrm{CS}(\mathfrak{A}, \mathcal{B})$? The answer is yes, Soskov (2013), using Marker's extentions