ω-Degree Spectra

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Outline

- Degree spectra and jump spectra
- ω-enumeration degrees
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Enumeration of a Structure

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_k, =, \neq)$ be a countable abstract structure.

- An enumeration f of $\mathfrak{A}$ is a total mapping from $\mathbb{N}$ onto $\mathbb{N}$.
- for any $A \subseteq \mathbb{N}^a$ let $f^{-1}(A) = \{\langle x_1 \ldots x_a \rangle : (f(x_1), \ldots, f(x_a)) \in A\}$.
- $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq)$.

Degree Spectra

Definition (L. Richter, 1981)

The Turing degree spectrum of $\mathfrak{A}$

$$\mathrm{DS}_{\mathrm{T}}(\mathfrak{A}) = \{d_{\mathrm{T}}(f^{-1}(\mathfrak{A})) \mid f \text{ is an injective enumeration of } \mathfrak{A}\}$$

- J. Knight, Ash, Jockush, Downey, Slaman.

Degree Spectra and Co-spectra

Definition (Soskov, 2004)

- The degree spectrum of $\mathfrak{A}$

$$\mathrm{DS}(\mathfrak{A}) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) \mid f \text{ is an enumeration of } \mathfrak{A}\}.$$

- The co-spectrum of $\mathfrak{A}$

$$\mathrm{CS}(\mathfrak{A}) = \{\mathbf{b} : (\forall \mathbf{a} \in \mathrm{DS}(\mathfrak{A}))(\mathbf{b} \leq \mathbf{a})\}.$$

Degree Spectra and Co-spectra

Definition

Let $\mathcal{A} \subseteq \mathcal{D}_e$. $\mathcal{A}$ is *upwards closed with respect to total enumeration degrees*, if

$$\mathbf{a} \in \mathcal{A}, \mathbf{b} \text{ is total and } \mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{b} \in \mathcal{A}.$$

The degree spectra are upwards closed with respect to total enumeration degrees.

Properties of upwards closed sets

Let $\mathcal{A} \subseteq \mathcal{D}_e$ be upwards closed with respect to total enumeration degrees. Denote by

$$co(\mathcal{A}) = \{b : b \in \mathcal{D}_e \;\&\; (\forall a \in \mathcal{A})(b \leq_e a)\}.$$

- $\mathcal{A}_t = \{\mathbf{a} : \mathbf{a} \in \mathcal{A} \;\&\; \mathbf{a} \text{ is total}\} \implies co(\mathcal{A}) = co(\mathcal{A}_t)$.
- Let $\mathbf{b} \in \mathcal{D}_e$ and $n > 0$.

$$\mathcal{A}_{\mathbf{b},n} = \{\mathbf{a} : \mathbf{a} \in \mathcal{A} \;\&\; \mathbf{b} \leq \mathbf{a}^{(n)}\} \implies co(\mathcal{A}) = co(\mathcal{A}_{\mathbf{b},n}).$$

Properties of degree spectra and co-spectra

- Let $\mathbf{c} \in \mathrm{DS}_n(\mathfrak{A})$ and $n > 0$. Then

$$\mathrm{CS}(\mathfrak{A}) = co(\{\mathbf{a} \mid \mathbf{a} \in \mathrm{DS}(\mathfrak{A}) \ \&\ \mathbf{a}^{(n)} = \mathbf{c}\}).$$

- A minimal pair theorem:
  There exist $\mathbf{f}$ and $\mathbf{g}$ in $\mathrm{DS}(\mathfrak{A})$:

$$(\forall \mathbf{a} \in \mathcal{D}_e)(\forall k)(\mathbf{a} \leq_e \mathbf{f}^{(k)} \ \&\ \mathbf{a} \leq_e \mathbf{g}^{(k)} \Rightarrow \mathbf{a} \in \mathrm{CS}_k(\mathfrak{A})).$$

- Quasi-minimal degree:
  There exists $\mathbf{q_0}$ quasi-minimal for $\mathrm{DS}(\mathfrak{A})$
  - $\mathbf{q_0} \notin \mathrm{CS}(\mathfrak{A})$;
  - for every total *e*-degree $\mathbf{a}$: $\mathbf{a} \geq_e \mathbf{q_0} \Rightarrow \mathbf{a} \in \mathrm{DS}(\mathfrak{A})$ and $\mathbf{a} \leq_e \mathbf{q_0} \Rightarrow \mathbf{a} \in \mathrm{CS}(\mathfrak{A})$.
- Every countable ideal can be represented as a co-spectrum of some structure $\mathfrak{A}$.

ω-Enumeration Degrees

- Uniform reducibility on sequences of sets
- $\mathcal{S}$ the set of all sequences of sets of natural numbers
- For $\mathcal{B} = \{B_n\}_{n<\omega} \in \mathcal{S}$ call *the jump class of* $\mathcal{B}$ the set

$$J_{\mathcal{B}} = \{d_{\mathrm{T}}(X) \mid (\forall n)(B_n \text{ is c.e. in } X^{(n)} \text{ uniformly in } n)\} \ .$$

- $\mathcal{A} \leq_\omega \mathcal{B}$ ($\mathcal{A}$ is *ω-enumeration reducible to* $\mathcal{B}$) if $J_{\mathcal{B}} \subseteq J_{\mathcal{A}}$
- $\mathcal{A} \equiv_\omega \mathcal{B}$ if $J_{\mathcal{A}} = J_{\mathcal{B}}$.

ω-Enumeration Degrees

- $\equiv_\omega$ is an equivalence relation on $\mathcal{S}$.
- $d_\omega(\mathcal{B}) = \{\mathcal{A} \mid \mathcal{A} \equiv_\omega \mathcal{B}\}$
- $\mathcal{D}_\omega = \{d_\omega(\mathcal{B}) \mid \mathcal{B} \in \mathcal{S}\}$.
- If $A \subseteq \mathbb{N}$ denote by $A \uparrow \omega = \{A, \emptyset, \emptyset, \dots\}$.
- For every $A, B \subseteq \mathbb{N}$:

$$A \leq_{\mathrm{e}} B \iff A \uparrow \omega \leq_\omega B \uparrow \omega.$$

- The mapping $\kappa(d_{\mathrm{e}}(A)) = d_\omega(A \uparrow \omega)$ gives an isomorphic embedding of $\mathcal{D}_{\mathrm{e}}$ to $\mathcal{D}_\omega$.

ω-Enumeration Degrees

Let $\mathcal{B} = \{B_n\}_{n<\omega} \in \mathcal{S}$.
A jump sequence $\mathcal{P}(\mathcal{B}) = \{\mathcal{P}_n(\mathcal{B})\}_{n<\omega}$:

1. $\mathcal{P}_0(\mathcal{B}) = B_0$
2. $\mathcal{P}_{n+1}(\mathcal{B}) = (\mathcal{P}_n(\mathcal{B}))' \oplus B_{n+1}$

Theorem (Soskov, Kovachev)

$\mathcal{A} \leq_\omega \mathcal{B}$, if $A_n \leq_{\mathrm{e}} \mathcal{P}_n(\mathcal{B})$ uniformly in n.

ω-Enumeration Jump

- For every $\mathcal{A} \in \mathcal{S}$ *the ω-enumeration jump of $\mathcal{A}$* is $\mathcal{A}' = \{\mathcal{P}_{n+1}(\mathcal{A})\}_{n<\omega}$
- $d_\omega(\mathcal{A})' = d_\omega(\mathcal{A}')$
- $\mathcal{A}^{(k+1)} = (\mathcal{A}^{(k)})'$
- $d_\omega(\mathcal{A})^{(k+1)} = d_\omega(\mathcal{A}^{(k+1)})$
- $\mathcal{A}^{(k)} = \{\mathcal{P}_{n+k}(\mathcal{A})\}_{n<\omega}$ for each k.

Relative Spectra

Let $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ be given structures.

Definition

The relative spectrum $\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1 \dots, \mathfrak{A}_n)$ of the structure $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ is the set

$$\{d_e(f^{-1}(\mathfrak{A})) \mid \quad f \text{ is an enumeration of } \mathfrak{A} \;\& \\ (\forall k \leq n)(f^{-1}(\mathfrak{A}_k) \leq_e f^{-1}(\mathfrak{A})^{(k)}).\}$$

It turns out that all properties of the degree spectra remain true for the relative spectra.

ω- Degree Spectra

Let $\mathcal{B} = \{B_n\}_{n<\omega}$ be a fixed sequence of sets.
The enumeration f of the structure $\mathfrak{A}$ is *acceptable with respect to* $\mathcal{B}$, if for every n,

$$f^{-1}(B_n) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(n)} \text{ uniformly in } n.$$

Denote by $\mathcal{E}(\mathfrak{A}, \mathcal{B})$ - the class of all acceptable enumerations.

Definition

The ω*- degree spectrum of* $\mathfrak{A}$ *with respect to* $\mathcal{B} = \{B_n\}_{n<\omega}$ is the set

$$\mathrm{DS}(\mathfrak{A}, \mathcal{B}) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) \mid f \in \mathcal{E}(\mathfrak{A}, \mathcal{B}).\}$$

ω- Degree Spectra

- It is easy to find a structure $\mathfrak{A}$ and a sequence $\mathcal{B}$ such that $\mathrm{DS}(\mathfrak{A}, \mathcal{B}) \neq \mathrm{DS}(\mathfrak{A})$.
- The notion of the ω-degree spectrum is a generalization of the relative spectrum: $\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \mathrm{DS}(\mathfrak{A}, \mathcal{B})$, where $\mathcal{B} = \{B_k\}_{k<\omega}$,
  - $B_0 = \emptyset$,
  - B_k is the positive diagram of the structure $\mathfrak{A}_k$, $k \leq n$
  - $B_k = \emptyset$ for all $k > n$.

ω- Degree Spectra and Jump Spectra

Proposition

$\mathrm{DS}(\mathfrak{A}, \mathcal{B})$ *is upwards closed with respect to total e-degrees.*

Definition

The kth ω-jump spectrum of $\mathfrak{A}$ with respect to $\mathcal{B}$ is the set

$$\mathrm{DS}_k(\mathfrak{A}, \mathcal{B}) = \{\mathbf{a}^{(\mathbf{k})} \mid \mathbf{a} \in \mathrm{DS}(\mathfrak{A}, \mathcal{B})\}.$$

Proposition

$\mathrm{DS}_k(\mathfrak{A}, \mathcal{B})$ *is upwards closed with respect to total e-degrees.*

ω-Co-Spectra

For every $\mathcal{A} \subseteq \mathcal{D}_\omega$ let
$co(\mathcal{A}) = \{\mathbf{b} \mid \mathbf{b} \in \mathcal{D}_\omega \ \& \ (\forall \mathbf{a} \in \mathcal{A})(\mathbf{b} \leq_\omega \mathbf{a})\}$.

Definition

The ω-co-spectrum of $\mathfrak{A}$ with respect to $\mathcal{B}$ is the set

$$\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = co(\mathrm{DS}(\mathfrak{A}, \mathcal{B})).$$

Definition

The kth ω-co-spectrum of $\mathfrak{A}$ with respect to $\mathcal{B}$ is the set

$$\mathrm{CS}_k(\mathfrak{A}, \mathcal{B}) = co(\mathrm{DS}_k(\mathfrak{A}, \mathcal{B})).$$

Normal Form Theorem

Let $\mathcal{L}$ be the language of the structure $\mathfrak{A}$. For each n let P_n be a new unary predicate representing the set B_n.

- An elementary Σ_0^+ formula is an existential formula of the form $\exists Y_1 \dots \exists Y_m \Phi(W_1, \dots, W_r, Y_1, \dots, Y_m)$, where Φ is a finite conjunction of atomic formulae in $\mathcal{L} \cup \{P_0\}$;
- A Σ_n^+ formula is a c.e. disjunction of elementary Σ_n^+ formulae;
- An elementary Σ_{n+1}^+ formula is a formula of the form $\exists Y_1 \dots \exists Y_m \Phi(W_1, \dots, W_r, Y_1, \dots, Y_m)$, where Φ is a finite conjunction of atoms of the form $P_{n+1}(Y_j)$ or $P_{n+1}(W_i)$ and Σ_n^+ formulae or negations of Σ_n^+ formulae in $\mathcal{L} \cup \{P_0\} \cup \dots \cup \{P_n\}$.

Normal Form Theorem

Definition

The sequence $\mathcal{A} = \{A_n\}_{n<\omega}$ is *formally k-definable* on $\mathfrak{A}$ with respect to $\mathcal{B}$ if there exists a computable sequence $\{\Phi^{\gamma(n,x)}(W_1, \dots, W_r)\}_{n,x<\omega}$ of Σ^+_{n+k} formulae and elements $t_1, \dots, t_r$ of $\mathbb{N}$ such that for every $x \in \mathbb{N}$, the following equivalence holds:

$$x \in A_n \iff (\mathfrak{A}, \mathcal{B}) \models \Phi^{\gamma(n,x)}(W_1/t_1, \dots, W_r/t_r).$$

Theorem

The sequence $\mathcal{A}$ is formally k-definable *on $\mathfrak{A}$ with respect to $\mathcal{B}$ iff $d_\omega(\mathcal{A}) \in \mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$.*

Properties of upwards closed sets

Let $\mathcal{A} \subseteq \mathcal{D}_e$ be an upwards closed set with respect to total e-degrees.

Proposition

$co(\mathcal{A}) = co(\{\mathbf{a} : \mathbf{a} \in \mathcal{A} \ \& \ \mathbf{a} \text{ is total}\})$.

Corrolary

$\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = co(\{\mathbf{a} \mid \mathbf{a} \in \mathrm{DS}(\mathfrak{A}, \mathcal{B}) \ \& \ \mathbf{a} \text{ is a total e-degree}\})$.

Negative results (Vatev)

Let $\mathcal{A} \subseteq \mathcal{D}_e$ be an upwards closed set with respect to total e-degrees and $k > 0$.

- There exists $\mathbf{b} \in \mathcal{D}_e$ such that

$$co(\mathcal{A}) \neq co(\{\mathbf{a} : \mathbf{a} \in \mathcal{A} \ \& \ \mathbf{b} \leq \mathbf{a}^{(k)}\}).$$

- Let $n > 0$. There is a structure $\mathfrak{A}$, a sequence $\mathcal{B}$ and $\mathbf{c} \in \mathrm{DS}_n(\mathfrak{A}, \mathcal{B})$ such that

$$\mathrm{CS}(\mathfrak{A}, \mathcal{B}) \neq co(\{\mathbf{a} \in \mathrm{DS}(\mathfrak{A}, \mathcal{B}) \mid \mathbf{a}^{(n)} = \mathbf{c}\}).$$

Minimal pair theorem

Theorem

For every structure $\mathfrak{A}$ and every sequence $\mathcal{B} \in \mathcal{S}$ there exist total enumeration degrees **f** *and* **g** *in* $\mathrm{DS}(\mathfrak{A}, \mathcal{B})$ *such that for every ω-enumeration degree* **a** *and $k \in \mathbb{N}$:*

$$\mathbf{a} \leq_\omega \mathbf{f}^{(k)} \ \& \ \mathbf{a} \leq_\omega \mathbf{g}^{(k)} \Rightarrow \mathbf{a} \in \mathrm{CS}_k(\mathfrak{A}, \mathcal{B}) \ .$$

Countable ideals of ω-enumeration degrees

Corrolary

$\mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$ *is the least ideal containing all kth ω-jumps of the elements of* $\mathrm{CS}(\mathfrak{A}, \mathcal{B})$.

- $I = \mathrm{CS}(\mathfrak{A}, \mathcal{B})$ is a countable ideal;
- $\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = I(\mathbf{f}) \cap I(\mathbf{g})$;
- $I^{(k)}$ - the least ideal, containing all kth ω-jumps of the elements of I;
- (Ganchev)
  $I = I(\mathbf{f}) \cap I(\mathbf{g}) \Longrightarrow I^{(k)} = I(\mathbf{f}^{(k)}) \cap I(\mathbf{g}^{(k)})$ for every k;
- $I(\mathbf{f}^{(k)}) \cap I(\mathbf{g}^{(k)}) = \mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$ for each k
- Thus $I^{(k)} = \mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$.

Countable ideals of ω-enumeration degrees

There is a countable ideal I of ω-enumeration degrees for which there is no structure $\mathfrak{A}$ and sequence $\mathcal{B}$ such that $I = \mathrm{CS}(\mathfrak{A}, \mathcal{B})$.

- $\mathcal{A} = \{\mathbf{0}, \mathbf{0}', \mathbf{0}'', \dots, \mathbf{0}^{(n)}, \dots\}$;
- $I = I(\mathcal{A}) = \{\mathbf{a} \mid \mathbf{a} \in \mathcal{D}_\omega \ \& \ (\exists n)(\mathbf{a} \leq_\omega \mathbf{0}^{(n)})\}$ - a countable ideal generated by $\mathcal{A}$.
- Assume that there is a structure $\mathfrak{A}$ and a sequence $\mathcal{B}$ such that $I = \mathrm{CS}(\mathfrak{A}, \mathcal{B})$
- Then there is a minimal pair $\mathbf{f}$ and $\mathbf{g}$ for $\mathrm{DS}(\mathfrak{A}, \mathcal{B})$, so $I^{(n)} = I(\mathbf{f}^{(n)}) \cap I(\mathbf{g}^{(n)})$ for each n.
- $\mathbf{f} \geq \mathbf{0}^{(n)}$ and $\mathbf{g} \geq \mathbf{0}^{(n)}$ for each n.
- Then by Enderton and Putnam [1970], Sacks [1971]: $\mathbf{f}'' \geq \mathbf{0}^{(\omega)}$ and $\mathbf{g}'' \geq \mathbf{0}^{(\omega)}$.
- Hence $I'' \neq I(\mathbf{f}'') \cap I(\mathbf{g}'')$. A contradiction.

Quasi-Minimal Degree

Theorem

For every structure $\mathfrak{A}$ and every sequence $\mathcal{B}$, there exists $F \subseteq \mathbb{N}$, such that $\mathbf{q} = d_\omega(F \uparrow \omega)$ and:

1. $\mathbf{q} \notin \mathrm{CS}(\mathfrak{A}, \mathcal{B})$;
2. *If $\mathbf{a}$ is a total e-degree and $\mathbf{a} \geq_\omega \mathbf{q}$ then $\mathbf{a} \in \mathrm{DS}(\mathfrak{A}, \mathcal{B})$*
3. *If $\mathbf{a}$ is a total e-degree and $\mathbf{a} \leq_\omega \mathbf{q}$ then $\mathbf{a} \in \mathrm{CS}(\mathfrak{A}, \mathcal{B})$.*

ω-degree spectra

- Questions:
  - Is it true that for every structure $\mathfrak{A}$ and every sequence $\mathcal{B}$ there exists a structure $\mathfrak{B}$ such that $\mathrm{DS}(\mathfrak{B}) = \mathrm{DS}(\mathfrak{A}, \mathcal{B})$?
  - If for a countable ideal $I \subseteq \mathcal{D}_\omega$ there is an exact pair then are there a structure $\mathfrak{A}$ and a sequence $\mathcal{B}$ so that $\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = I$?

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