A Jump Inversion Theorem for the Degree Spectra

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05.05.2008
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NOCAS - RISC

Outline

- Enumeration degrees
- Degree spectra and jump spectra
- Representing the countable ideals as co-spectra
- Properties of upwards closed set of degrees
- The minimal pair theorem
- Quasi-minimal degrees
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Computable Sets

Definition

A set $A \subseteq \mathbb{N}$ is computable if there is a computer program that, on input n, decides whether $n \in A$.

Church-Turing thesis: This definition is independent of the programming language chosen.

Example

The following sets are computable:

- The set of even numbers.
- The set of prime numbers.
- The set of stings that correspond to well-formed programs.

Recall that any finite object can be encoded by a natural number.

Basic definitions

Given sets $A, B \subseteq \mathbb{N}$ we say that **A is computable in B**, and we write $A \leq_T B$, if there is a computable procedure that can tell whether an element is in A or not, using B as an oracle.
We say that **A is Turing equivalent to B**, and we write $A \equiv_T B$ if $A \leq_T B$ and $B \leq_T A$.
We let $\mathbf{D} = (\mathcal{P}(\mathbb{N})/\equiv_T)$, and $\mathcal{D}_T = (\mathbf{D}, \leq_T)$.

There is a least degree $\mathbf{0}$.
The degree of the computable sets.

Operations on $\mathcal{D}_T$

Turing Join

Given $A, B \subseteq \mathbb{N}$, we let $A \oplus B = \{2n : n \in A\} \cup \{2n+1 : n \in B\}$.
Clearly $A \leq_T A \oplus B$ and $B \leq_T A \oplus B$,
and if both $A \leq_T C$ and $B \leq_T C$ then $A \oplus B \leq_T C$.

Turing Jump

Given $A \subseteq \mathbb{N}$, we let A' be the Turing jump of A, that is,

$A' =$ {programs, with oracle A, that **HALT** }.

$A' = \{x \mid P_x^A(x) \text{ halts } \} = K_A$.

For $\mathbf{a} \in \mathbf{D}$, let $\mathbf{a}'$ be the degree of the Turing jump of any set in $\mathbf{a}$

- $\mathbf{a} <_\mathbf{T} \mathbf{a}'$
- If $\mathbf{a} \leq_\mathbf{T} \mathbf{b}$ then $\mathbf{a}' \leq_\mathbf{T} \mathbf{b}'$.

Enumeration degrees

A set A is enumeration reducible to a set B, denoted by $A \leq_{\mathrm{e}} B$, if there is an effective procedure to enumerate A given any enumeration of B.

Definition (Enumeration operator)

$\Gamma_z : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$:

$$x \in \Gamma_z(B) \iff \exists v(\langle v, x\rangle \in W_z \;\&\; D_v \subseteq B).$$

D_v – the finite set having canonical code v,
$W_0, \ldots, W_z, \ldots$ – the Gödel enumeration of the c.e. sets.

- A is enumeration reducible to B, $A \leq_e B$, if $A = \Gamma_z(B)$ for some enumeration operator Γ_z.
- $A \equiv_e B \iff A \leq_e B \;\&\; B \leq_e A$.
- $\mathrm{d}_e(A) = \{B : B \equiv_e A\}$
- The least degree $\mathbf{0}_e$ is he degree of the computable enumerable sets.
- $\mathcal{D}_e = (\mathcal{D}_e, \leq_e, \mathbf{0}_e)$ – the structure of e-degrees.

Definition (A total set)

- $A^+ = A \oplus (\mathbb{N} \setminus A)$.
- A is **total** iff $A \equiv_e A^+$.
- A degree is **total** if it contains a total set.

The substructure $\mathcal{D}_T$ of $\mathcal{D}_e$ consisting of all total degrees is isomorphic of the structure of the Turing degrees.

- $A \leq_T B$ iff $A^+ \leq_e B^+$.
- $A \leq_{c.e.} B$ iff $A \leq_e B^+$.

The enumeration jump operator is defined by **Cooper**:

Definition (Enumeration jump)

Given a set A, let

- $L_A = \{\langle x, z\rangle : x \in \Gamma_z(A)\}$.
- $A' = (L_A)^+$.
- $A^{(n+1)} = (A^{(n)})'$.

- If $A \leq_e B$, then $A' \leq_e B'$.
- A is Σ^0_{n+1} relatively to B iff $A \leq_e (B^+)^{(n)}$.

- (Selman) If for all total X $(B \leq_e X^{(n)} \Rightarrow A \leq_e X^{(n)})$, then $A \leq_e B \oplus 0_e^{(n)}$.

Enumeration of a Structure

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_k, =)$ be a countable abstract structure.

- An enumeration f of $\mathfrak{A}$ is a total mapping from $\mathbb{N}$ onto $\mathbb{N}$.
- For each predicate R of $\mathfrak{A}$:

$$f^{-1}(R) = \{\langle x_1, \ldots, x_r \rangle \mid R(f(x_1), \ldots, f(x_r))\}.$$

- $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k) \oplus f^{-1}(=)$.

Degree Spectra

Definition

The degree spectrum of $\mathfrak{A}$ is the set

$$\mathrm{DS}(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) \mid f \text{ is an enumeration of } \mathfrak{A}\}.$$

- L. Richter [1981], J. Knight [1986].
- Let ι be the Roger's embedding of the Turing degrees into the enumeration degrees and $\mathfrak{A}$ is a total structure. Then

$$\mathrm{DS}(\mathfrak{A}) = \{\iota(d_T(f^{-1}(\mathfrak{A}))) \mid f \text{ is an enumeration of } \mathfrak{A}\}.$$

- The n-th jump spectrum of $\mathfrak{A}$ is the set $\mathrm{DS}_n(\mathfrak{A}) = \{\mathbf{a}^{(n)} \mid \mathbf{a} \in \mathrm{DS}(\mathfrak{A})\}$.

Co-spectra of structures

Definition

Let $\emptyset \neq \mathcal{A} \subseteq \mathcal{D}_e$.
The co-set of $\mathcal{A}$ is the set $co(\mathcal{A})$ of all lower bounds of $\mathcal{A}$:

$$co(\mathcal{A}) = \{\mathbf{b} : \mathbf{b} \in \mathcal{D}_e \ \& \ (\forall \mathbf{a} \in \mathcal{A})(\mathbf{b} \leq \mathbf{a})\}.$$

Example

Fix a $\mathbf{d} \in \mathcal{D}_e$ and let $\mathcal{A}_\mathbf{d} = \{\mathbf{a} : \mathbf{a} \geq \mathbf{d}\}$. Then $co(\mathcal{A}_\mathbf{d}) = \{\mathbf{b} : \mathbf{b} \leq \mathbf{d}\}$.

- $co(\mathcal{A})$ is a countable ideal.

Definition

The co-spectrum of $\mathfrak{A}$ is the co-set of $\mathrm{DS}(\mathfrak{A})$:

$$\mathrm{CS}(\mathfrak{A}) = \{\mathbf{b} : (\forall \mathbf{a} \in \mathrm{DS}(\mathfrak{A}))(\mathbf{b} \leq \mathbf{a})\}.$$

Definition

The n-th co-spectrum of $\mathfrak{A}$ is the set $\mathrm{CS}_n(\mathfrak{A}) = co(\mathrm{DS}_n(\mathfrak{A}))$.

- If $\mathrm{DS}(\mathfrak{A})$ contains a least element **a**, then **a** is called the degree of $\mathfrak{A}$.
- If $\mathrm{DS}_n(\mathfrak{A})$ contains a least element **a**, then **a** is called the n-th jump degree of $\mathfrak{A}$.
- If $\mathrm{CS}(\mathfrak{A})$ contains a greatest element **a**, then **a** is called the co-degree of $\mathfrak{A}$.
- If $\mathrm{CS}_n(\mathfrak{A})$ contains a greatest element **a**, then **a** is called the n-th jump co-degree of $\mathfrak{A}$.
- **Observation:** If $\mathfrak{A}$ has n-th jump degree **a**, then **a** is also n-th jump co-degree of $\mathfrak{A}$. The opposite is not always true.

Some Examples

1981 (Richter) Let $\mathfrak{A} = (\mathbb{N}; <, =, \neq)$ be a linear ordering.

- $\mathrm{DS}(\mathfrak{A})$ contains a minimal pair of degrees, $\mathrm{CS}(\mathfrak{A}) = \{\mathbf{0}_e\}$.
- If $\mathrm{DS}(\mathfrak{A})$ has a degree $\mathbf{a}$, then $\mathbf{a} = \mathbf{0}_e$.

1986 (Knight 1986) Consider again a linear ordering $\mathfrak{A}$.

- $\mathrm{CS}_1(\mathfrak{A})$ consists of all Σ^0_2 sets.
- The first jump co-degree of $\mathfrak{A}$ is $\mathbf{0}'_e$.

1998 (Slaman, Wehner)
$\mathrm{DS}(\mathfrak{A}) = \{\mathbf{a} : \mathbf{a} \text{ is total and } \mathbf{0}_e < \mathbf{a}\}$,

- $\mathrm{CS}(\mathfrak{A}) = \{\mathbf{0}_e\}$.
  The structure $\mathfrak{A}$ has co-degree $\mathbf{0}_e$ but has not a degree.

1998 (Coles, Downey, Slaman, Soskov) Let G be a subgroup of Q. There exists an e-degree $\mathbf{s}_G$:

$$\mathrm{DS}(G) = \{\mathbf{b} : \mathbf{b} \text{ is total and } \mathbf{s}_G \leq \mathbf{b}\}.$$

- The co-degree of G is $\mathbf{s}_G$.
- G has a degree iff $\mathbf{s}_G$ is total
- If $1 \leq n$, then $\mathbf{s}_G^{(n)}$ is the n-th jump degree of G.

For every $\mathbf{d} \in \mathcal{D}_e$ there exists a G, s.t. $\mathbf{s}_G = \mathbf{d}$. Hence every principle ideal of enumeration degrees is $\mathrm{CS}(G)$ for some G.

2002 (Soskov) Every countable ideals is CS of structures. Let $B_0, \ldots, B_n, \ldots$ be a sequence of sets of natural numbers. Set $\mathfrak{A} = (\mathbb{N}; G_\varphi; \sigma, =, \neq)$,

$$\varphi(\langle i, n\rangle) = \langle i+1, n\rangle;$$
$$\sigma = \{\langle i, n\rangle : n = 2k+1 \vee n = 2k \ \&\ i \in B_k\}.$$

- $\mathrm{CS}(\mathfrak{A}) = I(d_e(B_0), \ldots, d_e(B_n), \ldots)$
  - $I \subseteq \mathrm{CS}(\mathfrak{A}) : B_k \leq_e f^{-1}(\mathfrak{A})$ for each k;
  - $\mathrm{CS}(\mathfrak{A}) \subseteq I$: if $d_e(A) \in \mathrm{CS}(\mathfrak{A})$, then $A \leq_e B_k$ for some k.

Properties of the degree spectra

Let $\mathcal{A} \subseteq \mathcal{D}_e$. Then $\mathcal{A}$ is upwards closed if

$$\mathbf{a} \in \mathcal{A}, \mathbf{b} \text{ is total and } \mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{b} \in \mathcal{A}.$$

- The degree spectra are upwards closed.
- General properties of upwards closed sets of degrees.

Theorem

Let $\mathcal{A}$ be an upwards closed set of degrees. Then

(1) $co(\mathcal{A}) = co(\{\mathbf{b} \in \mathcal{A} : \mathbf{b} \text{ is total}\})$.

(2) *Let* $1 \leq n$ *and* $\mathbf{c} \in \mathcal{D}_e$. *Then*

$$co(\mathcal{A}) = co(\{\mathbf{b} \in \mathcal{A} : \mathbf{c} \leq \mathbf{b}^{(n)}\}).$$

Specific properties

Theorem

Let $\mathfrak{A}$ *be a structure,* $1 \leq n$*, and* $\mathbf{c} \in \mathrm{DS}_n(\mathfrak{A})$*. Then*

$$\mathrm{CS}(\mathfrak{A}) = co(\{\mathbf{b} \in \mathrm{DS}(\mathfrak{A}) : \mathbf{b}^{(n)} = \mathbf{c}\}).$$

Example

Let $B \not\leq_e A$ and $A \not\leq_e B'$. Set

$$\mathcal{D} = \{\mathbf{a} : \mathbf{a} \geq d_e(A)\} \cup \{\mathbf{a} : \mathbf{a} \geq d_e(B)\}.$$

$$\mathcal{A} = \{\mathbf{a} : \mathbf{a} \in \mathcal{D} \ \& \ \mathbf{a}' = d_e(B)'\}.$$

- $d_e(B)$ is the least element of $\mathcal{A}$ and hence $d_e(B) \in co(\mathcal{A})$.
- $d_e(B) \not\leq d_e(A)$ and hence $d_e(B) \notin co(\mathcal{D})$.

Minimal Pair Type Theorems

Theorem

There exist elements $\mathbf{f}_0$ *and* $\mathbf{f}_1$ *of* $\mathrm{DS}(\mathfrak{A})$ *such that for every n*

- $\mathbf{f}_0^{(n)}$ *and* $\mathbf{f}_1^{(n)}$ *do not belong to* $\mathrm{CS}_n(\mathfrak{A})$.
- $co(\{\mathbf{f}_0^{(n)}, \mathbf{f}_1^{(n)}\}) = \mathrm{CS}_n(\mathfrak{A})$.

Example

Finite lattice $L = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{a} \wedge \mathbf{b}, \mathbf{a} \wedge \mathbf{c}, \mathbf{b} \wedge \mathbf{c}, \top, \bot\}$.

$$\mathcal{A} = \{\mathbf{d} \in \mathcal{D}_e : \mathbf{d} \geq \mathbf{a} \vee \mathbf{d} \geq \mathbf{b} \vee \mathbf{d} \geq \mathbf{c}\}.$$

The Quasi-minimal degree

Definition

Let $\mathcal{A}$ be a set of enumeration degrees. The degree **q** is quasi-minimal with respect to $\mathcal{A}$ if:

- **q** $\notin co(\mathcal{A})$.
- If **a** is total and **a** $\geq$ **q**, then **a** $\in \mathcal{A}$.
- If **a** is total and **a** $\leq$ **q**, then **a** $\in co(\mathcal{A})$.

Theorem

*If **q** is quasi-minimal with respect to $\mathcal{A}$, then **q** is an upper bound of $co(\mathcal{A})$.*

Theorem

For every structure $\mathfrak{A}$ there exists a quasi-minimal with respect to $\mathrm{DS}(\mathfrak{A})$ degree.

For any countable structures $\mathfrak{A}$ and $\mathfrak{B}$ define the relation

$$\mathfrak{B} \preceq \mathfrak{A} \iff \mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}(\mathfrak{B}) .$$

- $\mathfrak{A} \equiv \mathfrak{B}$ if $\mathfrak{A} \preceq \mathfrak{B}$ and $\mathfrak{B} \preceq \mathfrak{A}$.
- $\mathfrak{B}' \preceq \mathfrak{A}$ if $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_1(\mathfrak{B})$.
- $\mathfrak{A} \preceq \mathfrak{B}'$ if $\mathrm{DS}_1(\mathfrak{B}) \subseteq \mathrm{DS}(\mathfrak{A})$.
- $\mathfrak{A} \equiv \mathfrak{B}'$ if $\mathfrak{A} \preceq \mathfrak{B}'$ and $\mathfrak{B}' \preceq \mathfrak{A}$.

Theorem

Each jump spectrum is degree spectrum of a structure, i.e. for every structure $\mathfrak{A}$ there exists a structure $\mathfrak{B}$ such that $\mathfrak{A}' \equiv \mathfrak{B}$.

Definition
Moschovakis' extension

- $\bar{0} \notin \mathbb{N}$, $\mathbb{N}_0 = \mathbb{N} \cup \{\bar{0}\}$.
- A pairing function $\langle .,. \rangle$, $\mathrm{range}(\langle .,. \rangle) \cap \mathbb{N}_0 = \emptyset$.
- The least set $\mathbb{N}^* \supseteq \mathbb{N}_0$, closed under $\langle .,. \rangle$.
- Moschovakis' extension of $\mathfrak{A}$ is the structure $\mathfrak{A}^* = (\mathbb{N}^*, R_1, \ldots, R_n, =, \mathbb{N}_0, G_{\langle .,. \rangle})$.
- $\mathfrak{A} \equiv \mathfrak{A}^*$.
- A new predicate $K_{\mathfrak{A}}$ (analogue of Kleene's set).
- For $e, x \in \mathbb{N}$ and finite part τ, let $\tau \Vdash F_e(x) \iff x \in \Gamma_e(\tau^{-1}(\mathfrak{A}))$.
- $K_{\mathfrak{A}} = \{\langle \delta^*, e, x \rangle : (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}$.
- $\mathfrak{B} = (\mathfrak{A}^*, K_{\mathfrak{A}})$.
- $\mathrm{DS}_1(\mathfrak{A}) = \mathrm{DS}(\mathfrak{B})$.

Question (Inverting the jump)

Given a set of enumeration degrees $\mathcal{A}$ does there exist a structure $\mathfrak{C}$ such that $\mathrm{DS}_1(\mathfrak{C}) = \mathcal{A}$?

1. Each element of $\mathcal{A}$ should be a jump of a degree.
2. $\mathcal{A}$ should be upwards closed (since each jump spectrum is a spectrum and the spectrum is upwards closed).

Problem

Not any upwards closed set of enumeration degrees is a spectrum of a structure and hence a jump spectrum.

A subset $\mathcal{B}$ of $\mathcal{A}$ is called base of $\mathcal{A}$ if for every element **a** of $\mathcal{A}$ there exists an element $\mathbf{b} \in \mathcal{B}$ such that $\mathbf{b} \leq \mathbf{a}$.

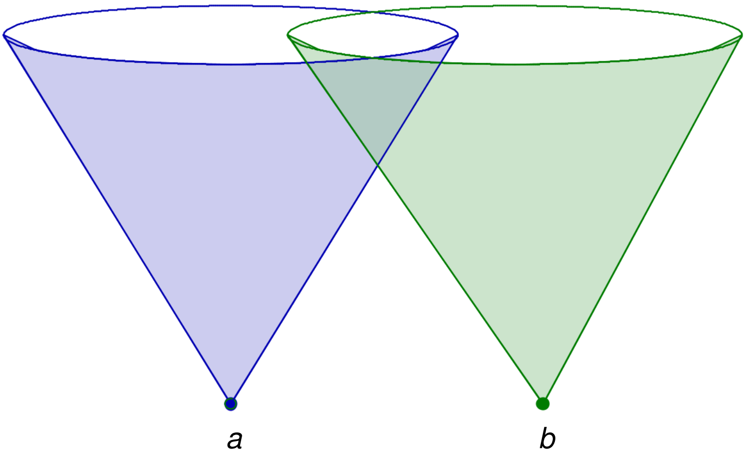
Proposition

If $\mathrm{DS}(\mathfrak{A})$ *has a countable base of total enumeration degrees, then* $\mathrm{DS}(\mathfrak{A})$ *has a least element.*

Example

Let **a** and **b** be incomparable enumeration degrees. Then there does not exist a structure $\mathfrak{A}$ such that:

$$\mathrm{DS}(\mathfrak{A}) = \{\mathbf{c} : \mathbf{c} \text{ is total } \& \ \mathbf{c} \geq \mathbf{a}\} \cup \{\mathbf{c} : \mathbf{c} \text{ is total } \& \ \mathbf{c} \geq \mathbf{b}\}.$$


a
b

- The set $\mathcal{A}$ should be a degree spectrum of a structure $\mathfrak{A}$.
- $\mathrm{DS}(\mathfrak{A})$ should contain only jumps of enumeration degrees.

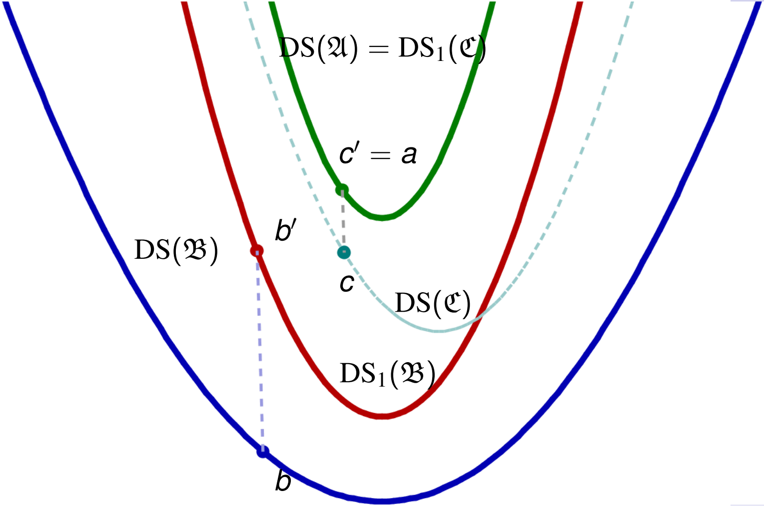
More generally:

Theorem (Jump Inversion Theorem)

If $\mathfrak{A}$ and $\mathfrak{B}$ are structures and $\mathfrak{B}' \preceq \mathfrak{A}$ then there exists a structure $\mathfrak{C}$ such that $\mathfrak{B} \preceq \mathfrak{C}$ and $\mathfrak{C}' \equiv \mathfrak{A}$.

- The structure $\mathfrak{C}$ we shall construct as a Marker's extension of $\mathfrak{A}$.
- We code the structure $\mathfrak{B}$ in $\mathfrak{C}$.
- In our construction we use also the relativized representation lemma for Σ^0_2 sets proved by Goncharov and Khoussainov

Theorem $\mathfrak{B}' \preceq \mathfrak{A} \Longrightarrow (\exists \mathfrak{C})(\mathfrak{B} \preceq \mathfrak{C} \;\&\; \mathfrak{C}' \equiv \mathfrak{A})$.



Marker's Extensions

Let $\mathfrak{A} = (A; R_1, \ldots, R_s, =)$.
$R^{\exists}$ — Marker's $\exists$-extension of R:

- $\exists$-fellow for R — $X = \{x_{\langle a_1,\ldots,a_r\rangle} \mid R(a_1, \ldots, a_r)\}$.
- $R^{\exists}(a_1, \ldots, a_r, x) \iff$
  $a_1, \ldots, a_r \in A \;\&\; x \in X \;\&\; x = x_{\langle a_1,\ldots,a_r\rangle}$.
- $\mathfrak{A}^{\exists} = (A \cup \bigcup_{i=1}^{s} X_i, R_1^{\exists}, \ldots, R_s^{\exists}, \bar{X}_1, \ldots, \bar{X}_s, =)$.

$R^{\forall}$ — Marker's $\forall$-extension of R:

- $\forall$-fellow for R — $Y = \{y_{\langle a_1,\ldots,a_r\rangle} \mid \neg R(a_1, \ldots, a_r)\}$.
-
  1. If $R^{\forall}(a_1, \ldots, a_r, y)$ then $a_1, \ldots, a_r \in A$ and $y \in Y$;
  2. If $a_1, \ldots, a_r \in A \;\&\; y \in Y$ then
     $\neg R^{\forall}(a_1, \ldots, a_r, y) \iff y = y_{\langle a_1,\ldots,a_r\rangle}$.
- $\mathfrak{A}^{\forall} = (A \cup \bigcup_{i=1}^{s} Y_i, R_1^{\forall}, \ldots, R_s^{\forall}, \bar{Y}_1, \ldots, \bar{Y}_s, =)$.

Properties of $\mathfrak{A}^{\exists\forall}$

Definition

The structure $\mathfrak{A}^{\exists\forall}$ is obtained from $\mathfrak{A}$ as $(\mathfrak{A}^{\exists})^{\forall}$.

1. $R(a_1, \ldots, a_r) \iff (\exists x \in X)(\forall y \in Y) R^{\exists\forall}(a_1, \ldots, a_r, x, y)$;
2. $(\forall y \in Y)(\exists$ a unique sequence $a_1, \ldots, a_r \in A \;\&\; x \in X)(\neg R^{\exists\forall}(a_1, \ldots, a_r, x, y))$;
3. $(\forall x \in X)(\exists$ a unique sequence $a_1, \ldots, a_r \in A)(\forall y \in Y) R^{\exists\forall}(a_1, \ldots, a_r, x, y)$.

Join of Two Structures

Let $\mathfrak{A} = (A; R_1, \dots, R_s, =)$ and $\mathfrak{B} = (B; P_1, \dots, P_t, =)$ be countable structures.

The join of the structures $\mathfrak{A}$ and $\mathfrak{B}$ is the structure $\mathfrak{A} \oplus \mathfrak{B} = (A \cup B; R_1, \dots, R_s, P_1, \dots, P_t, \bar{A}, \bar{B}, =)$

(a) the predicate $\bar{A}$ is true only over the elements of A and similarly $\bar{B}$ is true only over the elements of B;

(b) the predicate R_i is defined on the elements of A as in the structure $\mathfrak{A}$ and false on all elements not in A and the predicate P_j is defined similarly.

Lemma

$\mathfrak{A} \preceq \mathfrak{A} \oplus \mathfrak{B}$ *and* $\mathfrak{B} \preceq \mathfrak{A} \oplus \mathfrak{B}$.

One-to-one Representation of $\Sigma^0_2(D)$ Sets

Let $D \subseteq \mathbb{N}$.
A set $M \subseteq \mathbb{N}$ is in $\Sigma^0_2(D)$ if there exists a computable in D predicate Q such that
$n \in M \iff \exists a \forall b Q(n, a, b)$.

Definition

If $M \in \Sigma^0_2(D)$ then M is one-to-one representable if there is a computable in D predicate Q with the following properties:

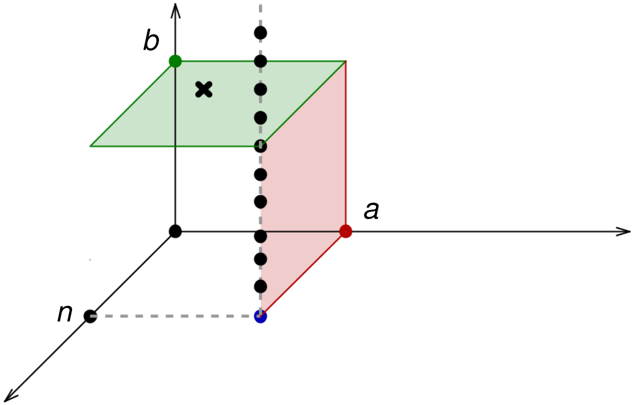
1. $n \in M \Leftrightarrow (\exists$ a unique $a)(\forall b)Q(n, a, b)$;
2. $(\forall b)(\exists$ a unique pair $\langle n, a\rangle)(\neg Q(n, a, b))$;
3. $(\forall a)(\exists$ a unique $n)(\forall b)Q(n, a, b)$.

Lemma (Goncharov and Khoussainov)

If M is a co-infinite $\Sigma^0_2(D)$ subset of $\mathbb{N}$ which has an infinite computable in D subset S such that $M \setminus S$ is infinite then M has an one-to-one representation.

One-to-one Representation of $\Sigma^0_2(D)$ Sets

1. $n \in M \Leftrightarrow (\exists \text{ a unique } a)(\forall b)Q(n, a, b)$;
2. $(\forall b)(\exists \text{ a unique pair } \langle n, a\rangle)(\neg Q(n, a, b))$;
3. $(\forall a)(\exists \text{ a unique } n)(\forall b)Q(n, a, b)$.



Theorem (Jump Inversion Theorem)

Let $\mathfrak{B}' \preceq \mathfrak{A}$. Then there exists a structure $\mathfrak{C}$ such that $\mathfrak{B} \preceq \mathfrak{C}$ and $\mathfrak{C}' \equiv \mathfrak{A}$.

- The structure $\mathfrak{C}$ is constructed as

$$\mathfrak{C} = \mathfrak{B} \oplus \mathfrak{A}^{\exists\forall}.$$

- $\mathrm{DS}_1(\mathfrak{C}) \subseteq \mathrm{DS}(\mathfrak{A})$.
- $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_1(\mathfrak{C})$.
- We use the one-to-one representation lemma.
- We use the fact that the degree spectra and the jump spectra are upwards closed with respect to total degrees

Some Applications

Definition

If **a** is the least element of $\mathrm{DS}(\mathfrak{A})$ then **a** is called the *degree of* $\mathfrak{A}$.

Proposition

Let $\mathfrak{B}' \preceq \mathfrak{A}$ *and suppose that the structure* $\mathfrak{A}$ *has a degree. Then there exists a torsion free abelian group* $\mathfrak{G}$ *of rank 1 which has a degree as well and such that* $\mathfrak{B} \preceq \mathfrak{G}$ *and* $\mathfrak{G}' \equiv \mathfrak{A}$*.*

Some Applications

$\mathrm{DS}_0(\mathfrak{A}) = \mathrm{DS}(\mathfrak{A})$ $\mathrm{DS}_{n+1}(\mathfrak{A}) = \{\mathbf{a}' : \mathbf{a} \in \mathrm{DS}_n(\mathfrak{A})\}$.
By induction on n we show that for each n there is a structure $\mathfrak{A}^{(n)}$ such that $\mathrm{DS}_n(\mathfrak{A}) = \mathrm{DS}(\mathfrak{A}^{(n)})$.

Theorem

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures such that $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_n(\mathfrak{B})$. Then there is a structure $\mathfrak{C}$such that $\mathrm{DS}(\mathfrak{C}) \subseteq \mathrm{DS}(\mathfrak{B})$ è $\mathrm{DS}_n(\mathfrak{C}) = \mathrm{DS}(\mathfrak{A})$.

Some Applications

(C1) $\mathrm{DS}(\mathfrak{A}) \subseteq \{\mathbf{a} : \mathbf{0^{(n)}} \leq \mathbf{a}\}$.

(C2) $\mathrm{DS}(\mathfrak{A})$ has no least element.

(C3) $\mathfrak{A}$ has a first jump degree $= \mathbf{0}^{(n+1)}$.

- $\mathfrak{B} = (N; =)$
- $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_n(\mathfrak{B})$.

JIT There is a structure $\mathfrak{C}$ such that $\mathrm{DS}_n(\mathfrak{C}) = \mathrm{DS}(\mathfrak{A})$

- $\mathfrak{C}$ has no n-th jump degree and hence no k-th jump degree, $k \leq n$
- But $\mathrm{DS}_{n+1}(\mathfrak{C}) = \mathrm{DS}_1(\mathfrak{A})$ and hence the $(n+1)$-th jump degree of $\mathfrak{C}$ is $\mathbf{0}^{(n+1)}$.

Some Applications

Fact

For each set $A \subseteq \mathbb{N}$ there is a group $G_A \subseteq Q$ such that that:

1. $\mathrm{DS}(G_A) = \{d_{\mathrm{T}}(X) : A \text{ is c.e. in } X\}$
2. $d_{\mathrm{T}}(J_e(A))$ *is the first jump degree of* G_A ($J_e(A) = \{x : x \in W_x(A)\}$)

From the relativized variant of JIT of McEvoy, there is a set A:

1. $(\emptyset^{(n)})^+ <_e A$;
2. $(\forall X)(X^+ \leq_e A \Rightarrow X \leq_T \emptyset^{(n)})$;
3. $\emptyset^{(n+1)} \equiv_{\mathrm{T}} J_e(A)$.

Some Applications

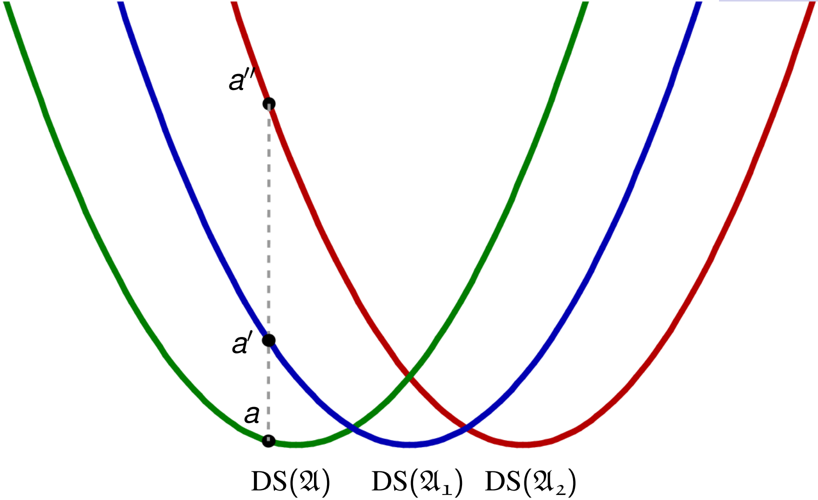
Let $\mathfrak{A} = G_A$.

(C1) $d_{\mathrm{T}}(X) \in \mathrm{DS}(\mathfrak{A}) \Rightarrow A$ is c.e. in $X \Rightarrow (\emptyset^{(n)})^+$ is c.e. in X. So $\emptyset^{(n)} \leq_{\mathrm{T}} X$.

(C2) $\mathrm{DS}(\mathfrak{A})$ has no minimal degree.
Assume that $d_{\mathrm{T}}(X)$ is the minimal element of $\mathrm{DS}(\mathfrak{A})$.
Then by Selman's theorem $X^+ \leq_e A$ and $X \leq_T \emptyset^{(n)}$.
So A is c.e. in $\emptyset^{(n)}$. It follows that $A \leq_e (\emptyset^{(n)})^+$. A contradiction.

(C3) $\mathfrak{A}$ has a first jump degree $= \mathbf{0}^{(n+1)}$.

The joint spectrum of $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2$



The Joint Spectrum of Structures

Let $\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$ be countable structures.

Definition

The joint spectrum of $\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is the set
$\mathrm{DS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) =$
$\{\mathbf{a} \mid \mathbf{a} \in \mathrm{DS}(\mathfrak{A}), \mathbf{a}' \in \mathrm{DS}(\mathfrak{A}_1), \ldots, \mathbf{a}^{(\mathbf{n})} \in \mathrm{DS}(\mathfrak{A}_n)\}$.

Corrolary

Let $\mathfrak{B}' \preceq \mathfrak{A}$. There exists a structure $\mathfrak{C} \succeq \mathfrak{B}$ such that
$\mathrm{DS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \mathrm{DS}_1(\mathfrak{C}, \mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

Relative Spectra of Structures

Definition

An enumeration f of $\mathfrak{A}$ is n-acceptable with respect to the structures $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$, if $f^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} (f^{-1}(\mathfrak{A}))^{(i)}$ for each $i \leq n$.

Definition

The relative spectrum of the structure $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is the set
$\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) =$
$\{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) \mid f$ is a n-acceptable enumeration of $\mathfrak{A}\}$.

Proposition

Let $\mathfrak{B}' \preceq \mathfrak{A}$. There exists a structure $\mathfrak{C} \succeq \mathfrak{B}$ such that $\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \mathrm{RS}_1(\mathfrak{C}, \mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

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