Effective coding and decoding in graphs and linear ordering

National Coding Theory Workshop “Professor Stefan Dodunekov” 2019

Alexandra A. Soskova [1]

Joint work with J. Knight and S. Vatev

[1] Supported by Bulgarian National Science Fund DN 02/16 /19.12.2016 and NSF grant DMS 1600625/2016

Borel embedding

Definition (Friedman-Stanley,1989)

We say that a class $\mathcal{K}$ of structures is *Borel embeddable* in a class of structures $\mathcal{K}'$, and we write $\mathcal{K} \leq_B \mathcal{K}'$, if there is a Borel function $\Phi : \mathcal{K} \to \mathcal{K}'$ such that for $\mathcal{A}, \mathcal{B} \in \mathcal{K}$, $\mathcal{A} \cong \mathcal{B}$ iff $\Phi(\mathcal{A}) \cong \Phi(\mathcal{B})$.

Theorem

The following classes lie on top under $\leq_B$.

1. undirected graphs [Lavrov,1963; Nies,1996; Marker,2002]
2. fields of any fixed characteristic [Friedman-Stanley,1989; R. Miller-Poonen-Schoutens-Shlapentokh,2018]
3. 2-step nilpotent groups [Mal'tsev,1949; Mekler,1981]
4. linear orderings [Friedman-Stanley,1989]

Graphs $\leq_B$ ACF(0)

- There are familiar ways of coding one structure in another, and for coding members of one class of structures in those of another class.
- Sometimes the coding is effective.
- Assuming this, it is interesting when there is effective decoding, and it is also interesting when decoding is very difficult.

Turing computable embeddings

Definition (Calvert-Cummins-Knight-S. Miller,2004)

We say that a class $\mathcal{K}$ is *Turing computably embedded* in a class $\mathcal{K}'$, and we write $\mathcal{K} \leq_{tc} \mathcal{K}'$, if there is a Turing operator $\Phi : \mathcal{K} \to \mathcal{K}'$ such that for all $\mathcal{A}, \mathcal{B} \in \mathcal{K}$, $\mathcal{A} \cong \mathcal{B}$ iff $\Phi(\mathcal{A}) \cong \Phi(\mathcal{B})$.

A Turing computable embedding represents an effective coding procedure.

Theorem

The following classes lie on top under $\leq_{tc}$.

1. undirected graphs
2. fields of any fixed characteristic
3. 2-step nilpotent groups
4. linear orderings

Directed graphs $\leq_{tc}$ undirected graphs

Example (Marker)

For a directed graph G the undirected graph $\Theta(G)$ consists of the following:

1. For each point a in G, $\Theta(G)$ has a point b_a connected to a triangle.
2. For each ordered pair of points $(a; a')$ from G, $\Theta(G)$ has a special point $p_{(a,a')}$ connected directly to b_a and with one stop to b'_a .
3. The point $p_{(a,a')}$ is connected to a square if there is an arrow from a to a', and to a pentagon otherwise.

For structures $\mathcal{A}$ with more relations, the same idea works.

Medvedev reducibility

A *problem* is a subset of 2^ω or ω^ω.
Problem P is Medvedev reducible to problem Q if there is a Turing operator Φ that takes elements of Q to elements of P.

Definition

We say that $\mathcal{A}$ is *Medvedev reducible* to $\mathcal{B}$, and we write $\mathcal{A} \leq_s \mathcal{B}$, if there is a Turing operator that takes copies of $\mathcal{B}$ to copies of $\mathcal{A}$.

Supposing that $\mathcal{A}$ is coded in $\mathcal{B}$, a Medvedev reduction of $\mathcal{A}$ to $\mathcal{B}$ represents an effective decoding procedure.

Effective interpretability

Definition (Montlbán,2017)

A structure $\mathcal{A} = (A, R_i)$ is *effectively interpreted* in a structure $\mathcal{B}$ if there is a set $D \subseteq \mathcal{B}^{<\omega}$, computable Σ_1-definable over $\emptyset$, and there are relations $\sim$ and R_i^* on D, computable Δ_1-definable over $\emptyset$, such that $(D, R_i^*)/_{\sim} \cong \mathcal{A}$.

Definition (R. Miller,2017)

A *computable functor* from $\mathcal{B}$ to $\mathcal{A}$ is a pair of Turing operators Φ, Ψ such that Φ takes copies of $\mathcal{B}$ to copies of $\mathcal{A}$ and Ψ takes isomorphisms between copies of $\mathcal{B}$ to isomorphisms between the corresponding copies of $\mathcal{A}$, so as to preserve identity and composition.

Equivalence

The main result gives the equivalence of the two definitions.

Theorem (Harrison-Trainor, Melnikov, R. Miller and Montalbán,2017)

For structures $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}$ is effectively interpreted in $\mathcal{B}$ iff there is a computable functor Φ, Ψ from $\mathcal{B}$ to $\mathcal{A}$.

Corollary

If $\mathcal{A}$ is effectively interpreted in $\mathcal{B}$, then $\mathcal{A} \leq_s \mathcal{B}$.

Example

The usual definition of the ring of integers $\mathbb{Z}$ involves an interpretation in the semi-ring of natural numbers $\mathbb{N}$. Let D be the set of ordered pairs (m, n) of natural numbers. We think of the pair (m, n) as representing the integer $m - n$. We can easily give finitary existential formulas that define ternary relations of addition and multiplication on D, and the complements of these relations, and a congruence relation $\sim$ on D, and the complement of this relation, such that $(D, +, \cdot)/_{\sim} \cong \mathbb{Z}$.

Coding and Decoding

Proposition (Kalimullin,2010)

There exist $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \leq_s \mathcal{B}$ but $\mathcal{A}$ is not effectively interpreted in $\mathcal{B}$.

Proposition

If $\mathcal{A}$ is computable, then it is effectively interpreted in all structures $\mathcal{B}$.

Proof.

Let $D = \mathcal{B}^{<\omega}$. Let $\bar{b} \sim \bar{c}$ if $\bar{b}, \bar{c}$ are tuples of the same length. For simplicity, suppose $\mathcal{A} = (\omega, R)$, where R is binary. If $\mathcal{A} \models R(m, n)$, then $R^*(\bar{b}, \bar{c})$ for all $\bar{b}$ of length m and $\bar{c}$ of length n. Thus, $(D, R^*)/_\sim \cong \mathcal{A}$. □

Graphs and linear orderings

Graphs and linear orderings both lie on top under Turing computable embeddings.

Graphs also lie on top under effective interpretation.

Question: What about linear orderings under effective interpretation?

And under using $L_{\omega_1\omega}$-formulas?

Interpreting graphs in linear orderings

Proposition

There is a graph G such that for all linear orderings L, $G \not\leq_s L$.

Proof.

Let S be a non-computable set. Let G be a graph such that every copy computes S.
We may take G to be a "daisy" graph", consisting of a center node with a "petal" of length $2n+3$ if $n \in S$ and $2n+4$ if $n \notin S$.
Now, apply:

Proposition (Richter,1981)

For a linear ordering L, the only sets computable in all copies of L are the computable sets.

□

Interpreting a graph in the jump of linear ordering

We are identifying a structure $\mathcal{A}$ with its atomic diagram. We may consider an interpretation of $\mathcal{A}$ in the jump $\mathcal{B}'$ of $\mathcal{B}$. Note that the relations definable in $\mathcal{B}'$ by computable Σ_1 relations are the ones definable in $\mathcal{B}$ by computable Σ_2 relations.

Proposition

There is a graph G such that for all linear orderings L, $G \not\leq_s L'$.

Proof.

Let S be a non-Δ^0_2 set. Let G be a graph such that every copy computes S. Then apply:

Proposition (Knight,1986)

For a linear ordering L, the only sets computable in all copies of L' (or in the jumps of all copies of L), are the Δ^0_2 sets.

□

Interpreting a graph in the second jump of linear ordering

Proposition

For any set S, there is a linear ordering L such that for all copies of L, the second jump of L computes S.

Proof.

We may take L to be a "shuffle sum" of $n+1$ for $n \in S \oplus S^c$ and ω. □

Proposition

For any graph G, there is a linear ordering L such that $G \leq_s L''$. In fact, G is interpreted in L using computable Σ_3 formulas.

Proof.

Let S be the diagram of a specific copy G_0 of G and let L be a linear order such that $S \leq_s L''$. We have computable functor that takes the second jump of any copy of L to G_0, and takes all isomorphisms between copies of L to the identity isomorphism on G_0. □

Friedman-Stanley embedding of graphs in orderings

Friedman and Stanley determined a Turing computable embedding $L : G \to L(G)$, where $L(G)$ is a sub-ordering of $\mathbb{Q}^{<\omega}$ under the lexicographic ordering.

1. Let $(A_n)_{n\in\omega}$ be an effective partition of $\mathbb{Q}$ into disjoint dense sets.
2. Let $(t_n)_{1\leq n}$ be a list of the atomic types in the language of directed graphs.

Definition

For a graph G, the elements of $L(G)$ are the finite sequences $r_0 q_1 r_1 \ldots r_{n-1} q_n r_n k \in \mathbb{Q}^{<\omega}$ such that for $i < n$, $r_i \in A_0$, $r_n \in A_1$, and for some $a_1, \ldots, a_n \in G$, satisfying t_m, $q_i \in A_{a_i}$ and $k < m$.

No uniform interpretation of G in $L(G)$

Theorem

There are not $L_{\omega_1\omega}$ formulas that, for all graphs G, interpret G in $L(G)$.

The idea of Proof: We may think of an ordering as a directed graph. It is enough to show the following.

Proposition

1. ω_1^{CK} is not interpreted in $L(\omega_1^{CK})$ using computable infinitary formulas.
2. For all X, ω_1^X is not interpreted in $L(\omega_1^X)$ using X-computable infinitary formulas.

Proof of (1)

The Harrison ordering H has order type $\omega_1^{CK}(1+\eta)$. It has a computable copy.

Let I be the initial segment of H of order type ω_1^{CK}. Thinking of H as a directed graph, we can form the linear ordering $L(H)$. We consider $L(I) \subseteq L(H)$.

Proposition (Main)

There do not exist computable infinitary formulas that define an interpretation of H in $L(H)$ and an interpretation of I in $L(I)$.

To prove (1), we suppose that there are computable infinitary formulas interpreting ω_1^{CK} in $L(\omega_1^{CK})$. Using Barwise Compactness theorem, we get essentially H and I with these formulas interpreting H in $L(H)$ and I in $L(I)$.

Conjecture

We believe that Friedman and Stanley did the best that could be done.

Conjecture. For any Turing computable embedding Θ of graphs in orderings, there do not exist $L_{\omega_1\omega}$ formulas that, for all graphs G, define an interpretation of G in $\Theta(G)$.

M. Harrison-Trainor and A. Montlbán came to a similar result very recently by a totally different construction. Their result is that there exist structures which cannot be computably recovered from their tree of tuples. They proved :

1. There is a structure $\mathcal{A}$ with no computable copy such that $T(\mathcal{A})$ has a computable copy.
2. For each computable ordinal α there is a structure $\mathcal{A}$ such that the Friedman and Stanley Borel interpretation $L(\mathcal{A})$ is computable but $\mathcal{A}$ has no Δ^0_α copy.

Mal'tsev embedding of fields in groups,1960

If F is a field, we denote by $H(F)$ the multiplicative group of matrices of kind

$$h(a,b,c) = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

where $a, b, c \in F$. Note that $h(0,0,0) = 1$.
Groups of kind $H(F)$ are known as *Heisenberg groups*.

Theorem (Mal'tsev,1960)

There is a copy of F defined in $H(F)$ with parameters.

Definition of F in $H(F)$

Let u, v be a non-commuting pair in $H(F)$.
Then $(D, +, \cdot_{(u,v)})$ is a copy of F, where

1. D is the group center – $x \in D \iff [x, u] = 1$ and $[x, v] = 1$,
2. $x + y = z$ if $x * y = z$, where $*$ is the group operation,
3. $x \cdot_{(u,v)} y = z$ if there exist x', y' such that $[x', u] = [y', v] = 1, [x', v] = x, [u, y'] = y$, and $[x', y'] = z$.

Here $[x, y] = x^{-1}y^{-1}xy$.

Definability: We have finitary existential formulas that define D and the relation $+$ and its complement. For any non-commuting pair (u, v), we have finitary existential formulas, with parameters (u, v) that define the relation $\cdot$ and its complement.

Natural isomorphisms

For a non-commuting pair (u, v), where $u = h(u_1, u_2, u_3)$ and $v = h(v_1, v_2, v_3)$, let

$$\Delta_{(u,v)} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

Theorem

The function f that takes $x \in F$ to $h(0, 0, \Delta_{(u,v)} \cdot_F x)$ is an isomorphism.

Morozov's isomorphism

Lemma (Morozov)

Let (u, v) and (u', v') be non-commuting pairs in $G = H(F)$. Let $F_{(u,v)}$ and $F_{(u',v')}$ be the copies of F defined in G with these pairs of parameters. There is an isomorphism g from $F_{(u,v)}$ onto $F_{(u',v')}$ defined in G by an existential formula with parameters u, v, u', v'.

Note that $\Delta_{(u,v)}$ is the multiplicative identity in $F_{(u,v)}$.
Let $g(x) = y \iff x = \Delta_{(u,v)} \cdot_{(u',v')} y$.

Computable functor

Theorem

There is a computable functor Φ, Ψ from $H(F)$ to F.

- For $G \cong H(F)$, $\Phi(G)$ is the copy of F obtained by taking the first non-commuting pair (u, v) in G and forming $(D; +; \cdot_{(u,v)})$.
- Take (G_1, f, G_2), where $G_i = H(F)$, and $G_1 \cong_f G_2$. Let (u, v), (u', v') be the first non-commuting pairs in G_1, G_2, respectively.
  - Let h be the isomorphism from $F_{(f(u),f(v))}$ onto $F_{(u',v')}$ defined in G_2 with parameters $f(u), f(v), u', v'$.
  - Let f' be the restriction of f to the center of G_1.
  - Then $\Psi(G_1, f, G_2) = h \circ f'$.

Finitely existential interpretation and generalizing

Corollary (Alvir,Calvert,Harizanov,Knight,Miller,Morozov,S,Weisshaar, 2019)

F is effectively interpreted in $H(F)$.

$(u, v, x) \sim (u', v', x')$ holds if Morozov's isomorphism from $F_{(u,v)}$ to $F_{(u',v')}$ takes x to x'.

Proposition

Suppose $\mathcal{A}$ has a copy $\mathcal{A}_{\bar{b}}$ defined in $(\mathcal{B}, \bar{b})$, using computable Σ_1 formulas, where the orbit of $\bar{b}$ is defined by a computable Σ_1 formula $\varphi(\bar{x})$. Suppose also that there is a computable Σ_1 formula $\psi(\bar{b}, \bar{b}', u, v)$ that, for any tuples $\bar{b}$, $\bar{b}'$ satisfying $\varphi(\bar{x})$, defines a specific isomorphism $f_{\bar{b},\bar{b}'}$ from $\mathcal{A}_{\bar{b}}$ onto $\mathcal{A}_{\bar{b}'}$. We suppose that for each $\bar{b}$ satisfying φ, $f_{\bar{b},\bar{b}}$ is the identity isomorphism, and for any $\bar{b}$, $\bar{b}'$, and $\bar{b}''$ satisfying φ, $f_{\bar{b}',\bar{b}''} \circ f_{\bar{b},\bar{b}'} = f_{\bar{b},\bar{b}''}$. Then there is an effective interpretation of $\mathcal{A}$ in $\mathcal{B}$.

$SL_2(C)$

Let C be an algebraically closed field of characteristic 0 and of infinite transcendence degree.
We consider $SL_2(C)$ for the group of 2×2 matrices over C with determinant 1.

Proposition

F is interpreted in $SL_2(F)$ with parameters.

Let A be the set of matrices of form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$.

Let M be the set of matrices of form $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$.

$SL_2(C)$

Let T consist of the pairs (X, Y) such that $X \in A$ and $Y \in M$ and Y has a square root Z such that $Z * P * Z^{-1} = X$.
For $(X, Y) \in T$, we define addition and multiplication relations as follows:

1. $(X, Y) \oplus (X', Y') = (U, V)$ if $X * X' = U$ and $(U, V) \in T$,
2. $(X, Y) \otimes (X', Y') = (U, V)$ if $Y * Y' = V$ and $(U, V) \in T$.

We define the set T with parameters.
Possibly, we can show model completeness of the theory of $SL_2(C)$.
This, together with the result, according to Pillay, saying that C is interpreted in $SL_2(C)$ by elementary first order formulas with no parameters, we could show that it is interpreted with existential formulas.

W. Calvert, D. Cummins, J. F. Knight, and S. Miller
Comparing classes of finite structures
Algebra and Logic, vo. 43(2004), pp. 374-392.

H. Friedman and L. Stanley
A Borel reducibility theory for classes of countable structures
JSL, vol. 54(1989), pp. 894-914.

J. Knight, A. Soskova, and S. Vatev
Coding in graphs and linear orderings
https://arxiv.org/abs/1903.06948

M. Harrison-Trainor, A. Melnikov, R. Miller, and A. Montalbán
Computable functors and effective interpretability,
JSL, vol. 82(2017), pp. 77-97.

THANK YOU