Sofia University "St. Kliment Ohridski"
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Mathematical Logic and Applications

Logics for relational geometric structures:

distributive mereotopology, extended contact algebras and related quantifier-free logics

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DISSERTATION
for degree PhD in Mathematics
for scientific specialty Mathematical logic

Scientific supervisor: Prof. Dimiter Vakarelov, Dr. Habil

February 11, 2017

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1. Introduction

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In the classical Euclidean geometry the notion of point is taken as one of the basic primitive notions. In contrast the region-based theory of space (RBTS) has as primitives the more realistic notion of region as an abstraction of physical body, together with some basic relations and operations on regions. Some of these relations are mereological - part-of ($x \leq y$), overlap (xOy), its dual underlap ($x\widehat{O}y$). Other relations are topological - contact (xCy), nontangential part-of ($x \ll y$), dual contact ($x\widehat{C}y$) and some others definable by means of the contact and part-of relations. This is one of the reasons that the extension of mereology with these new relations is commonly called *mereotopology*. There is no clear difference in the literature between RBTS and mereotopology, and by some authors RBTS is related rather to the so called *mereogeometry*, while mereotopology is considered only as a kind of point-free topology, considering mainly topological properties of things. The origin of RBTS goes back to Whitehead [**46**] and de Laguna [**28**]. According to Whitehead points, as well as the other primitive notions in Euclidean geometry like lines and planes, do not have separate existence in reality and because of this are not appropriate for primitive notions; but points have to be definable by the other primitive notions.

Survey papers about RBTS are [**40, 5, 17, 32**] (see also the handbook [**1**] and [**4**] for some logics of space). Surveys concerning various applications are [**6, 7**] and the book [**20**] (see also special issues of Fundamenta Informaticæ[**9**] and the Journal of Applied Nonclassical Logics [**3**]). RBTS has applications in computer science because of its more simple way of representing qualitative spatial information and it initiated a special field in Knowledge Representation (KR) called Qualitative Spatial Representation and Reasoning (QSRR). One of the most popular systems in QSRR is the Region Connection Calculus (RCC) introduced in [**33**].

The notion of contact algebra is one of the main tools in RBTS. This notion appears in the literature under different names and formulations as an extension of Boolean algebra with some mereotopological relations [**43, 35, 41, 42, 5, 13, 8, 10**]. The simplest system, called just contact algebra was introduced in [**8**] as an extension of Boolean algebra $\underline{B} = (B, 0, 1, \cdot, +, *)$ with a binary relation C called contact and satisfying several simple axioms:

(C1) If aCb, then $a \neq 0$ and $b \neq 0$,
(C2) If aCb and $a \leq c$ and $b \leq d$, then cCd,
(C3) If $aC(b + c)$, then aCb or aCc,
(C4) If aCb, then bCa,
(C5) If $a \cdot b \neq 0$, then aCb.

The elements of the Boolean algebra are called regions and are considered analogs of physical bodies. Boolean operations are considered as operations for constructing new regions from given ones. The unit element 1 symbolizes the region containing as its parts all regions, and the zero region 0 symbolizes the empty region. The contact relation is used also to define some other important mereotopological relations like non-tangential inclusion, dual contact and others.

The standard model of Boolean algebra is the algebra of subsets of a given universe. This model cannot express all kinds of contact, for example, the external contact in which the regions share only a boundary point. Because of this standard models of contact algebras are topological and are the contact algebras of regular closed sets in a given topological space.

Non-tangential inclusion and dual contact are defined by the operation of Boolean complementation. But there are some problems related to the motivation of this operation. A question arises: if the region a represents a physical body, then what kind of body represents a^*? To avoid this problem, we can drop the operation of complement and replace the Boolean part of a contact algebra with distributive lattice. First steps in this direction were made in [**11, 12**], introducing the notion of distributive contact lattice. In a distributive contact lattice the only mereotopological relation is the contact relation. In the first part of the first chapter we extend the language of distributive contact lattices by considering as non-definable primitives the relations of contact, nontangential inclusion and dual contact. We obtain an axiomatization of the theory consisting of the formulas in the language $\mathcal{L}(0, 1; +, \cdot; \leq, C, \widehat{C}, \ll)$ true in all contact algebras. The structures in $\mathcal{L}$, satisfying the axioms in question, are called extended distributive contact lattices (EDC-lattices). A representation theorem is proved, stating that each EDC-lattice can be isomorphically embedded into a contact algebra. Relations of EDC-lattices with other mereotopological systems are also considered: EDC-lattices are relational mereotopological systems in the sense of [**29**], and the well known RCC-8 system of mereotopological relations is definable in the language of EDC-lattices.

Part II of chapter 1 is devoted to the topological representation theory of EDC-lattices and some of their axiomatic extensions yielding representations in T_1 and T_2 spaces. Special attention is given to dual dense and dense representations (defined in Section 4.1) in contact algebras of regular closed and regular open subsets of topological spaces. The method is an extension of the representation theory of distributive contact lattices [**12**] and adaptation of some constructions from the representation theory of contact algebras [**8, 10**].

Since the investigations in chapter 1 form a special subfield of mereotopology based on distributive lattices, we introduce for this subfield a special name - *distributive mereotopology*, which is included in the title of the chapter. Having in mind this terminology, then the subarea of mereotopology based on Boolean algebras should be named *Boolean mereotopology*. Similar special names for other subfields of mereotopology depending on the corresponding mereological parts also can be suggested: for instance the mereotopology considered in [**19, 44, 45**] is based on some non-distributive lattices - hence *non-distributive mereotopology*, and the mereotopological structures considered, for instance, in [**29, 16**] are pure relational and without any algebraic lattice-structure in the set of regions - hence *relational mereotopology*. Another way of obtaining various new mereotopologies is considered in [**18**] by means different generalizations of Boolean complementation.

In [**38**] is presented a complete quantifier-free axiomatization of several logics on region-based theory of space, based on contact relation and connectedness predicates c and $c^{\leq n}$, and completeness theorems for the logics in question are proved. It was shown in [**38**] that c and $c^{\leq n}$ are definable in contact algebras by the contact C. The predicates c and $c^{\leq n}$ were studied for the first time in [**30, 31**] (see also [**40**]). The expressiveness and complexity of spatial logics containing c and $c^{\leq n}$ has been investigated in [**23, 24, 25, 26, 27**]. In chapter 2 we consider the predicate c^o - internal connectedness. Let X be a topological space and $x \in RC(X)$. Let $c^o(x)$ means that $Int(x)$ is a connected topological space in the subspace topology. We prove that the predicate internal connectedness cannot be defined in the language of contact algebras. Because of this we add to the language a new ternary predicate symbol $\vdash$ which has the following sense: in the contact algebra of regular closed sets of some topological space $a, b \vdash c$ iff $a \cap b \subseteq c$. It turns out that the predicate c^o can be defined in the new language. We define *extended contact algebras* - Boolean algebras with added relations $\vdash$, C and c^o, satisfying some axioms, and prove that every extended contact algebra can be isomorphically embedded in the contact algebra of the regular closed subsets of some compact, semiregular, T_0 topological space with added relations $\vdash$ and c^o. So extended contact algebra can be considered an axiomatization of the theory, consisting of the universal formulas true in all topological contact algebras with added relations $\vdash$ and c^o.

In chapter 3 we consider a first-order language without quantifiers corresponding to EDCL. We give completeness theorems with respect to both algebraic and topological semantics for several logics for this language. It turns out that all these logics are decidable. We also consider a quantifier-free first-order language corresponding to ECA and a logic for ECA which is decidable.

Acknowledgements. I thank all members of Department of Mathematical Logic and Applications at the Faculty of Mathematics and Computer Science of the Sofia University for their understanding to my health problems which led to significant delay of my work. I am grateful to all my teachers from this department for the considerable amounts of knowledge I have obtained from them. I thank my supervisors Prof. Dimiter Vakarelov and Prof. Tinko Tinchev for their mentorship and for the interesting mathematical tasks they gave me to solve.

CHAPTER 1

Distributive mereotopology: extended distributive contact lattices

In the first part of this chapter we extend the language of distributive contact lattices ([**11, 12**]) by considering as non-definable primitives also the relations of nontangential inclusion and dual contact. We obtain an axiomatization of the theory consisting of the formulas in the language $\mathcal{L}(0,1;+,\cdot;\leq,C,\widehat{C},\ll)$ true in all contact algebras. The structures in $\mathcal{L}$, satisfying the axioms in question, are called extended distributive contact lattices (EDC-lattices). A representation theorem is proved, stating that each EDC-lattice can be isomorphically embedded into a contact algebra. Relations of EDC-lattices with other mereotopological systems are also considered: EDC-lattices are relational mereotopological systems in the sense of [**29**], and the well known RCC-8 system of mereotopological relations is definable in the language of EDC-lattices.

Part II of chapter 1 is devoted to the topological representation theory of EDC-lattices and some of their axiomatic extensions yielding representations in T_1 and T_2 spaces. Special attention is given to dual dense and dense representations (defined in Section 4.1) in contact algebras of regular closed and regular open subsets of topological spaces. The method is an extension of the representation theory of distributive contact lattices [**12**] and adaptation of some constructions from the representation theory of contact algebras [**8, 10**].

PART I: EXTENDED DISTRIBUTIVE CONTACT LATTICES: AXIOMATIZATION AND EMBEDDING IN CONTACT ALGEBRAS

1. Extended distributive contact lattices. Choosing the right axioms

1.1. Contact algebras, distributive contact lattices and extended distributive contact lattices. As it was mention in the introduction, *contact algebra* is a Boolean algebra $\underline{B}=(B,\leq,0,1,\cdot,+,*,C)$ with an additional binary relation C called *contact*, and satisfying the following axioms:

(C1) If aCb, then $a\neq 0$ and $b\neq 0$,
(C2) If aCb and $a\leq a'$ and $b\leq b'$, then $a'Cb'$,
(C3) If $aC(b+c)$, then aCb or aCc,
(C4) If aCb, then bCa,
(C5) If $a\cdot b\neq 0$, then aCb.

Let us note that on the base of (C4) we have (C3') $(a+b)Cc$ implies aCc or bCc.

REMARK 1.1. Observe that the above axioms are universal first-order conditions on the language of Boolean algebra with the C-relation and not containing the Boolean complementation *. This fact says that the axioms of C will be true in any distributive sublattice of $\underline{B}$. □

The Remark 1.1 was one of the formal motivations for the definition of *distributive contact lattice* introduced in [**11, 12**]: the definition is obtained just by replacing the underlying Boolean algebra by a bounded distributive lattice $(D, \leq, 0, 1, \cdot, +)$ and taking for the relation C the same axioms. This makes possible to consider the main standard models of contact algebras, namely the algebras of regular closed or regular open sets of a topological space, also as the main models for distributive contact lattices, just by ignoring the Boolean complementation * in this models. This was guaranteed by Theorem 7 from [**12**] stating that every distributive contact lattice can be isomorphically embedded into a contact algebra, which fact indicates also that the choice of the set of axioms for distributive contact lattice is sufficient for proving this theorem. Since our main goal in the present paper is to obtain a definition of distributive contact lattice extended with relations of dual contact $\widehat{C}$ and nontangential part-of $\ll$, we will follow here the above strategy, namely to choose universal first-order statements for the relations $C, \widehat{C}, \ll$ as additional axioms which are true in arbitrary contact algebras and which guarantee the embedding into a contact algebra. The obtained algebraic system will be called *extended distributive contact lattice*. The next definition is a result of several preliminary experiments for fulfilling the above program.

DEFINITION 1.1. **Extended distributive contact lattice.** *Let* $\underline{D} = (D, \leq, 0, 1, \cdot, +, C, \widehat{C}, \ll)$ *be a bounded distributive lattice with three additional relations* $C, \widehat{C}, \ll$*, called respectively* ***contact, dual contact*** *and* ***nontangential part-of****. The obtained system, denoted shortly by* $\underline{D} = (D, C, \widehat{C}, \ll)$*, is called* ***extended distributive contact lattice*** *(EDC-lattice, for short) if it satisfies the axioms listed below.*

Notations: if R *is one of the relations* $\leq, C, \widehat{C}, \ll$*, then its complement is denoted by* $\overline{R}$*. We denote by* $\geq$ *the converse relation of* $\leq$ *and similarly* $\gg$ *denotes the converse relation of* $\ll$*.*

Axioms for C ***alone:*** *The axioms (C1)-(C5) mentioned above.*

Axioms for $\widehat{C}$ ***alone:***

($\widehat{C}1$) *If* $a\widehat{C}b$*, then* $a, b \neq 1$,

($\widehat{C}2$) *If* $a\widehat{C}b$ *and* $a' \leq a$ *and* $b' \leq b$*, then* $a'\widehat{C}b'$,

($\widehat{C}3$) *If* $a\widehat{C}(b \cdot c)$*, then* $a\widehat{C}b$ *or* $a\widehat{C}c$,

($\widehat{C}4$) *If* $a\widehat{C}b$*, then* $b\widehat{C}a$,

($\widehat{C}5$) *If* $a + b \neq 1$*, then* $a\widehat{C}b$.

Axioms for $\ll$ ***alone:***

($\ll 1$) $0 \ll 0$,

($\ll 2$) $1 \ll 1$,

($\ll 3$) *If* $a \ll b$*, then* $a \leq b$,

($\ll 4$) *If* $a' \leq a \ll b \leq b'$*, then* $a' \ll b'$,

($\ll 5$) *If* $a \ll c$ *and* $b \ll c$*, then* $(a + b) \ll c$,

($\ll 6$) *If* $c \ll a$ *and* $c \ll b$*, then* $c \ll (a \cdot b)$,

($\ll 7$) *If* $a \ll b$ *and* $(b \cdot c) \ll d$ *and* $c \ll (a + d)$*, then* $c \ll d$.

Mixed axioms:

$(MC1)$ *If* aCb *and* $a \ll c$, *then* $aC(b \cdot c)$,

$(MC2)$ *If* $a\overline{C}(b \cdot c)$ *and* aCb *and* $(a \cdot d)\overline{C}b$, *then* $d\widehat{C}c$,

$(M\widehat{C}1)$ *If* $a\widehat{C}b$ *and* $c \ll a$, *then* $a\widehat{C}(b + c)$,

$(M\widehat{C}2)$ *If* $a\overline{\widehat{C}}(b + c)$ *and* $a\widehat{C}b$ *and* $(a + d)\overline{\widehat{C}}b$, *then* dCc,

$(M \ll 1)$ *If* $a\overline{\widehat{C}}b$ *and* $(a \cdot c) \ll b$, *then* $c \ll b$,

$(M \ll 2)$ *If* $a\overline{C}b$ *and* $b \ll (a + c)$, *then* $b \ll c$.

REMARK 1.2. **(i) About the axioms.** As one can see the list of axioms is quite long and one can ask about the leading intuition to accept these axioms. We invite the reader to read again the text after 1.1. Namely, we followed the next 3 principles: (1) the axioms to be first-order sentences true in contact algebras, (2) the principle of duality to be true (see the next Observation) and (3) the axioms to be sufficient to prove the embedding theorem of EDC-latices in contact algebras. The most difficult was the last one. In order to fulfil it we proceeded as follows: we started to prove the embedding theorem having in mind some construction and during this process we look for the needed axioms satisfying (1). Then we polished the obtained set of axioms several times in order to obtain a more elegant set and accordingly reedited the proof.

(ii) Duality principle. For the language of EDCL we can introduce the following principle of duality: dual pairs $(0, 1), (\cdot, +), (\leq, \geq), (C, \widehat{C}), (\ll, \gg)$. The motivation to consider the first three dual pairs comes from the corresponding notion of duality in lattice theory. But why we consider $(C, \widehat{C}), (\ll, \gg)$ as dual pairs? The motivation comes from the following facts about duality principle for operations and relations in Boolean algebras:

If $f(a_1, \ldots, a_n)$ is a (definable) Boolean operation then its dual $\widehat{f}$ satisfies the equation $\widehat{f}(a_1, \ldots, a_n) = f(a_1^*, \ldots, a_n^*)^*$,

If $R(a_1, \ldots, a_n)$ is a (definable) Boolean relation then its dual $\widehat{R}$ satisfies the equivalence $\widehat{R}(a_1, \ldots, a_n) \leftrightarrow R(a_1^*, \ldots, a_n^*)$.

For instance the dual of Boolean ordering $a \leq b$ is $a \geq b$ which is equivalent to $a^* \leq b^*$. Extending this observation to the contact relation C we define its dual $a\widehat{C}b$ in the contact algebra as a^*Cb^*. In contact algebras non-tangential part-of has the following definition $a \ll b \leftrightarrow_{def} a\overline{C}b^*$. Then its dual should be $a^* \ll b^*$ which is equivalent to $b \ll a$ $(a \gg b)$.

By means of dual pairs for each statement (definition) A of the language we can define in an obvious way its dual $\widehat{A}$. For each axiom Ax from the list of axioms of EDCL its dual $\widehat{Ax}$ is also an axiom. On the base of this observation the proofs of dual statements will be omitted. Note, for instance, that each axiom from the first group (axioms for C alone) is dually equivalent to the corresponding axiom from the second group (axioms for $\widehat{C}$ alone) and vice versa, the third and fourth groups of axioms (axioms for $\ll$ alone and mixed axioms) are closed under duality. For instance for the mixed axioms we have: axiom $(M\widehat{C}1)$ is dually equivalent to the axiom $(MC1)$, $(M\widehat{C}2)$ is dually equivalent to $(MC2)$ and $(M \ll 2)$ is dually equivalent to $(M \ll 1)$. □

1.2. Relational models of EDC-lattices. In order to prove that the axioms of EDC-lattices are true in contact algebras we will introduce a relational models of EDCL which are slight modifications of the relational models of contact algebras introduced in [**10**] and called there *discrete contact algebras*. The model is defined as follows.

Let (W, R) be a relational system where W is a nonempty set and R is a reflexive and symmetric relation in W and let a, b be arbitrary subsets of W. Define a contact relation between a and b as follows

(Def C_R) aC_Rb iff $\exists x \in a$ and $\exists y \in b$ such that xRy.

Then any Boolean algebra of subsets of W with thus defined contact is a contact algebra, and moreover, every contact algebra is isomorphic to a contact algebra of such a kind [**10**].

We will modify this model for EDCL as follows: instead of Boolean algebras of sets we consider only families of subsets containing the empty set $\varnothing$ and the set W and closed under the set-union and set-intersection which are bounded distributive lattices of sets. Hence we interpret lattice constants and operations as follows: $0 = \varnothing$, $1 = W$, $a \cdot b = a \cap b$, $a + b = a \cup b$. For the contact relation we preserve the definition (Def C_R). This modification is just a model of distributive contact lattice studied in [**12**].

Having in mind the definitions $a\widehat{C}b \leftrightarrow_{def} a^*Cb^*$ and $a \ll b \leftrightarrow_{def} a\overline{C}b^*$ in Boolean contact algebras, we introduce the following definitions for $\widehat{C}$ and $\ll$ (for some convenience we present the definition of the negation of $\ll$):

(Def $\widehat{C}_R$) $a\widehat{C}_Rb$ iff $\exists x \notin a$ and $\exists y \notin b$ such that xRy, and

(Def $\not\ll_R$) $a \not\ll_R b$ iff $\exists x \in a$ and $\exists y \notin b$ such that xRy.

Lemma 1.1. *Let (W, R) be a relational system with reflexive and symmetric relation R and let $\underline{D}$ be any collection of subsets of W which is a bounded distributive set-lattice with relations $C, \widehat{C}$ and $\ll$ defined as above. Then $(\underline{D}, C_R, \widehat{C}_R, \ll_R)$ is an EDC-lattice.*

Proof. Routine verification that all axioms of EDC-lattice are true. □

EDC-lattice $\underline{D} = (D, C_R, \widehat{C}_R, \ll_R)$ over a relational system (W, R) will be called *discrete EDC-lattice*. If D is a set of all subsets of W then $\underline{D}$ is called a *full discrete EDC-lattice*.

Corollary 1.1. *The axioms of the relations $C, \widehat{C}$ and $\ll$ are true in contact algebras.*

Proof. The proof follows by Lemma 1.1 and the fact that every contact algebra can be isomorphically embedded into a discrete contact algebra over some relational system (W, R) with reflexive and symmetric relation R [**10**]. □

2. Embedding EDC-lattices into contact algebras

The main aim of this section is the proof of a theorem stating that every EDC-lattice can be embedded into a full discrete EDC-lattice, which, of course is a Boolean contact algebra. As a consequence this will show that the axiomatization program for EDCL is fulfilled successfully. Since all axioms of EDC-lattice are universal first-order conditions, the axiomatization can be considered also as a characterization of the universal fragment of complement-free contact algebras based on the three relations. We will use in the representation theory a Stone like technique developed in [**36**] for the representation theory of distributive lattices.

2.1. Preliminary facts about filters and ideals in distributive lattices. We remind the reader of some basic facts about filters and ideals in distributive lattices, for details see [**2, 36**].

Let $\underline{D}$ be a distributive lattice. A subset F of D is called a filter in D if it satisfies the following conditions: (f1) $1 \in F$, (f2) if $a \in F$ and $a \leq b$ then $b \in F$, (f3) if $a, b \in F$ then $a.b \in F$. F is a proper filter if $0 \notin F$, F is a prime filter if it is a proper filter and $a + b \in F$ implies $a \in F$ or $b \in F$.

Dually, a subset I of D is an ideal if (i1) $0 \in I$, (i2) if $a \in I$ and $b \leq a$ then $b \in I$, (i3) if $a, b \in I$ then $a + b \in I$. I is a proper ideal if $1 \notin I$, I is a prime ideal if it is a proper ideal and $a.b \in I$ implies $a \in I$ or $b \in I$.

We will use later on some of the following facts without explicit mentioning.

FACTS 2.1. *Let $\underline{D}$ be a bounded distributive lattice and Let F, F_1, F_2 be filters and I, I_1, I_2 be ideals.*

(1) *The complement of a prime filter is a prime ideal and vice-versa.*
(2) $[a) = \{x \in D : a \leq x\}$ *is the smallest filter containing* a*;*
    $(a] = \{x \in D : x \leq a\}$ *is the smallest ideal containing* a*.*
(3) $F_1 \oplus F_2 = \{c \in D : (\exists a \in F_1, b \in F_2)(a \cdot b \leq c)\} = \{a \cdot b : a \in F_1, b \in F_2\}$ *is the smallest filter containing* F_1 *and* F_2*.*
    $[a) \oplus F = \{x \cdot y : a \leq x,\ y \in F\}$
    $I_1 \oplus I_2 = \{c \in D : (\exists a \in I_1, b \in I_2)(c \leq a+b)\} = \{a+b : a \in I_1, b \in I_2\}$ *is the smallest ideal containing* I_1 *and* I_2*.*
    $(a] \oplus I = \{x + y : x \leq a,\ y \in I\}$.
    In both cases the operation $\oplus$ *is associative and commutative.*
(4) $[a) \cap I = \varnothing$ *iff* $a \notin I$
    If $(F \oplus [a)) \cap I \neq \varnothing$ *then* $(\exists x \in F)(a \cdot x \in I)$*,*
    $(a] \cap F = \varnothing$ *iff* $a \notin F$
    If $F \cap (I \oplus (a]) \neq \varnothing$ *then* $(\exists x \in I)(a + x \in F)$*.*

The following three statements are well known in the representation theory of distributive lattices.

LEMMA 2.1. *Let F_0 be a filter, I_0 be an Ideal and $F_0 \cap I_0 = \varnothing$. Then:*

(1) **Filter-extension Lemma.** *There exists a prime filter F such that $F_0 \subseteq F$ and $F \cap I_0 = \varnothing$.*
(2) **Ideal-extension Lemma.** *There exists a prime ideal I such that $I_0 \subseteq I$ and $F_0 \cap I = \varnothing$.*
(3) **Separation Lemma for filters and ideals.** *There exist a (prime) filter F and an (prime) ideal I such that $F_0 \subseteq F$, $I_0 \subseteq I$, $F \cap I = \varnothing$, and $F \cup I = D$.*

REMARK 2.2. Note that *Filter-extension Lemma* is dual to the *Ideal-extension Lemma* and that each of the three statement easily implies the other two. Normally they can be proved by application of the Zorn Lemma. The proof, for instance, of Filter-extension Lemma goes as follows. Apply the Zorn Lemma to the set $M = \{G : G$ is a filter, $F_0 \subseteq G$ and $G \cap I_0 = \varnothing\}$ and denote by F one of its maximal elements. Then it can be proved that F is a prime filter, and this finishes the proof. The sketched proof gives, however, an additional property of the filter F, namely

$(\forall x \notin F)(\exists y \in F)(x \cdot y \in I_0)$, which added to the formulation of the lemma makes it stronger. Since we will need later on this stronger version let us prove this property.

Suppose that $x \notin F$ and consider the filter $F \oplus [x)$. Since F is a maximal element of M, then $F \oplus [x)$ does not belong to M and consequently $F \oplus [x) \cap I_0 \neq \varnothing$. By the Fact 2.1, 4, there exists $y \in F$ such that $x \cdot y \in I_0$. We formulate this new statement below as *Strong filter-extension Lemma* and its dual as *Strong ideal-extension Lemma*. We do not know if these two statements for distributive lattices are new, but we will use them in the representation theorem in the next section. □

LEMMA 2.2. *Let F_0 be a filter, I_0 be an Ideal and $F_0 \cap I_0 = \varnothing$. Then:*

(1) **Strong filter-extension Lemma.** *There exists a prime filter F such that $F_0 \subseteq F$, $(\forall x \in F)(x \notin I_0)$ and $(\forall x \notin F)(\exists y \in F)(x \cdot y \in I_0)$.*

(2) **Strong ideal-extension Lemma.** *There exists a prime ideal I such that $I_0 \subseteq I$, $(\forall x \in I)(x \notin F_0)$ and $(\forall x \notin I)(\exists y \in I)(x + y \in F_0)$.*

2.2. Filters and Ideals in EDC-lattices. In the next two lemmas we list some constructions of filters and ideals in EDCL which will be used in the representation theory of EDC-lattices.

LEMMA 2.3. *Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be an EDC-lattice. Then:*

(1) The set $I(x\overline{C}b) = \{x \in D : x\overline{C}b\}$ is an ideal,

(2) the set $F(x\overline{\widehat{C}}b) = \{x \in D : x\overline{\widehat{C}}b\}$ is a filter,

(3) the set $I(x \ll b) = \{x \in D : x \ll b\}$ is an ideal,

(4) the set $F(x \gg b) = \{x \in D : x \gg b\}$ is a filter.

PROOF. *1.* By axiom (C1) $0\overline{C}b$, so $0 \in I(x\overline{C}b)$. Suppose $x \in I(x\overline{C}b)$ (hence $x\overline{C}b$) and $y \leq x$. Then by axiom (C2) $y\overline{C}b$). Let $x, y \in I(x\overline{C}b)$, hence $x\overline{C}b$ and $y\overline{C}b$. Then by axiom (C3) and (C4) we get $(x + y)\overline{C}b$ which shows that $x + y \in I(x\overline{C}b)$, which ends the proof of this case.

In a similar way one can proof *3.* The cases *2.* and *4.* follow from *1.* and *3.* respectively by duality.

□

LEMMA 2.4. *Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be an EDC-lattice and Let Γ be a prime filter in $\underline{D}$. Then:*

(1) The set $I(x\overline{C}\Gamma) = \{x \in D : (\exists y \in \Gamma)(x\overline{C}y)\}$ is an ideal,

(2) the set $F(x\overline{\widehat{C}}\overline{\Gamma}) = \{x \in D : (\exists y \in \overline{\Gamma})(x\overline{\widehat{C}}y)\}$ is a filter,

(3) the set $I(x \ll \overline{\Gamma}) = \{x \in D : (\exists y \in \overline{\Gamma})(x \ll y)\}$ is an ideal,

(4) the set $F(x \gg \Gamma) = \{x \in D : (\exists y \in \Gamma)(x \gg y)\}$ is a filter.

PROOF. Note that the Lemma remains true if we replace Γ by a filter and $\overline{\Gamma}$ by an ideal.

1. The proof that $I(x\overline{C}\Gamma)$ satisfies the conditions (i1) and (i2) from the definition of ideal is easy. For the condition (i3) suppose $x_1, x_2 \in I(x\overline{C}\Gamma)$. Then $\exists y_1, y_2 \in \Gamma$ such that $x_1\overline{C}y_1$ and $x_2\overline{C}y_2$, Since Γ is a filter then $y = y_1 \cdot y_2 \in \Gamma$. Since $y \leq y_1$ and $y \leq y_2$, then by axiom (C2) we get $x_1\overline{C}y$ and $x_2\overline{C}y$. Then applying (C3') we obtain $(x_1 + x_2)\overline{C}y$, which shows that $x_1 + x_2 \in I(x\overline{C}\Gamma)$.

In a similar way one can prove *3.* The proofs of *2* and *4* follow by duality from *1* and *3*, taking into account that $\overline{\Gamma}$ is a prime ideal.

□

2.3. Relational representation theorem for EDC-lattices. Throughout this section we assume that $\underline{D} = (D, C, \widehat{C}, \ll)$ is an EDC-lattice and let $PF(D)$ and $PI(D)$ denote the set of prime filters of $\underline{D}$ and the set of prime ideals of D. Let $h(a) = \{\Gamma \in PF(D) : a \in \Gamma\}$ be the well known Stone embedding mapping. We shall construct a canonical relational structure (W^c, R^c) related to $\underline{D}$ putting $W^c = PF(D)$ and defining the canonical relation R^c for $\Gamma, \Delta \in PF(D)$ as follows:

$\Gamma R^c \Delta \leftrightarrow_{def} (\forall a, b \in D)((a \in \Gamma, b \in \Delta \to aCb) \& (a \notin \Gamma, b \notin \Delta \to a\widehat{C}b) \& (a \in \Gamma, b \notin \Delta \to a \not\ll b) \& (a \notin \Gamma, b \in \Delta \to b \not\ll a))$

For some technical reasons and in order to use duality we introduce also the dual canonical structure $(\widehat{W}^c, \widehat{R}^c)$ putting $\widehat{W}^c = PI(D)$ and for $\Gamma, \Delta \in PI(D)$, $\Gamma \widehat{R}^c \Delta \leftrightarrow_{def} \overline{\Gamma} R^c \overline{\Delta}$.

Our aim is to show that the Stone mapping h is an embedding from $\underline{D}$ into the EDC-lattice over (W^c, R^c) (see Section 1.1). First we need several technical lemmas.

LEMMA 2.5. *The canonical relations R^c and $\widehat{R}^c$ are reflexive and symmetric.*

PROOF. **(For R^c)** Symmetry is obvious by the definition of R^c and axioms (C4) and ($\widehat{C}$4). In order to prove that $\Gamma R^c \Gamma$ suppose $a \in \Gamma$ and $b \in \Gamma$. Then $a \cdot b \in \Gamma$ and since Γ is a prime filter, then $a.b \neq 0$. Then by axiom (C5) we obtain aCb, which proves the first conjunct of the definition of R^c. For the second conjunct suppose that $a \notin \Gamma$ and $b \notin \Gamma$, then, since Γ is a prime filter, $a + b \notin \Gamma$ and hence $a + b \neq 1$. Then by axiom ($\widehat{C}$5) we get $a\widehat{C}b$. For the third conjunct suppose $a \in \Gamma$ and $b \notin \Gamma$, which implies that $a \not\leq b$. Then by axiom ($\ll$ 3) we obtain $a \not\ll b$. The proof of the last conjunct is similar.

(For $\widehat{R}^c$) - by the definition of $\widehat{R}^c$.

□

LEMMA 2.6. *(i) aCb iff $(\exists \Gamma, \Delta \in PF(D))(a \in \Gamma$ and $b \in \Delta$ and $\Gamma R^c \Delta)$.*
(ii) $a \not\ll b$ iff $(\exists \Gamma, \Delta \in PF(D))(a \in \Gamma$ and $b \notin \Delta$ and $\Gamma R^c \Delta)$.

PROOF. (i) Note that the proof is quite technical, so we will present it with full details. The reasons for this are twofold: first to help the reader to follow it more easily, and second, to skip the details in a similar proofs.

($\Leftarrow$) If $a \in \Gamma$ and $b \in \Delta$ and $\Gamma R^c \Delta$), then by the definition of R^c we obtain aCb.

($\Rightarrow$) Suppose aCb.

The proof will go on several steps.

Step 1: construction of Γ. Consider the ideal $I(x\overline{C}b) = \{x \in D : x\overline{C}b\}$ (Lemma 2.3). Since aCb, $a \notin \{x \in D : x\overline{C}b\}$. Then $[a) \cap \{x \in D : x\overline{C}b\} = \varnothing$ and $[a)$ is a filter (see Facts 2.1). By the Strong filter-extension lemma (see Lemma 2.2) there exists a prime filter Γ such that $[a) \subseteq \Gamma$ and $(\forall x \in \Gamma)(x \notin \{x \in D : x\overline{C}b\}$ and $(\forall x \notin \Gamma)(\exists y \in \Gamma)(x \cdot y \in \{x \in D : x\overline{C}b\}$. From here we conclude that Γ satisfies the following three properties:

(#0) $a \in \Gamma$,

(#1) If $x \in \Gamma$, then xCb, and

(#2) If $x \notin \Gamma$, then there exists $y \in \Gamma$ such that $(x \cdot y)\overline{C}b$.

Step 2: construction of Δ. This will be done in two sub-steps.

Step 2.1 Consider the filters and ideals definable by Γ as in Lemma 2.4

$I(x\overline{C}\Gamma) = \{x \in D : (\exists y \in \Gamma)(x\overline{C}y)\}$, $F(x\overline{\widehat{C}}\overline{\Gamma}) = \{x \in D : (\exists y \in \overline{\Gamma})(x\overline{\widehat{C}}y)\}$, $I(x \ll \overline{\Gamma}) = \{x \in D : (\exists y \in \overline{\Gamma})(x \ll y)\}$, and $F(x \gg \Gamma) = \{x \in D : (\exists y \in \Gamma)(x \gg y\}$. In order to apply the Separation Lemma we will prove the following condition:

(#3) $(F(x \gg \Gamma) \oplus F(x\overline{\widehat{C}}\overline{\Gamma}) \oplus [b)) \cap (I(x\overline{C}\Gamma) \oplus I(x \ll \overline{\Gamma})) = \varnothing$.

Suppose that (#3) is not true, then for some $t \in D$ we have

(1) $t \in F(x \gg \Gamma) \oplus F(x\overline{\widehat{C}}\overline{\Gamma}) \oplus [b)$ and
(2) $t \in I(x\overline{C}\Gamma) \oplus I(x \ll \overline{\Gamma})$.

It follows from (2) that $\exists k_1, k_2$ such that

(3) $k_1 \in I(x \ll \overline{\Gamma})$ and
(4) $k_2 \in I(x\overline{C}\Gamma)$ and
(5) $t = k_1 + k_2$. (Here we use Facts 2.1 (3).)

It follows from (1) that $\exists k_4, k_5, k_6 \in D$ such that

(6) $k_4 \in F(x \gg \Gamma)$ and
(7) $k_5 \in F(x\overline{\widehat{C}}\overline{\Gamma})$ and
(8) $k_6 \in [b)$ and
(9) $t = k_4 \cdot k_5 \cdot k_6$. (Here we use Facts 2.1 (3).)

From (5) and (9) we get

(10) $k_1 + k_2 = k_4 \cdot k_5 \cdot k_6$.

It follows from (3), (4), (6) and (7) that

(11) $\exists x_1 \in \overline{\Gamma}$ such that $k_1 \ll x_1$,
(12) $\exists x_2 \in \Gamma$ such that $k_2\overline{C}x_2$,
(13) $\exists x_3 \in \Gamma$ such that $x_3 \ll k_4$,
(14) $\exists x_4 \in \overline{\Gamma}$ such that $k_5\overline{\widehat{C}}x_4$.

Let $x = x_1 + x_4$. Since $\overline{\Gamma}$ is an ideal, we obtain by (11) and (14) that

(15) $x \in \overline{\Gamma}$ and $x \notin \Gamma$. Then by (#2) we get
(16) $\exists y \in \Gamma$ such that $(x \cdot y)\overline{C}b$.

Let $z = x_2 \cdot x_3 \cdot y$. Then by (12), (13) and (16) we obtain that

(17) $z \in \Gamma$
and by (#1) that
(18)zCb.

From $x_1 \leq x$ and (11) by axiom ($\ll 4$) we get

(19) $k_1 \ll x$.

From $x_4 \leq x$ and (14) by axiom ($\widehat{C}2$) we obtain

(20) $k_5\overline{\widehat{C}}x$.

From $z \leq x_2$ and (12) by axiom ($C2$) we get

(21) $k_2\overline{C}z$.

From $z \leq x_3$ and (13) by axiom ($\ll 4$) we obtain

(22) $z \ll k_4$.

We shall show that the following holds

(23) $z\overline{C}(b \cdot k_1)$.

Suppose for the sake of contradiction that $zC(b \cdot k_1)$. From $b \cdot k_1 \leq k_1$ and (19) by axiom ($\ll 4$) we get $(b \cdot k_1) \ll x$. From this fact and $zC(b \cdot k_1)$ by axiom ($MC1$) we obtain $(b \cdot k_1)C(z \cdot x)$. But we also have $b \cdot k_1 \leq b$, $z \cdot x \leq y \cdot x$, so by axiom ($C2$) we get $bC(y \cdot x)$ - a contradiction with (16).

The following condition holds

(24) $z\overline{C}(b \cdot k_2)$.

To prove this suppose for the sake of contradiction that $zC(b \cdot k_2)$. We also have $b \cdot k_2 \leq k_2$, so by axiom $(C2)$ we get zCk_2 - a contradiction with (21).

Suppose that $zC(b \cdot (k_1 + k_2))$. By axiom (C3) we have $zC(b \cdot k_1)$ or $zC(b \cdot k_2)$ - a contradiction with (23) and (24). Consequently $z\overline{C}(b \cdot (k_1 + k_2))$ and by (10) we obtain $z\overline{C}(b \cdot k_4 \cdot k_5 \cdot k_6)$. But $b \leq k_6$ (from (8)), so $b \cdot k_4 \cdot k_5 \cdot k_6 = b \cdot k_4 \cdot k_5$. Consequently

(25) $z\overline{C}(b \cdot k_4 \cdot k_5)$.

From (18) and (22) by axiom $(MC1)$ we get

(26) $zC(b \cdot k_4)$.

We shall show that the following condition holds

(27) $(z \cdot x)\overline{C}(b \cdot k_4)$

For to prove this suppose the contrary $(z \cdot x)C(b \cdot k_4)$. We also have $z \cdot x \leq y \cdot x$, $b \cdot k_4 \leq b$, so by axiom $(C2)$ we get $(y \cdot x)Cb$ - a contradiction with (16).

From (25), (26) and (27) by axiom $(MC2)$ we obtain $x\widehat{C}k_5$ - a contradiction with (20). Consequently (#3) is true.

Step 2.2: the construction of Δ. Applying the Filter extension Lemma to (#3) we obtain a prime filter Δ (and this is just the required Δ) such that:

(1) $F(x \gg \Gamma) = \{x \in D : (\exists y \in \Gamma)(x \gg y\} \subseteq \Delta$,
(2) $F(x\overline{\widehat{C}}\Gamma) = \{x \in D : (\exists y \in \overline{\Gamma})(x\overline{\widehat{C}}y)\} \subseteq \Delta$,
(3) $b \in \Delta$,
(4) $I(x\overline{C}\Gamma) \cap \Delta = \varnothing$,
(5) $I(x \ll \overline{\Gamma}) \cap \Delta = \varnothing$.

Step 3: proof of $\Gamma R^c \Delta$. We will verify the four cases of the definition of R^c.

- **Case 1:** $y \in \Gamma$ **and** $x \in \Delta$**.** We have to show yCx. Suppose $y\overline{C}x$. Then $x\overline{C}y$ and by $y \in \Gamma$ we get $x \in I(x\overline{C}\Gamma)$. Then by 4. $x \notin \Delta$ - a contradiction, hence yCx.
- **Case 2:** $y \in \Gamma$ **and** $x \notin \Delta$**.** Suppose $y \ll x$. Then $x \gg y$ and $y \in \Gamma$ implies $x \in F(x \gg \Gamma)$. By (1) $x \in \Delta$ - a contradiction, hence $y \not\ll x$.
- **Case 3:** $y \notin \Gamma$ **and** $x \in \Delta$**.** Suppose $x \ll y$. Then $x \in I(x \ll \overline{\Gamma})$ and by 5. $x \notin \Delta$ - a contradiction. Hence $x \not\ll y$.
- **Case 4:** $y \notin \Gamma$ **and** $x \notin \Delta$**.** Suppose $y\overline{\widehat{C}}x$. Then $x\overline{\widehat{C}}y$ and by 2. we obtain $x \in \Delta$ - a contradiction. Hence $y\widehat{C}x$.

Thus we have constructed prime filters Γ and Δ such that: $a \in \Gamma$, $b \in \Delta$ (item 3 from Step 2.2) and $\Gamma R^c \Delta$ (Step 3).

Proof of (ii). $(\Leftarrow)$ If $a \in \Gamma$ and $b \notin \Delta$ then by the definition of R^c we obtain $a \not\ll b$.

$(\Rightarrow)$ Suppose $a \not\ll b$. The proof, as in (i), will go on several steps.

Step 1: construction of Γ. Consider the ideal $I(x \ll b) = \{x \in D : x \ll b\}$ (Lemma 2.3).

Since $a \not\ll b$, $a \notin \{x \in D : x \ll b\}$. Then $[a) \cap \{x \in D : x \ll b\} = \varnothing$ and $[a)$ is a filter (see FACTS 2.1). By the Strong filter-extension lemma (Lemma 2.2) there exists a prime filter Γ such that $[a) \subseteq \Gamma$ and $(\forall x \in \Gamma)(x \notin \{x \in D : x \ll b\})$ and $(\forall x \notin \Gamma)(\exists y \in \Gamma)(x \cdot y \in \{x \in D : x \ll b\})$. From here we conclude that Γ satisfies the following properties:

(#0) $a \in \Gamma$,

(#1) If $x \in \Gamma$, then $x \not\ll b$, and

(#2) If $x \notin \Gamma$, then there exists $y \in \Gamma$ such that $(x \cdot y) \ll b$.

Step 2: construction of Δ. This will be done in two sub-steps.

Step 2.1 Consider the filters and ideals definable by Γ as in Lemma 2.4 $I(x\overline{C}\Gamma) = \{x \in D : (\exists y \in \Gamma)(x\overline{C}y)\}$, $F(x\overline{\widehat{C}}\overline{\Gamma}) = \{x \in D : (\exists y \in \overline{\Gamma})(x\overline{\widehat{C}}y)\}$, $I(x \ll \overline{\Gamma}) = \{x \in D : (\exists y \in \overline{\Gamma})(x \ll y)\}$, and $F(x \gg \Gamma) = \{x \in D : (\exists y \in \Gamma)(x \gg y\}$. In order to apply the Filter-extension Lemma (Lemma 2.1) we will prove the following condition:

(#3) $(F(x \gg \Gamma) \oplus F(x\overline{\widehat{C}}\overline{\Gamma})) \cap (I(x \ll \overline{\Gamma}) \oplus I(x\overline{C}\Gamma) \oplus (b]) = \varnothing$

Suppose that (#3) is not true. Consequently $\exists t$ such that
(1) $t = k_1 \cdot k_2 = k_4 + k_5 + k_6$ for some $k_1, k_2, k_4, k_5, k_6 \in D$ and
(2) $\exists x_1 \in \Gamma$ such that $x_1 \ll k_1$,
(3) $\exists x_2 \in \overline{\Gamma}$ such that $k_2\overline{\widehat{C}}x_2$,
(4) $\exists x_3 \in \overline{\Gamma}$ such that $k_4 \ll x_3$,
(5) $\exists x_4 \in \Gamma$ such that $k_5\overline{C}x_4$,
(6) $k_6 \leq b$.

Let $z = x_2 + x_3$. Then by (3) and (4) we obtain $z \in \overline{\Gamma}$. By axiom $(\widehat{C}2)$ we get
(7) $k_2\overline{\widehat{C}}z$.

By (4) and axiom $(\ll 4)$ we get
(8) $k_4 \ll z$.

By $z \notin \Gamma$ and (#2) we have
(9) $\exists y \in \Gamma$ such that $(z \cdot y) \ll b$.

Let $x = x_1 \cdot x_4 \cdot y \cdot a$. Then by (#0), (2), (5) and (9) we get $x \in \Gamma$. By axiom $(\ll 4)$ we get
(10) $x \ll k_1$.

By (5), $x \leq x_4$ and axiom $(C2)$ we get
(11) $k_5\overline{C}x$.

From $x \in \Gamma$ by (#1) we obtain
(12) $x \not\ll b$.

From (10) by axiom $(\ll 4)$ we get
(13) $x \ll (b + k_1)$

From (7) by axiom $(\widehat{C}2)$ we obtain
(14) $z\overline{\widehat{C}}(b + k_2)$.

From (9) by axiom $(\ll 4)$ we get
(15) $(z \cdot y) \ll (b + k_2)$.

From (14) and (15) by axiom $(M \ll 1)$ we obtain $y \ll (b + k_2)$. We also have $x \leq y$ and by axiom $(\ll 4)$ we get
(16) $x \ll (b + k_2)$.

From (13) and (16) by axiom $(\ll 6)$ we get $x \ll (b + k_1) \cdot (b + k_2)$. We have $(b + k_1) \cdot (b + k_2) = b + k_1 \cdot k_2 = b + k_4 + k_5 + k_6 = b + k_4 + k_5$ (since $k_6 \leq b$ from (6)). Thus:
(17) $x \ll (b + k_4 + k_5)$.

Suppose (in order to obtain a contradiction) that $x \ll (b + k_4)$. From (9) and $x \cdot z \leq z \cdot y$ (which follows from the definitions of x and z) by axiom $(\ll 4)$ we obtain $(x \cdot z) \ll b$. Using this fact, (8), $x \ll (b + k_4)$ and axiom $(\ll 7)$ we get $x \ll b$ - a contradiction with (12). Consequently
(18) $x \not\ll (b + k_4)$.

From (11) and (17) by axiom ($M \ll 2$) we obtain $x \ll (b+k_4)$ - a contradiction with (18). Consequently (#3) is true.

Step 2.2: the construction of Δ. Applying the Filter-extension Lemma to (#3) we obtain a prime filter Δ (and this is just the required Δ) such that:

(1) $F(x \gg \Gamma) = \{x \in D : (\exists y \in \Gamma)(x \gg y\} \subseteq \Delta$,
(2) $F(x\overline{\widehat{C}}\Gamma) = \{x \in D : (\exists y \in \overline{\Gamma})(x\overline{\widehat{C}}y)\} \subseteq \Delta$,
(3) $b \notin \Delta$,
(4) $I(x\overline{C}\Gamma) \cap \Delta = \varnothing$,
(5) $I(x \ll \overline{\Gamma}) \cap \Delta = \varnothing$.

Step 3: proof of $\Gamma R^c \Delta$. The proof is the same as in the corresponding step in (i).

To conclude: we have constructed prime filters Γ, Δ such that $\Gamma R^c \Delta$, $a \in \Gamma$ and $b \notin \Delta$, which finishes the proof of the lemma.

□

LEMMA 2.7. *(i) $a\widehat{C}b$ iff $(\exists \Gamma, \Delta \in PI(D))(a \in \Gamma$ and $b \in \Delta$ and $\Gamma \widehat{R}^c \Delta)$.*
(ii) $a\widehat{C}b$ iff $(\exists \Gamma, \Delta \in PF(D))(a \notin \Gamma$ and $b \notin \Delta$ and $\Gamma R^c \Delta)$.
(iii) $a \not\gg b$ iff $(\exists \Gamma, \Delta \in PI(D))(a \in \Gamma$ and $b \notin \Delta$ and $\Gamma \widehat{R}^c \Delta)$.
(iv) $a \not\gg b$ iff $(\exists \Gamma, \Delta \in PF(D))(a \notin \Gamma$ and $b \in \Delta$ and $\Gamma R^c \Delta)$.

PROOF. (i) by duality from Lemma 2.6. Note that in this case Strong ideal-extension Lemma is used. The proof can follow in a "dual way" the steps of the proof of Lemma 2.6 (i).
(ii) is a corollary from (i).
(iii) by duality from Lemma 2.6 (ii) with the same remark as above.
(iv) is a corollary from (iii).

□

LEMMA 2.8. *Let (W^c, R^c) be the canonical structure of $\underline{D} = (D, C, \widehat{C}, \ll)$ and $h(a) = \{U \in PF(D) : a \in U\}$ be the Stone mapping from D into the distributive lattice of all subsets of W^c. Then h is an embedding of $\underline{D}$ into the EDC-lattice over (W^c, R^c).*

PROOF. It is a well known fact that h is an embedding of distributive lattice into the distributive lattice of all subsets of the set of prime filters $PF(D)$ (see, [**36, 2**]). The only thing which have to be done is to show the following equivalences for all $a, b \in D$:
(i) aCb iff $h(a)C_{R^c}h(b)$,
(ii) $a\widehat{C}b$ iff $h(a)\widehat{C}_{R^c}h(b)$
(iii) $a \ll b$ iff $h(a) \ll_{R^c} h(b)$.
Note that these equivalences are another equivalent reformulation of Lemma 2.6 (i) and (ii) and Lemma 2.7 (ii) and (iv).

□

THEOREM 2.3. **Relational representation theorem of EDC-latices.** *Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be an EDC-lattice. Then there is a relational system $\underline{W} = (W, R)$ with reflexive and symmetric R and an embedding h into the EDC-lattice of all subsets of W.*

Proof. The theorem is a corollary of Lemma 2.8. □

Corollary 2.1. *Every EDC-lattice can be isomorphically embedded into a contact algebra.*

Proof. Since the lattice of all subsets of a given set is a Boolean algebra, then this is a corollary of Theorem 2.3. □

The following theorem states that the axiom system of EDC-lattice can be considered as an axiomatization of the universal fragment of contact algebras in the language of EDC-lattices.

Theorem 2.4. *Let A be an universal first-order formula in the language of EDC-lattices. Then A is a consequence from the axioms of EDC-lattice iff A is true in all contact algebras.*

Proof. The proof is a consequence from Corollary 2.1 and the fact that all axioms of EDC-lattice are universal first-order conditions and that A is also an universal first-order condition. □

3. Relations to other mereotopologies

In order to see the expressivity power of EDC-lattices compared to distributive contact lattices from [**11, 12**] we will compare them with other two mereotopologies: the *relational mereotopology* and *RCC-8*. We show that the mentioned two mereotopologies are expressible in the language of EDC-lattices but not expressible in the distributive contact lattices from [**11, 12**].

3.1. Relational mereotopology.

3.1. Relational mereotopology. Relational mereotopology is based on *mereotopological structures* introduced in [**29**] (Definition 7, page 254). These are relational structures in the form $(W, \leq, O, \widehat{O}, \ll, C, \widehat{C})$ axiomatizing the basic mereological relations part-of $\leq$, overlap O and dual overlap (underlap) $\widehat{O}$, and the basic mereotopological relations non-tangential part-of $\ll$, contact C and dual contact $\widehat{C}$. These relations satisfy the following list of universal first-order axioms:

(≤ 0)	$a \leq b$ and $b \leq a \rightarrow a = b$	(≤ 1)	$a \leq a$,
(≤ 2)	$a \leq b$ and $b \leq c \rightarrow a \leq c$		
$(O1)$	$aOb \rightarrow bOa$	$(\widehat{O}1)$	$a\widehat{O}b \rightarrow b\widehat{O}a$
$(O2)$	$aOb \rightarrow aOa$	$(\widehat{O}2)$	$a\widehat{O}b \rightarrow a\widehat{O}a$
$(\overline{O} \leq)$	$a\overline{O}a \rightarrow a \leq b$	$(\overline{\widehat{O}} \leq)$	$b\overline{\widehat{O}}b \rightarrow a \leq b$
$(O \leq)$	aOb and $b \leq c \rightarrow aOc$	$(\widehat{O} \leq)$	$c \leq a$ and $a\widehat{O}b \rightarrow c\widehat{O}b$
$(O\widehat{O})$	aOa or $a\widehat{O}a$	$(\leq O\widehat{O})$	$c\overline{O}a$ and $c\overline{\widehat{O}}b \rightarrow a \leq b$
(C)	$aCb \rightarrow bCa$	$(\widehat{C})$	$a\widehat{C}b \rightarrow b\widehat{C}a$
$(CO1)$	$aOb \rightarrow aCb$	$(\widehat{C}\widehat{O}1)$	$a\widehat{O}b \rightarrow a\widehat{C}b$
$(CO2)$	$aCb \rightarrow aOa$	$(\widehat{C}\widehat{O}2)$	$a\widehat{C}b \rightarrow a\widehat{O}a$
$(C \leq)$	aCb and $b \leq c \rightarrow aCc$	$(\widehat{C} \leq)$	$a\widehat{C}b$ and $c \leq b \rightarrow a\widehat{C}c$
$(\ll\leq 1)$	$a \ll b \rightarrow a \leq b$		
$(\ll\leq 2)$	$a \leq b$ and $b \ll c \rightarrow a \ll c$	$(\ll\leq 3)$	$a \ll b$ and $b \leq c \rightarrow a \ll c$
$(\ll O)$	$a\overline{O}a \rightarrow a \ll b$	$(\ll \widehat{O})$	$b\overline{\widehat{O}}b \rightarrow a \ll b$
$(\ll CO)$	aCb and $b \ll c \rightarrow aOc$	$(\ll \widehat{C}\widehat{O})$	$c \ll a$ and $a\widehat{C}b \rightarrow c\widehat{O}b$
$(\ll C\widehat{O})$	$c\overline{C}a$ and $c\overline{\widehat{O}}b \rightarrow a \ll b$	$(\ll \widehat{C}O)$	$c\overline{O}a$ and $c\overline{\widehat{C}}b \rightarrow a \ll b$.

Note that all axioms of mereotopological structures are universal first-order conditions which are true in contact algebras under the standard definitions of the basic mereological relations [**29**] ($aOb \overset{\leftrightarrow}{def}\ a \cdot b \neq 0$, $a\widehat{O}b \overset{\leftrightarrow}{def}\ a + b \neq 1$). So a standard topological model of a mereotopological structure is any non-empty set of regular-closed subsets of a given topological space under the standard topological definitions of contact, dual contact, non-tangential part-of and the standard definitions of the mereological relations.

It is proved in [**29**] that each mereotopological structure is embeddable into a contact algebra (Theorem 26).

The following theorem relates EDC-lattices to mereotopological structures.

THEOREM 3.1. *Every EDC-lattice is a mereotopological structure under the standard definitions of the basic mereological relations.*

PROOF. Since all axioms of mereotopological structures are universal first-order sentences true in all contact algebras, then by Theorem 2.4 they follow from the axioms of EDC-lattice, which shows that they are true in all EDC-lattices. Another long and non-elegant, but direct proof of this theorem is to show one by one that all axioms of mereotopological structures are theorems of EDC-lattices. □

Let us note that mereotopological structures cannot be expressed in distributive contact lattices studied in [**11, 12**] just because dual contact and nontangential part-of are not expressible in them.

3.2. RCC-8 spatial relations. One of the most popular systems of topological relations in the community of QSRR is RCC-8. The system RCC-8 was

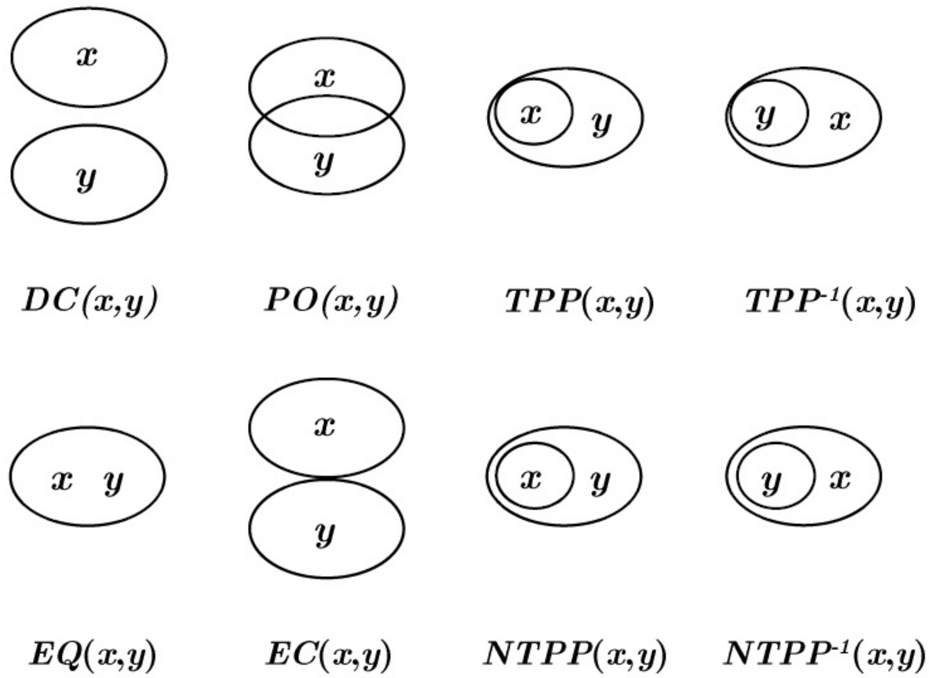

FIGURE 1. **RCC-8 relations**

introduced for the first time in [**14**]. It consists of 8 relations between non-empty regular closed subsets of arbitrary topological space. Having in mind the topological representation of contact algebras, it was given in [**40**] an equivalent definition of RCC-8 in the language of contact algebras:

DEFINITION 3.1. *The system* **RCC-8.**

- **disconnected** – $\mathbf{DC}(a,b)$**::** $a\overline{C}b$,
- **external contact** – $\mathbf{EC}(a,b)$**::** aCb *and* $a\overline{O}b$,
- **partial overlap** – $\mathbf{PO}(a,b)$**::** aOb *and* $a \not\leq b$ *and* $b \not\leq a$,
- **tangential proper part** – $\mathbf{TPP}(a,b)$**::** $a \leq b$ *and* $a \not\ll b$ *and* $b \not\leq a$,
- **tangential proper part**$^{-1}$ – $\mathbf{TPP}^{-1}(a,b)$**::** $b \leq a$ *and* $b \not\ll a$ *and* $a \not\leq b$,
- **nontangential proper part** $\mathbf{NTPP}(a,b)$**::** $a \ll b$ *and* $a \neq b$,
- **nontangential proper part**$^{-1}$ – $\mathbf{NTPP}^{-1}(a,b)$**::** $b \ll a$ *and* $a \neq b$,
- **equal** – $\mathbf{EQ}(a,b)$**::** $a = b$.

Looking at this definition it can be easily seen that the RCC-8 relations are expressible in the language of EDC-lattices. Let us note that RCC-8 relations are not expressible in the language of distributive contact algebras from [**11, 12**] just because dual contact and nontangential part-of are not expressible in them.

4. Additional axioms

In this Section we will formulate several additional axioms for EDC-lattices which are adaptations for the language of EDC-lattices of some known axioms considered in the context of contact algebras. First we will formulate some new lattice axioms for EDC-lattices - the so called extensionality axioms for the definable predicates of overlap - $aOb \leftrightarrow_{def} a \cdot b \neq 0$ and underlap - $a\widehat{O}b \leftrightarrow_{def} a + b \neq 1$.

(Ext O) $a \not\leq b \rightarrow (\exists c)(a \cdot c \neq 0$ and $b \cdot c = 0)$ - *extensionality of overlap*,

(Ext $\widehat{O}$) $a \not\leq b \rightarrow (\exists c)(a + c = 1$ and $b + c \neq 1)$ - *extensionality of underlap.*

We say that a lattice is *O-extensional* if it satisfies (Ext O) and *U-extensional* if it satisfies (Ext $\widehat{O}$). Note that the conditions (Ext O) and (Ext $\widehat{O}$) are true in Boolean algebras but not always are true in distributive lattices (see **[12]** for some examples, references and additional information about these axioms).

We will study also the following extensionality axioms.

(Ext C) $a \neq 1 \rightarrow (\exists b \neq 0)(a\overline{C}b)$ - *C-extensionality,*

(Ext $\widehat{C}$) $a \neq 0 \rightarrow (\exists b \neq 1)(a\overline{\widehat{C}}b)$ - *$\widehat{C}$-extensionality.*

In contact algebras these two axioms are equivalent. It is proved in **[12]** that (Ext $\widehat{O}$) implies that (Ext C) is equivalent to the following extensionality principle considered by Whitehead **[46]**

(EXT C) $a \not\leq b \rightarrow (\exists c)(aCc$ and $b\overline{C}c)$.

Just in a dual way one can show that (Ext O) implies that (Ext $\widehat{C}$) is equivalent to the following condition

(EXT $\widehat{C}$) $a \not\leq b \rightarrow (\exists c)(b\widehat{C}c$ and $a\overline{\widehat{C}}c)$.

Let us note that (EXT C) and (EXT $\widehat{C}$) are equivalent in contact algebras.

(Con C) $a \neq 0, b \neq 0$ and $a + b = 1 \rightarrow aCb$ - *C-connectedness axiom* and

(Con $\widehat{C}$) $a \neq 1, b \neq 1$ and $a \cdot b = 0 \rightarrow a\widehat{C}b$ - *$\widehat{C}$-connectedness axiom* .

In contact algebras these axioms are equivalent and guarantee topological representation in connected topological spaces.

(Nor 1) $a\overline{C}b \rightarrow (\exists c, d)(c + d = 1, a\overline{C}c$ and $b\overline{C}d)$,

(Nor 2) $a\overline{\widehat{C}}b \rightarrow (\exists c, d)(c \cdot d = 0, a\overline{\widehat{C}}c$ and $b\overline{\widehat{C}}d)$,

(Nor 3) $a \ll b \rightarrow (\exists c)(a \ll c \ll b)$.

Let us note that the above three axioms are equivalent in contact algebras and are known by different names. For instance (Nor 1) comes from the proximity theory **[37]** as *Efremovich axiom*, (Nor 3) sometimes is called *interpolation axiom.* We adopt the name *normality axioms* for (Nor 1), (Nor 2) and (Nor 3) because in topological representations they imply some normality conditions in the corresponding topological spaces. It is proved in **[10]** that (Nor 1) is true in the relational models (W, R) (see Section 1.2) if and only if the relation R is transitive and that (Nor 1) implies representation theorem in transitive models. In the next lemma we shall prove similar result using all normality axioms.

LEMMA 4.1. **Transitivity lemma.** *Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be a EDC-lattice satisfying the axioms (Nor1), (Nor 2) and (Nor 3) and let (W^c, R^c) be the canonical structure of $\underline{D}$ (see Section 2.3) Then:*

(i) R^c is a transitive relation.

(ii) $\underline{D}$ is representable in EDC-lattice over some system (W, R) with an equivalence relation R.

Proof. (i) Let Γ, Δ and Θ be prime filters in D such that
(1) $\Gamma R^c \Delta$ and
(2) $\Delta R^c \Theta$
and suppose for the sake of contradiction that
(3) $\Gamma \overline{R}^c \Theta$. By the definition of R^c we have to consider four cases.

Case 1: $\exists a \in \Gamma, b \in \Theta$ such that $a\overline{C}b$.

Then by (Nor 1) there exists c, d such that $c + d = 1$, $a\overline{C}c$ and $b\overline{C}d$. Since $c + d = 1$ then either $c \in \Delta$ or $d \in \Delta$. The case $c \in \Delta$ together with $a \in \Gamma$ imply by (1) aCc - a contradiction. The case $d \in \Delta$ together with $b \in \Theta$ imply by (2) bCd - again a contradiction.

Case 2: $\exists a \in \Gamma, b \notin \Theta$ such that $a \ll b$.

Then by (Nor 3) $\exists c$ such that $a \ll c$ and $c \ll b$. Consider the case $c \notin \Delta$. Then $a \in \Gamma$ and (1) imply $a \not\ll c$ a contradiction. Consider now $c \in \Delta$. Then $b \notin \Theta$ imply $c \not\ll b$ - again a contradiction.

In a similar way one can obtain a contradiction in the remaining two cases:
Case 3: $\exists a \notin \Gamma, b \in \Theta$ such that $b \ll a$ and
Case 4: $\exists a \notin \Gamma, b \notin \Theta$ such that $b\overline{\widehat{C}}a$.
(ii) The proof follows from (i) analogous to the proof of Theorem 2.3. □

Another kind of axioms which will be used in the topological representation theory in PART II are the so called rich axioms.

(U-rich $\ll$) $a \ll b \rightarrow (\exists c)(b + c = 1$ and $a\overline{C}c)$,

(U-rich $\widehat{C}$) $a\overline{\widehat{C}}b \rightarrow (\exists c, d)(a + c = 1, b + d = 1$ and $c\overline{C}d)$.

(O-rich $\ll$) $a \ll b \rightarrow (\exists c)(a \cdot c = 0$ and $c\overline{\widehat{C}}b)$,

(O-rich C) $a\overline{C}b \rightarrow (\exists c, d)(a \cdot c = 0, b \cdot d = 0$ and $c\overline{\widehat{C}}d)$.

Let us note that U-rich axioms will be used always with the U-extensionality axiom and that O-rich axioms will be used always with O-extensionality axiom.

The following lemma is obvious.

Lemma 4.2. *The axioms (U-rich $\ll$), (U-rich $\widehat{C}$), (O-rich $\ll$) and (O-rich C) are true in all contact algebras.*

4.1. Some good embedding properties. Let $(D_1, C_1, \widehat{C}_1, \ll_1)$ and $(D_2, C_2, \widehat{C}_2, \ll_2)$ be two EDC-lattices. We will write $D_1 \preceq D_2$ if D_1 is a substructure of D_2, i.e., D_1 is a sublattice of D_2, and the relations $C_1, \widehat{C}_1, \ll_1$ are restrictions of the relations $C_2, \widehat{C}_2, \ll_2$ on D_1. Since we want to prove embedding theorems, it is valuable to know under what conditions we have equivalences of the form:

D_1 satisfies some additional axiom iff D_2 satisfies the same axiom.

Remark 4.1. The importance of such conditions is related to the representation theory of EDC-lattices satisfying some additional axioms. In general, if we have some embedding theorem for EDC-lattice D satisfying a given additional axiom A, it is not known in advance that the lattice in which D is embedded also satisfies A. That is why it is good to have such conditions which automatically guarantee this. Below we formulate several such "good conditions": dense and dual dense sublattice, C- and $\widehat{C}$-separable sublattice.

DEFINITION 4.1. **Dense and dual dense sublattice.** *Let D_1 be a distributive sublattice of D_2. D_1 is called a* dense *sublattice of D_2 if the following condition is satisfied:*

(Dense) $(\forall a_2 \in D_2)(a_2 \neq 0 \Rightarrow (\exists a_1 \in D_1)(a_1 \leq a_2 \text{ and } a_1 \neq 0))$.

If h is an embedding of the lattice D_1 into the lattice D_2 then we say that h is a dense *embedding if the sublattice $h(D_1)$ is a dense sublattice of D_2.*

Dually, D_1 is called a dual dense *sublattice of D_2 if the following condition is satisfied:*

(Dual dense) $(\forall a_2 \in D_2)(a_2 \neq 1 \Rightarrow (\exists a_1 \in D_1)(a_2 \leq a_1 \text{ and } a_1 \neq 1))$.

If h is an embedding of the lattice D_1 into the lattice D_2 then we say that h is a Dual dense *embedding if the sublattice $h(D_1)$ is a dually dense sublattice of D_2.*

Note that in Boolean algebras, dense and dually dense conditions are equivalent; in distributive lattices this equivalence does not hold (see [**12**] for some known characterizations of density and dual density in distributive lattices).

For the case of contact algebras [**40**] and distributive contact lattices [**12**] we introduced the notion of *C-separability* as follows. Let $D_1 \preceq D_2$; we say that D_1 is a *C-separable sublattice of* D_2 if the following condition is satisfied:

(C-separable) $(\forall a_2, b_2 \in D_2)(a_2\overline{C}b_2 \Rightarrow (\exists a_1, b_1 \in D_1)(a_2 \leq a_1, b_2 \leq b_1, a_1\overline{C}b_1))$.

For the case of EDC-lattices we modified this notion adding two additional clauses corresponding to the relations $\widehat{C}$ and $\ll$ just having in mind the definitions of these relations in contact algebras. Namely

DEFINITION 4.2. **C-separability.** *Let $D_1 \preceq D_2$; we say that D_1 is a C-*separable EDC-sublattice of D_2 *if the following conditions are satisfied:*

(C-separability for C) -

$(\forall a_2, b_2 \in D_2)(a_2\overline{C}b_2 \Rightarrow (\exists a_1, b_1 \in D_1)(a_2 \leq a_1, b_2 \leq b_1, a_1\overline{C}b_1))$.

(C-separability for $\widehat{C}$) -

$(\forall a_2, b_2 \in D_2)(a_2\overline{\widehat{C}}b_2 \Rightarrow (\exists a_1, b_1 \in D_1)(a_2 + a_1 = 1, b_2 + b_1 = 1, a_1\overline{C}b_1))$.

(C-separability for $\ll$) -

$(\forall a_2, b_2 \in D_2)(a_2 \ll b_2 \Rightarrow (\exists a_1, b_1 \in D_1)(a_2 \leq a_1, b_2 + b_1 = 1, a_1\overline{C}b_1))$.

If h is an embedding of the lattice D_1 into the lattice D_2 then we say that h is a C-separable embedding *if the sublattice $h(D_1)$ is a C-separable sublattice of D_2.*

The notion of a $\widehat{C}$-separable embedding h is defined similarly. The following lemma is analogous to a similar result from [**40**] (Theorem 2.2.2) and from [**12**] (Lemma 5).

LEMMA 4.3. *Let D_1, D_2 be EDC-lattices and D_1 be a C-separable EDC-sublattice of D_2. Then:*

(i) If D_1 is a dually dense EDC-sublattice of D_2, then D_1 satisfies the axiom (Ext C) iff D_2 satisfies the axiom (Ext C),

(ii) D_1 satisfies the axiom (Con C) iff D_2 satisfies the axiom (Con C),

(iii) D_1 satisfies the axiom (Nor 1) iff D_2 satisfies the axiom (Nor 1),

(iv) D_1 satisfies the axiom (U-rich $\ll$) iff D_2 satisfies the axiom (U-rich $\ll$),

(v) D_1 satisfies the axiom (U-rich $\widehat{C}$) iff D_2 satisfies the axiom (U-rich $\widehat{C}$).

PROOF. Conditions (i), (ii) and (iii) have the same proof as in Theorem 2.2.2 from [**40**].

(iv) ($\Rightarrow$) Suppose that D_1 satisfies the axiom (U-rich $\ll$), $a_2, b_2 \in D_2$ and let $a_2 \ll b_2$. Then by (C-separability for $\ll$) we obtain: $(\exists a_1, b_1 \in D_1)(a_2 \leq a_1, b_2 + b_1 = 1, a_1\overline{C}b_1)$. Since D_1 is a sublattice of D_2 then $a_1, b_1 \in D_2$. From $a_2 \leq a_1$ and $a_1\overline{C}b_1$ we get $a_2\overline{C}b_1$. Thus we have just proved: $(a_2 \ll b_2 \rightarrow (\exists b_1 \in D_2)(b_2 + b_1 = 1$ and $a_2\overline{C}b_1)$ which shows that D_2 satisfies (U-rich $\ll$).

($\Leftarrow$) Suppose that D_2 satisfies the axiom (U-rich $\ll$), $a_1, b_1 \in D_1$ (hence $a_1, b_1 \in D_2$) and let $a_1 \ll b_1$. Then by (U-rich $\ll$) for D_2 we get: $(\exists c_2 \in D_2)(b_1 + c_2 = 1, a_1\overline{C}c_2)$. Since $a_1, c_2 \in D_2$ and $a_1\overline{C}c_2$, then by (C-separability for C) we get: $(\exists a_1', b_1' \in D_1)(a_1 \leq a_1', c_2 \leq b_1', a_1'\overline{C}b_1')$. Combining the above results we get: $1 = b_1 + c_2 \leq b_1 + b_1'$ and $a_1\overline{C}b_1'$. We have just proved the following: $a_1 \ll b_1 \rightarrow (\exists b_1' \in D_1)(b_1 + b_1' = 1, a_1\overline{C}b_1')$ which shows that D_1 satisfies (U-rich $\ll$).

(v) The proof is similar to that of (iv). □

The notion of $\widehat{C}$-separable sublattice can be defined in a dual way as follows:

DEFINITION 4.3. *Suppose that $D_1 \preceq D_2$; we say that D_1 is a $\widehat{C}$*-separable EDC-sublattice of D_2 *if the following condition is satisfied:*

($\widehat{C}$-separability for C) -

$(\forall a_2, b_2 \in D_2)(a_2Cb_2 \Rightarrow (\exists a_1, b_1 \in D_1)(a_1 + a_2 = 1, b_1 + b_2 = 1, a_1\overline{\widehat{C}}b_1))$,

($\widehat{C}$-separability for $\widehat{C}$) -

$(\forall a_2, b_2 \in D_2)(a_2\widehat{C}b_2 \Rightarrow (\exists a_1, b_1 \in D_1)(a_1 \leq a_2, b_1 \leq b_2, a_1\overline{\widehat{C}}b_1))$,

($\widehat{C}$-separability for $\ll$) -

$(\forall a_2, b_2 \in D_2)(a_2 \ll b_2 \Rightarrow (\exists a_1, b_1 \in D_1)(a_1 + a_2 = 1, b_1 \leq b_2, a_1\overline{\widehat{C}}b_1))$.

The notion of a $\widehat{C}$-separable embedding h is defined as in definition 4.2.

The following lemma is dual to Lemma 4.3 and can be proved in a dual way.

LEMMA 4.4. *Let D_1, D_2 be EDC-lattices and D_1 be a $\widehat{C}$-separable EDC-sublattice of D_2; then:*

(i) If D_1 is a dense EDC-sublattice of D_2, then D_1 satisfies the axiom (Ext $\widehat{C}$) iff D_2 satisfies the axiom (Ext $\widehat{C}$),

(ii) D_1 satisfies the axiom (Con $\widehat{C}$) iff D_2 satisfies the axiom (Con $\widehat{C}$),

(iii) D_1 satisfies the axiom (Nor 2) iff D_2 satisfies the axiom (Nor 2).

(iv) D_1 satisfies the axiom (O-rich $\ll$) iff D_2 satisfies the axiom (O-rich $\ll$).

(v) D_1 satisfies the axiom (O-rich $\widehat{C}$) iff D_2 satisfies the axiom (O-rich $\widehat{C}$).

COROLLARY 4.1. *Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be an EDC-lattice and $\underline{B} = (B, C)$ be a contact algebra. Then:*

(i) If h is a C-separable embedding of $\underline{D}$ into $\underline{B}$ then $\underline{D}$ must satisfy the axioms (U-rich $\ll$) and (U-rich $\widehat{C}$).

(ii) If h is a $\widehat{C}$-separable embedding of $\underline{D}$ into $\underline{B}$ then $\underline{D}$ must satisfy the axioms (O-rich $\ll$) and (O-rich $\widehat{C}$).

PROOF. (i) Note that by Lemma 4.2 $\underline{B}$ satisfies the axioms (U-rich $\ll$) and (U-rich $\widehat{C}$). Then by Lemma 4.3 (iv) and (v) $\underline{D}$ satisfies the axioms (U-rich $\ll$) and (U-rich $\widehat{C}$).

(ii) Similarly to (i) the proof follows from Lemma 4.2 and Lemma 4.4.

□

PART II: TOPOLOGICAL REPRESENTATIONS OF EXTENDED DISTRIBUTIVE CONTACT LATTICES

The aim of this second part of the paper is to investigate several kinds of topological representations of EDC-lattices. We concentrate our attention mainly on topological representations with some "good properties" in the sense of Section 4.1: dual density and C-separability, and their dual versions - density and $\widehat{C}$-separability.

5. Topological models of EDC-lattices

We assume some familiarity of the reader with the basic theory of topological spaces:(see [**15**]). First we recall some notions from topology. By a topological space we mean a set X provided with a family $C(X)$ of subsets, called closed sets, which contains the empty set $\varnothing$, the whole set X, and is closed with respect to finite unions and arbitrary intersections. Fixing $C(X)$ we say that X is endowed with a topology. A subset $a \subseteq X$ is called *open* if it is the complement of a closed set. A family of closed sets $\mathbf{CB}(X)$ is called a *closed basis* of the topology if every closed set can be represented as an intersection of sets from $\mathbf{CB}(X)$. In a similar way the topology of X can be characterized by the family $O(X)$ of open sets: it contains the empty set, X and is closed under finite intersections and arbitrary unions. A family $\mathbf{OB}(X)$ of open sets is called an *open basis* of the topology if every open set can be represented as an union of sets from $\mathbf{OB}(X)$. X is called *semiregular space* if it has a closed base of regular closed sets or an open base of regular open sets.

We remind the reader of the definitions of two important topological operations on sets - *closure operation* Cl, and *interior operation* Int. Namely $Cl(a)$ is the intersection of all closed sets of X containing a and $Int(a)$ is the union of all open sets included in a. Note that the operations Cl and Int are interdefinable: $Cl(a) = -Int(-a)$ and $Int(a) = -Cl(-a)$. Using the bases $\mathbf{CB}(X)$ and $\mathbf{OB}(X)$ the definitions of closure and interior operations have the following useful expressions:

$x \in Cl(a)$ iff $(\forall b \in \mathbf{CB}(X))(a \subseteq b \rightarrow x \in b)$,

$x \in Int(a)$ iff $(\exists b \in \mathbf{OB}(X))(b \subseteq a$ and $x \in b)$.

We say that a is a *regular closed set* if $a = Cl(Int(a))$ and a is a *regular open* set if $a = Int(Cl(a))$. It is a well known fact that the set $RC(X)$ of all regular closed subsets of X is a Boolean algebra with respect to the relations, operations and constants defined as follows: $a \leq b$ iff $a \subseteq b$, $0 = \varnothing$, $1 = X$, $a + b = a \cup b$, $a \cdot b = Cl(Int(a \cap b)$, $a^* = Cl(-a)$ where $-a = X \setminus a$. If we define a contact C by aCb iff $a \cap b \neq \varnothing$ then we obtain the standard topological model of contact algebra.

Another topological model of contact algebra is the set $RO(X)$ of regular open subsets of X. The relevant definitions are as follows: $a \leq b$ iff $a \subseteq b$, $0 = \varnothing$, $1 = X$, $a \cdot b = a \cap b$, $a + b = Int(Cl(a \cup b)$, $a^* = Int - a$. The contact relation is aCb iff $Cl(a) \cap Cl(b) \neq \varnothing$.

Note that these two models are isomorphic.

Topological model of EDC-lattice by regular-closed sets. Consider the contact algebra $RC(X)$ of regular closed subsets of X. Let us remove the operation a^* and define the relations $\widehat{C}$ and $\ll$ topologically according to their definitions in contact algebra as follows:

$a\widehat{C}b$ iff $Cl(-a) \cap Cl(-b) \neq \varnothing$ iff (equivalently) $Int(a) \cup Int(b) \neq X$.

$a \ll b$ iff $a \cap Cl(-b) = \varnothing$ iff (equivalently) $a \subseteq Int(b)$.

Obviously the obtained structure is a model of EDC-lattice. Also any distributive sublattice of $RC(X)$ with the same definitions of the relations C, $\widehat{C}$ and $\ll$ is a model of EDC-lattice. These models are considered as *standard topological models of EDC-lattice by regular closed sets.*

Topological model of EDC-lattice by regular-open sets. Consider the contact algebra $RO(X)$ of regular open subsets of X. Let us remove the operation a^* from the contact algebra $RO(X)$ and define the relations $\widehat{C}$ and $\ll$ topologically according to their definitions in the contact algebra as follows:

$a\widehat{C}b$ iff $Cl(Int(-a) \cap Cl(Int(-b)) \neq \varnothing$ iff (equivalently) $a \cup b \neq X$,

$a \ll b$ iff $Cl(a) \cap Cl(Int(-b)) = \varnothing$ iff (equivalently) $Cl(a) \subseteq b$.

Obviously the obtained structure is another standard topological model of EDC-lattice and any distributive sublattice of $RO(X)$ with the same relations C, $\widehat{C}$ and $\ll$ is also a model of EDC-lattice.

The main aim of PART II of the paper is the topological representation theory of EDC-lattices related to the above two standard models. The first simple result is the following representation theorem.

THEOREM 5.1. **Topological representation theorem for EDC-lattices.** *Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be an EDC-lattice. Then:*

(i) There exists a topological space X and an embedding of $\underline{D}$ into the contact algebra $RC(X)$ of regular closed subsets of X.

(ii) There exists a topological space Y and an embedding of $\underline{D}$ into the contact algebra $RO(Y)$ of regular open subsets of Y.

PROOF. It is shown in [**8**] that every contact algebra is isomorphic to a subalgebra of the contact algebra $RC(X)$ of regular closed subsets of some topological space X, and dually, that it is also isomorphic to a subalgebra of the contact algebra $RO(Y)$ of the regular open subsets of some topological space Y. Then the proof follows directly from this result and the Corollary 2.1.

□

The above theorem is not the best one, because it cannot be extended straightforwardly to EDC-lattices satisfying some of the additional axioms mentioned in Section 4. That is why we will study in the next sections representation theorems based on embeddings satisfying some of the good conditions described in Section 4.1. Before going on let us remind some other topological facts, which will be used later on.

A topological space X is called:

• *normal* if every pair of closed disjoint sets can be separated by a pair of open sets;

• *κ-normal* [**34**] if every pair of regular closed disjoint sets can be separated by a pair of open sets;

• *weakly regular* [**13**] if it is semiregular and for each nonempty open set a there exits a nonempty open set b such that $Cl(a) \subseteq b$;

• *connected* if it cannot be represented by a sum of two disjoint nonempty open sets;

• T_0 if for every pair of distinct points there is an open set containing one of them and not containing the other; X is called T_1 if every one-point set is a closed set, and X is called *Hausdorff* (or T_2) if each pair of distinct points can be separated by a pair of disjoint open sets.

• *compact* if it satisfies the following condition: let $\{A_i : i \in I\}$ be a non-empty family of closed sets of X such that for every finite subset $J \subseteq I$ the intersection $\bigcap\{A_i : i \in J\} \neq \varnothing$, then $\bigcap\{A_i : i \in I\} \neq \varnothing$.

The following lemma relates topological properties to the properties of the relations C, $\widehat{C}$ and $\ll$ and shows the importance of the additional axioms for EDC-lattices.

LEMMA 5.1. *(i) If X is semiregular, then X is weakly regular iff $RC(X)$ satisfies any of the axioms (Ext C), (Ext $\widehat{C}$).*

(ii) X is κ-normal iff $RC(X)$ satisfies any of the axioms (Nor 1), (Nor 2) and (Nor 3).

(iii) X is connected iff $RC(X)$ satisfies any of the axioms (Con C), (Con $\widehat{C}$).

(iv) If X is compact and Hausdorff, then $RC(X)$ satisfies (Ext C), (Ext $\widehat{C}$) and (Nor 1), (Nor 2) and (Nor 3) .

PROOF. A variant of the above lemma concerning only axioms (Ext C), (Nor 1) and (Con C) was proved, for instance, in [**13**]. Having in mind the equivalence of some of the mentioned axioms in $RC(X)$, it is obvious that the present formulation is equivalent to the cited result from [**13**]. □

5.1. Looking for good topological representations of EDC-lattices.

In topological representation theory of lattices the following three problems have to be solved: (1) for a given lattice L to associate to L a set X of points, (2) to define an embedding h into the set of subsets of X, and (3) to define in X a suitable topology. Very often the topology of X is determined by the embedding h considering the set $\{h(a) : a \in L\}$ as a base (closed or open) of the topology of X. Let us note that this construction sometimes yields good properties of the obtained topology - for instance compactness and some desirable topological separation properties. That is why we call in this paper such embeddings "good topological representations". However, good representations require sometimes some special properties of the lattice L, and this is just the subject of the present section.

The following topological theorem proved in [**12**] (Theorem 4) gives necessary and sufficient conditions for a closed base of a topology to be semiregular.

THEOREM 5.2. **First characterization theorem for semiregularity.** *Let X be a topological space and let $\mathbf{CB}(X)$ be a closed basis for X. Suppose that "$\cdot$" is a binary operation defined on the set $\mathbf{CB}(X)$ such that $(\mathbf{CB}(X), \varnothing, X, \cup, \cdot)$ is a lattice. Then:*

1. *The following conditions are equivalent:*
   - *(a) $\mathbf{CB}(X)$ is U-extensional.*
   - *(b) $\mathbf{CB}(X) \subseteq RC(X)$.*

  (c) *For all* $a, b \in \mathbf{CB}(X)$, $a \cdot b = Cl(Int(a \cap b))$.
  (d) $(\mathbf{CB}(X), \varnothing, X, \cup, \cdot)$ *is a dually dense sublattice of the Boolean algebra* $RC(X)$.
(2) *If any of the (equivalent) conditions (a),(b),(c) or (d) of 1. is fulfilled then:*
  (a) $(\mathbf{CB}(X), \varnothing, X, \cup, \cdot)$ *is a U-extensional distributive lattice.*
  (b) X *is a semiregular space.*

The following is a corollary of the above theorem.

COROLLARY 5.1. **[12]** *Let X be a topological space, let $L = (L, 0, 1, +, \cdot)$ be a lattice and let h be an embedding of the upper semi-lattice $(L, 0, 1, +)$ into the lattice $C(X)$ of closed sets of X. Suppose that the set $\mathbf{CB}(X) = \{h(a) : a \in L\}$ forms a closed basis for the topology of X. Then:*

(1) *The following conditions are equivalent:*
  (a) L *is U-extensional.*
  (b) $\mathbf{CB}(X) \subseteq RC(X)$.
  (c) *For all* $a, b \in L$, $h(a \cdot b) = Cl(Int(h(a) \cap h(b)))$.
  (d) h *is a dually dense embedding of* L *into the Boolean algebra* $RC(X)$.
(2) *If any of the (equivalent) conditions (a),(b),(c) or (d) of 1. is fulfilled then:*
  (a) L *is a U-extensional distributive lattice.*
  (b) X *is a semiregular space.*

A dual version of Theorem 5.2 is the following one.

THEOREM 5.3. **Second characterization theorem for semiregularity.** *Let X be a topological space and let $\mathbf{OB}(X)$ be an open basis for X. Suppose that $+$ is a binary operation defined on the set $\mathbf{OB}(X)$ such that $(\mathbf{OB}(X), \varnothing, X, \cap, +)$ is a lattice. Then:*

(1) *The following conditions are equivalent:*
  (a) $\mathbf{OB}(X)$ *is O-extensional.*
  (b) $\mathbf{OB}(X) \subseteq RO(X)$.
  (c) *For all* $a, b \in \mathbf{OB}(X)$, $a + b = Int(Cl(a \cup b))$.
  (d) $(\mathbf{OB}(X), \varnothing, X, \cap, +)$ *is a dually dense sublattice of the Boolean algebra* $RO(X)$.
(2) *If any of the (equivalent) conditions (a),(b),(c) or (d) of 1. is fulfilled then:*
  (a) $(\mathbf{OB}(X), \varnothing, X, \cap, +)$ *is an O-extensional distributive lattice.*
  (b) X *is a semiregular space.*

The following is a corollary of the above theorem.

COROLLARY 5.2. *Let X be a topological space, let $L = (L, 0, 1, +, \cdot)$ be a lattice and let h be an embedding of the lower semi-lattice $(L, 0, 1, \cdot)$ into the lattice $O(X)$ of open sets of X. Suppose that the set $\mathbf{OB}(X) = \{h(a) : a \in L\}$ forms an open basis for the topology of X. Then:*

(1) *The following conditions are equivalent:*
  (a) L *is O-extensional.*
  (b) $\mathbf{OB}(X) \subseteq RO(X)$.
  (c) *For all* $a, b \in L$, $h(a + b) = Int(Cl(h(a) \cup h(b)))$.
  (d) h *is a dense embedding of* L *into the Boolean algebra* $RO(X)$.
(2) *If any of the (equivalent) conditions (a),(b),(c) or (d) of 1. is fulfilled then:*
  (a) L *is a O-extensional distributive lattice.*

(b) X is a semiregular space.

REMARK 5.4. **(i) Dual dense representations.** Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be an EDC-lattice. Suppose that we want to represent $\underline{D}$ by an embedding h in the set $RC(X)$ of regular closed sets of some topological space X such that the topology of X is determined by the set $\mathbf{CB}(X) = \{h(a) : a \in D\}$ considered as a closed base for X. Then Corollary 5.1 say that h must be a dual dense embedding. The Corollary 5.1 states also that this fact is equivalent to U-extensionality of $\underline{D}$, which means that $\underline{D}$ must satisfy the axiom (Ext $\widehat{O}$) - extensionality of underlap. If in addition we want to apply the C-separability property from Lemma 4.3, then we must assume that h is also a C-separable embedding into $RC(X)$. But then Corollary 4.1 implies that $\underline{D}$ must satisfy also the axioms (U-rich $\ll$) and (U-rich $\widehat{C}$).

(ii) Dense representations. Similar to the above conclusion is the following. Suppose that we want to represent $\underline{D}$ by an embedding h into the set $RO(X)$ of regular open subsets of some topological space X such that the the topology of X to be determined by the set $\mathbf{OB}(X) = \{h(a) : a \in D\}$ considered as an open base for X. Then Corollary 5.2 say that h must be a dense embedding. The Corollary 5.2 states also that this fact is equivalent to O-extensionality of $\underline{D}$, which means that $\underline{D}$ must satisfy the axiom (Ext O) - extensionality of overlap. If in addition we want to apply the $\widehat{C}$-separability property of Lemma 4.4, then we must assume that h is also a $\widehat{C}$-separable embedding into $RO(X)$. But then Corollary 4.1 implies that $\underline{D}$ must satisfy also the axioms (O-rich $\ll$) and (O-rich $\widehat{C}$). □

DEFINITION 5.1. **U-rich and O-rich EDC-lattices.** *Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be an EDC-lattice. Then:*

(i) $\underline{D}$ is called U-rich EDC-lattice if it satisfies the axioms (Ext $\widehat{O}$), (U-rich $\ll$) and (U-rich $\widehat{C}$).

(ii) $\underline{D}$ is called O-rich EDC-lattice if it satisfies the axioms (Ext O), (O-rich $\ll$) and (O-rich $\widehat{C}$).

A question arises - aren't U-rich EDC-lattices (O-rich EDC-lattices) Boolean algebras? The answer is "no" as it can be seen from the next proposition.

PROPOSITION 5.1. *(i)There is an U-rich EDCL $\underline{D} = (D, C, \widehat{C}, \ll)$ such that $(\exists x \in D)(\forall y \in D)\neg(x + y = 1$ and $x.y = 0)$;*
(ii)There is an O-rich EDCL $\underline{D} = (D, C, \widehat{C}, \ll)$ such that $(\exists x \in D)(\forall y \in D)\neg(x + y = 1$ and $x.y = 0)$.

PROOF. (i) Let (W, R) be a relational structure, where $W = (-\infty; +\infty)$, $R = W \times W$. We consider the contact algebra of all subsets of W: $\underline{B_1} = (2^W, \subseteq, \emptyset, W, *, C_R, \widehat{C_R}, \ll_R)$. It turns out that $aC_Rb \leftrightarrow a, b \neq \emptyset$, $a\widehat{C_R}b \leftrightarrow a, b \neq 1$, $a \ll_R b \leftrightarrow a = \emptyset$ or $b = (-\infty; +\infty)$. $\underline{B_1} = (2^W, \subseteq, \emptyset, W, C_R, \widehat{C_R}, \ll_R)$ is an EDCL. We consider the substructure of $\underline{B_1}$ $\underline{B}$ with universe B, consisting of the following sets: $\emptyset$, W, $(-\infty; 1]$, $[0; +\infty)$, $[0; 1]$ and all sets of the kind:
1) $(-\infty; a_1) \cup (a_2; a_3) \cup \ldots \cup (a_{2n}; a_{2n+1}) \cup (a_{2n+2}; +\infty)$
2) $[0; a_1) \cup (a_2; a_3) \cup \ldots \cup (a_{2n}; a_{2n+1}) \cup (a_{2n+2}; +\infty)$
3) $(-\infty; a_1) \cup (a_2; a_3) \cup \ldots \cup (a_{2n}; a_{2n+1}) \cup (a_{2n+2}; 1]$
4) $[0; a_1) \cup (a_2; a_3) \cup \ldots \cup (a_{2n}; a_{2n+1}) \cup (a_{2n+2}; 1]$,
where $n \geq 0$, $0 < a_1 < a_2 < a_3 < \ldots < a_{2n} < a_{2n+1} < a_{2n+2} < 1$.

It can be easily verified that B is closed under $\cup$ and $\cap$. Consequently $\underline{B}$ is a distributive lattice of sets with 0 and 1. Since $\underline{B}$ is a substructure of $\underline{B_1}$, $\underline{B_1}$ is an EDCL, the axioms of EDCL are universal formulas, we have that $\underline{B}$ is an EDCL. For $(-\infty;1]$ it does not exist $x \in B$ such that $(-\infty;1] \cup x = (-\infty;+\infty)$, $(-\infty;1] \cap x = \emptyset$.
($\bullet$) We will prove that $\underline{B}$ satisfies (U-rich $\ll$). Let $a, b \in B$ and $a \ll b$. Then $a = \emptyset$ or $b = W$.
Case 1: $a = \emptyset$
We have $b \cup W = W$ and $a\overline{C_R}W$.
Case 2: $b = W$
We have $b \cup \emptyset = W$ and $a\overline{C_R}\emptyset$.
($\bullet$) We will prove that $\underline{B}$ satisfies (U-rich $\widehat{C}$). Let $a, b \in B$ and $a\overline{\widehat{C}}b$. Then $a = W$ or $b = W$. Without loss of generality $a = W$. We have $a \cup \emptyset = W$, $b \cup W = W$ and $\emptyset\overline{C_R}W$.
($\bullet$) We will prove that $\underline{B}$ satisfies (Ext $\widehat{O}$). Let $a, b \in B$ and $a \not\subseteq b$. There is A such that $A \in a$, $A \notin b$. We will prove that there is $c \in B$ such that $a \cup c = W$, $b \cup c \neq W$
Case 1: a is of the kind 1), 2), 3) or 4)
Case 1.1: b is of the kind 1), 2), 3) or 4)
Case 1.1.1: $A \in (0;1)$
There are x, y such that $0 < x < y < 1$, $A \in (x, y)$, $(x, y) \subseteq a$, $(x, y) \cap b = \emptyset$. Let x_1, y_1 are such that $x < x_1 < A < y_1 < y$. Let $c \stackrel{def}{=} (-\infty;x_1) \cup (y_1;+\infty)$.
Case 1.1.2: $A \notin (0;1)$
b is of the kind 1), 2), 3) or 4). Consequently $0 \in b$, $1 \in b$. Consequently $A \neq 0$, $A \neq 1$. Without loss of generality $A < 0$. We also have $A \in a$, so there is $a_1 \in (0;1)$ such that $(-\infty, a_1) \subseteq a$. Let x, y be such that $0 < x < y < a_1$. Let $c \stackrel{def}{=} [0;x) \cup (y;+\infty)$.
Case 1.2: $b = \emptyset$
There is $a_1 \in (0;1)$ such that $[0;a_1) \subseteq a$. Let x, y be such that $0 < x < y < a_1$. Let $c \stackrel{def}{=} (-\infty;x) \cup (y;+\infty)$.
Case 1.3: $b = (-\infty;1]$ or $b = [0;+\infty)$
Without loss of generality $b = (-\infty;1]$. Let $c = (-\infty;1]$. $A \notin b$; so $1 < A$ but $A \in a$ and a is of the kind 1), 2), 3) or 4), so there is $a' \in (0;1)$ such that $(a';+\infty) \subseteq a$. Consequently $a \cup c = W$. We also have $b \cup c \neq W$.
Case 1.4: $b = [0;1]$
$A \notin b$, so $A < 0$ or $A > 1$. Without loss of generality $A < 0$. $A \in a$ and a is of the kind 1), 2), 3) or 4), so there is $a_1 \in (0;1)$ such that $(-\infty;a_1) \subseteq a$. Let $c \stackrel{def}{=} [0;+\infty)$.
Case 2: $a = (-\infty;+\infty)$
We take $c \stackrel{def}{=} \emptyset$.
Case 3: $a = (-\infty;1]$
Case 3.1: $b = \emptyset$
We take $c \stackrel{def}{=} [0;+\infty)$.
Case 3.2: $b = [0;+\infty)$
We take $c \stackrel{def}{=} [0;+\infty)$.
Case 3.3: $b = [0;1]$

We take $c \stackrel{def}{=} [0;+\infty)$.
Case 3.4: b is of the kind 1), 2), 3) or 4).
Case 3.4.1: $A \in (0;1)$
The proof is similar to the proof in case 1.1.1.
Case 3.4.2: $A \notin (0;1)$
We have $A \neq 0$ because otherwise $A \in b$. Consequently $A < 0$. We take $c \stackrel{def}{=} [0;+\infty)$.
Case 4: $a = [0;+\infty)$
The proof is similar to the proof in case 3.
Case 5: $a = [0;1]$
Case 5.1: $b = \emptyset$
We take $c \stackrel{def}{=} (-\infty;\frac{1}{3}\Big) \cup \Big(\frac{2}{3};+\infty)$
Case 5.2: b is of the kind 1), 2), 3) or 4)
The proof is similar to the proof in case 1.1.1.
(ii) Dual proof. □

The aim of the next sections is to develop the topological representation theory of U-rich and O-rich EDC-lattices.

6. Topological representation theory of U-rich EDC-lattices

The aim of this section is to develop a topological representation theory for U-rich EDC-latices. According to Theorem 5.2 we will look for a dual dense representation with regular closed sets (see 5.4 (i)). To realize this we will follow the representation theory of contact algebras by regular closed sets developed in [**8, 40**], updating the results of Section 4 from [**12**] to the case of U-rich EDC-lattices. We will consider also extensions of U-rich EDC-lattices with some of the additional axioms mentioned in Section 4. The scheme of the representation procedure is the following: for each U-rich EDC-lattice D from a given class, determined by the additional axioms, we will do the following:

- Define a set $X(D)$ of "abstract points" of $\underline{D}$,
- define a topology in $X(D)$ by the set $\mathbf{CB}(X(D)) = \{h(a) : a \in D\}$, considered as a closed base of the topology, where h is the intended embedding of Stone type: $h(a) = \{\Gamma : \Gamma$ is "abstract point" and $a \in \Gamma\}$. $X(D)$ is called the *canonical topological space of* $\underline{D}$ and h is called *canonical embedding*,
- establish that h is a dual dense embedding of the lattice $\underline{D}$ into the Boolean algebra $RC(X(D))$ of regular closed sets of the space $X(D)$.

We will consider separately the cases of representations in T_0, T_1 and T_2 spaces which requires introducing different "abstract points".

6.1. Representations in T_0 spaces. Throughout this section we consider that $\underline{D} = (D, C, \widehat{C}, \ll)$ is a U-rich EDC-lattice.

6.1.1. *Abstract points of* $\underline{D}$. As in [**12**], we consider the abstract points of $\underline{D}$ to be clans (see [**8**] for the origin of this notion). The definition is the following. A subset $\Gamma \subseteq D$ is a *clan* if it satisfies the following conditions:
(Clan 1) $1 \in \Gamma$, $0 \notin \Gamma$,
(Clan 2) If $a \in \Gamma$ and $a \leq b$, then $b \in \Gamma$,
(Clan 3) If $a + b \in \Gamma$, then $a \in \Gamma$ or $b \in \Gamma$,

(Clan 4) If $a, b \in \Gamma$ then aCb.

These conditions are similar to the conditions for prime filters.

Γ is a *maximal clan* if it is maximal with respect to the set-inclusion. We denote by CLAN(D) (MaxCLAN(D)) the set of all (maximal) clans of $\underline{D}$.

The notion of clan is an abstraction from the following natural example. Let X be a topological space and $RC(X)$ be the contact algebra of regular-closed subsets of X and let $x \in X$. Then the set $\Gamma_x = \{a \in RC(X) : x \in a\}$ is a clan.

Now we will present a construction of clans which is similar to the constructions of clans in contact algebras. First we will introduce a new canonical relation between prime filters.

Definition 6.1. *Let U, V be prime filters. Define a new canonical relation R_C (R_C-*canonical relation*) between prime filters as follows:*
$UR_CV \leftrightarrow_{def} (\forall a \in U)(\forall b \in V)(aCb)$.

Let us note that the relation R_C depends only on C and can be defined also for filters. It is different from the canonical relation between prime filters defined in Section 2.3, but the presence of U-rich axioms makes it equivalent to R^c as it can be seen from the following lemma.

Lemma 6.1. *Let U, V be prime filters and R_C the relation defined as $UR_CV \leftrightarrow_{def}$ $(\forall a \in U)(\forall b \in V)(aCb)$.*
Then

(i) R_C is reflexive and symmetric relation.

(ii) If $\underline{D}$ satisfies the axioms (U-rich $\ll$) and (U-rich $\widehat{C}$) then $R_C = R^c$.

Proof. (i) follows from the axioms (C4) and (C5).

(ii) The inclusion $R^c \subseteq R_C$ follows directly by the definition of R^c. For the converse inclusion suppose UR_CV. To show UR^cV we have to inspect the four cases of the definition of R^c.

Claim 1: $a \in U$ and $b \in V$ implies aCb. This is just by the definition of R_C .

Claim 2: $a \in U$ and $b \notin V$ implies $a \not\ll b$. For the sake of contradiction suppose $a \in U$ and $b \notin V$ but $a \ll b$. Then by axiom (U-rich $\ll$) ($a \ll b \rightarrow (\exists c)(b + c = 1$ and $a\overline{C}c)$, we obtain $b + c = 1$ and $a\overline{C}c$. Conditions $b + c = 1$ and $b \notin V$ imply $c \in V$. But $a \in U$, so aCc - a contradiction.

Claim 3: $a \notin U$ and $b \in V$ implies $b \not\ll a$. The proof is similar to the proof of Claim 2.

Claim 4: $a \notin U$ and $b \notin V$ implies $a\widehat{C}b$. The proof is similar to the proof of Claim 2 by the use of axiom

(U-rich $\widehat{C}$) $a\overline{\widehat{C}}b \rightarrow (\exists c, d)(a + c = 1, b + c = 1$ and $c\overline{C}d)$.

□

The following statement lists some facts about the relation R_C.

Facts 6.1. [**10, 8, 12**].

1. *Let F, G be filters and FR_CG then there are prime filters U, V such that $F \subseteq U$, $G \subseteq V$ and UR_CV.*
2. *For all $a, b \in D$: aCb iff there exist prime filters U, V such that UR_CV, $a \in U$ and $b \in V$.*

In the following lemma we list some facts about clans (see, for instance, [**8, 12**]).

FACTS 6.2. *(1) Every prime filter is a clan.*

(2) The complement of every clan is an ideal.

(3) If Γ is a clan and F is a filter such that $F \subseteq \Gamma$, then there is a prime filter U such that $F \subseteq U \subseteq \Gamma$. In particular, if $a \in \Gamma$, then there exists a prime filter U such that $a \in U \subseteq \Gamma$.

(4) Every clan Γ is the union of all prime filters contained in Γ.

(5) Every clan is contained in a maximal clan.

(6) Let Σ be a nonempty set of prime filters such that for every $U, V \in \Sigma$ we have $UR_C V$ and let Γ be the union of the elements of Σ. Then Γ is a clan and every clan can be obtained in this way.

(7) Let U, V be prime filters, Γ be a clan and $U, V \subseteq \Gamma$,. Then $UR_C V$ and $UR^c V$.

LEMMA 6.2. *Let Γ be a clan and $a \in D$. Then the following two conditions are equivalent:*

(i) $(\forall c \in D)(a + c = 1 \rightarrow c \in \Gamma)$,

(ii) There exists a prime filter $U \subseteq \Gamma$ such that $a \notin U$.

PROOF. (i)$\rightarrow$ (ii). Suppose that (i) holds. It is easy to see that the set $F = \{c : a + c = 1\}$ is a filter. The complement $\overline{\Gamma}$ of Γ is an ideal (Facts 6.2) and hence $\overline{\Gamma} \oplus (a]$ is an ideal. We will show that $F \cap \overline{\Gamma} \oplus (a] = \varnothing$. Suppose the contrary. Then there is a c such that $a + c = 1$ (and hence by (i) $c \in \Gamma$) and $c \in \overline{\Gamma} \oplus (a]$. Then there is $x \in \overline{\Gamma}$ such that $c \leq x + a$. From here we get: $1 = a + c \leq a + x + a = x + a$, hence $x + a = 1$ and by (i) - $x \in \Gamma$, contrary to $x \in \overline{\Gamma}$. Now we can apply Filter-extension Lemma and obtain a prime filter U extending F such that $U \cap \overline{\Gamma} \oplus (a] = \varnothing$. It follows from here that $a \notin U$, $U \cap \overline{\Gamma} = \varnothing$ which implies $U \subseteq \Gamma$.

(ii)$\rightarrow$(i). Suppose (ii) holds: $U \subseteq \Gamma$ and $a \notin U$. Suppose $a + c = 1$. Then $c \in U \subseteq \Gamma$, so $c \in \Gamma$ - (i) is fulfilled.

□

6.1.2. *Defining the canonical topological space $X(D)$ of $\underline{D}$ and the canonical embedding h.* Define the Stone like embedding: $h(a) = \{\Gamma \in CLAN(D) : a \in \Gamma\}$ and consider the set $\mathbf{CB}(X) = \{h(a) : a \in D\}$ as a closed base of the topology in $X(D) = CLAN(D)$.

LEMMA 6.3. *The space $X(D)$ is semiregular and h is a dually dense embedding of $\underline{D}$ into the contact Boolean algebra $RC(X(D))$.*

PROOF. Using the properties of clans, one can easily check that $h(0) = \varnothing$, $h(1) = X$, and that $h(a + b) = h(a) \cup h(b)$. This shows that the set $\mathbf{CB}(X(D)) = \{h(a) : a \in D\}$ is closed under finite unions and, in fact, it is a closed basis for the topology of X. Also we have the implication: $a \leq b$ then $h(a) \subseteq h(b)$.

To show that h is an embedding we use the fact that prime filters are clans and prove that $a \not\leq b$ implies $h(a) \not\subseteq h(b)$. Indeed, from $a \not\leq b$ it follows by the theory of distributive lattices (see [**2**]) that there exists a prime filter U (which is also a clan) such that $a \in U$ (so $U \in h(a)$) and $b \notin U$ (so, $U \notin h(b)$), which proves that $h(a) \not\subseteq h(b)$. Consequently, h is an embedding of the upper semi-lattice $(D, 0, 1, +)$ into the lattice of closed sets of the space $X(D)$. By Corollary 5.1, $X(D)$ is a semiregular space and h is a dually dense embedding of D into the Boolean algebra

$RC(X)$. It remains to show that h preserves the relations $C, \widehat{C}$ and $\ll$. This follows from the following claim.

CLAIM 6.3. *(i) Let Γ be a clan and $a \in D$. The following equivalence holds: $\Gamma \in h(a)$ iff there exists a prime filter U such that $a \in U \subseteq \Gamma$.*
(ii) Let Γ be a clan and $a \in D$. Then following conditions are equivalent:

(I) $(\forall c \in D)(a + c = 1 \rightarrow c \in \Gamma)$,
(II) $\Gamma \in Cl(-h(a))$,
(III) There exists a prime filter U such that $a \notin U \subseteq \Gamma$.

(iii) aCb iff $h(a) \cap h(b) \neq \varnothing$,
(iv) $a \not\ll b$ iff $h(a) \cap Cl(-h(b)) \neq \varnothing$.
(v) $a\widehat{C}b$ iff $Cl(-h(a)) \cap Cl(-h(b)) \neq \varnothing$,

Proof of the claim. (i) follows easily from Facts 6.2 (3.).
(ii) The proof of $(I) \leftrightarrow (II)$ follows by the following sequence of equivalences:

$(\forall c \in D)(a + c = 1 \rightarrow c \in \Gamma)$ iff
$(\forall c \in D)(h(a) \cup h(c) = X(D) \rightarrow \Gamma \in h(c))$ iff
$(\forall c \in D)(-h(a) \subseteq h(c) \rightarrow \Gamma \in h(c))$ iff
$\Gamma \in Cl(-h(a))$

The first equivalence holds because h is an embedding of the upper semi-lattice $(D, 0, 1, +)$ into the lattice of closed sets of the space $X(D)$, the third equivalence uses the fact that the set $\{h(c) : c \in D\}$ is a closed base of the topology of $X(D)$.
The equivalence $(I) \leftrightarrow (III)$ is just the Lemma 6.2.

(iii) ($\Rightarrow$) Suppose aCb, then by Lemma 2.6 (i) there exist prime filters U, and V such that UR^cV, $a \in U$ and $b \in V$. Let $\Gamma = U \cup V$. By Facts 6.2 Γ is a clan, obviously containing a and b, which implies $h(a) \cap h(b) \neq \varnothing$.

($\Leftarrow$) Suppose $h(a) \cap h(b) \neq \varnothing$. Then there exists a clan Γ containing a and b, hence aCb.

(iv) ($\Rightarrow$) Suppose $a \not\ll b$. Then by Lemma 2.6 (ii) there exist prime filters U, V such that UR^cV, $a \in U$ and $b \notin V$. Let $\Gamma = U \cup V$, then Γ is a clan containing U and V. So, $a \in \Gamma$ and hence $\Gamma \in h(a)$. From the condition $b \notin V \subseteq \Gamma$ we obtain by (ii) that $\Gamma \in Cl(-h(b))$ and hence $h(a) \cap Cl(-h(b)) \neq \varnothing$.

($\Leftarrow$) Suppose $h(a) \cap Cl(-h(b)) \neq \varnothing$. Then there exists a clan $\Gamma \in h(a)$ and $\Gamma \in Cl(-h(b))$. It follows by (i) that there exists a prime filter U such that $a \in U \subseteq \Gamma$ and by (ii) we obtain that there exists a prime filter V such that $b \notin V \subseteq \Gamma$. Condition $U, V \subseteq \Gamma$ implies by Facts 6.2 (7.) that UR^cV. Using the properties of the relation R^c and $a \in U$ and $b \notin V$ we get $a \not\ll b$.

(v) The proof of (v) is similar to the proof of (iv) with the use of Lemma 2.7. This finishes the proof of Lemma 6.3 □

LEMMA 6.4. *The following conditions are true for the canonical space $X(D)$:*
(i) $X(D)$ is T_0.
(ii) $X(D)$ is compact.

PROOF. The proof is the same as the proof of Lemma 19 from [**12**].

□

LEMMA 6.5. *The mapping h is a C-separable embedding of D into $RC(X(D))$.*

PROOF. This lemma was proved in [**12**] by a special construction. Since the definition of C-separability for EDC-lattices uses an extended definition for which the special construction from [**12**] does not hold, in this paper we give a new proof deducing the statement from the compactness of the space $X(D)$.

We have to prove the following three statements, corresponding to the three clauses of the condition of C-separability (see Definition 4.2).

(C-separability for C) $(\forall\alpha,\beta\in RC(X(D)))(\alpha\cap\beta=\varnothing\rightarrow(\exists a,b\in D)(\alpha\subseteq h(a),\beta\subseteq h(b),h(a)\cap h(b))=\varnothing$.

(C-separability for $\widehat{C}$) $(\forall\alpha,\beta\in RC(X(D))(Cl(-\alpha)\cap Cl(-\beta)=\varnothing\rightarrow(\exists a,b\in D)(\alpha\cup h(a)=X(D),\beta\cup h(b)=X(D),h(a)\cap h(b)=\varnothing)$.

(C-separability for $\ll$) $(\forall\alpha,\beta\in RC(X(D))(\alpha\cap Cl(-\beta)=\varnothing\rightarrow(\exists a,b\in D)(\alpha\subseteq h(a),\beta\cup h(b)=X(D),h(a)\cap h(b)=\varnothing)$.

As an example we shall prove the condition (C-separability for C). The proofs for the other two conditions are similar.

Proof of (C-separability for C). Let $\alpha,\beta\in RC(X(D))$ and $\alpha\cap\beta=\varnothing$. Since α and β are closed sets they can be represented as intersections from the elements of the basis $\mathbf{CB}(X(D))=\{h(c):c\in D\}$ of $X(D)$. So there are subsets $A,B\subseteq\mathbf{CB}(X(D))$ such that $\alpha=\bigcap\{h(c):h(c)\in A\}$ and $\beta=\bigcap\{h(c):h(c)\in B\}$. Then $\alpha\cap\beta=\bigcap\{h(c):h(c)\in A\}\cap\bigcap\{h(c):h(c)\in B\}=\varnothing$. By the compactness of $X(D)$ (Lemma 6.4 (ii)), there are finite subsets $A_0\subseteq A$ and $B_0\subseteq B$ such that $\alpha\cap\beta=\bigcap\{h(c):h(c)\in A_0\}\cap\bigcap\{h(c):h(c)\in B_0\}=\varnothing$. Let $A_0=\{h(c_1),...,h(c_n)\}$ and $B_0=\{h(d_1),...,h(d_m)\}$ and let $a=c_1\cdot...\cdot c_n$ and $b=d_1\cdot...\cdot d_m$. Then $h(a)\subseteq h(c_i)$, $i=1...n$ and from here we get $h(a)\subseteq h(c_1)\cap...\cap h(c_n)$. Analogously we obtain that $h(b)\subseteq h(d_1)\cap...\cap h(d_m)$. Consequently $h(a)\cap h(b)\subseteq(h(c_1)\cap...\cap h(c_n))\cap(h(d_1)\cap...\cap h(d_m))=\varnothing$, so $h(a)\cap h(b)=\varnothing$. Also we have $\alpha\subseteq h(c)$ for all $h(c)\in A$ and consequently for all $h(c)\in A_0$. Hence $\alpha\subseteq h(c_1)\cdot...\cdot h(c_n)=h(c_1\cdot...\cdot c_n)=h(a)$, so $\alpha\subseteq h(a)$. Analogously we get $\beta\subseteq h(b)$. □

The following theorem is the main result of this section.

THEOREM 6.4. **Topological representation theorem for U-rich EDC-lattices**
Let $\underline{D}=(D,C,\widehat{C},\ll)$ be an U-rich EDC-lattice. Then there exists a compact semiregular T_0-space X and a dually dense and C-separable embedding h of $\underline{D}$ into the Boolean contact algebra $RC(X)$ of the regular closed sets of X. Moreover:

(i) $\underline{D}$ satisfies (Ext C) iff $RC(X)$ satisfies (Ext C); in this case X is weakly regular.

(ii) $\underline{D}$ satisfies (Con C) iff $RC(X)$ satisfies (Con C); in this case X is connected.

(iii) $\underline{D}$ satisfies (Nor 1) iff $RC(X)$ satisfies (Nor 1); in this case X is κ-normal.

PROOF. Let X be the canonical space $X(D)$ of $\underline{D}$ and h be the canonical embedding of $\underline{D}$. Then, the theorem is a corollary of Lemma 6.3, Lemma 6.4, Lemma 6.5 and Lemma 5.1. □

Note that Theorem 6.4 generalizes several results from [**8, 13**] to the distributive case.

6.2. Representations in T_1 spaces. The aim of this section is to obtain representations of some U-rich EDC-lattices in T_1-spaces extending the corresponding results from [**12**]. The constructions will be slight modifications of the corresponding constructions from the previous section, so we will be sketchy.

Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be an U-rich EDC-lattice. In the previous section the abstract points were clans and this guarantees that the representation space is T_0. To obtain representations in T_1 spaces we assume abstract points to be maximal clans, so for the canonical space of $\underline{D}$ we put $X(D) = MaxCLAN(D)$ and define the canonical embedding h to be $h(a) = \{\Gamma \in MaxCLAN(D) : a \in \Gamma\}$. The topology in $X(D)$ is defined considering the set $\mathbf{CB}(X(D)) = \{h(a) : a \in D\}$ to be the closed base for the space. Note that in general, without additional axioms we cannot prove in this case that h is an embedding. In order to guarantee this we will assume that $\underline{D}$ satisfies additionally the axiom of C-extensionality

(Ext C) $a \neq 1 \to (\exists b \neq 0)(a\overline{C}b)$.

Note that in this case, due to U-extensionality (see Section 4), the lattice $\underline{D}$ satisfies also the axiom

(EXT C) $a \not\leq b \to (\exists c)(aCc$ and $b\overline{C}c)$,

which is essential in the proof that h is an embedding.

LEMMA 6.6. *The space $X(D)$ is semiregular and h is a dually dense embedding of D into the contact Boolean algebra $RC(X(D))$.*

PROOF. The proof is similar to the proof of Lemma 6.3, so we will indicate only the differences. First we show that h is an embedding of the upper semi-lattice $(D, 0, 1, +)$ into the lattice of closed sets of the space $X(D)$. The only new thing which we have to show is: If $a \not\leq b$ then $h(a) \not\subseteq h(b)$. To do this suppose $a \not\leq b$. Then by axiom (EXT C) there exists $c \in D$ such that aCc but $b\overline{C}c$. Condition aCc implies that there exist prime filters U, V such that UR^cV, $a \in U$ and $c \in V$. Let $\Gamma_0 = U \cup V$. Γ_0 is a clan and by Facts 6.2 it is contained in a maximal clan Γ. Obviously $a, c \in \Gamma$, so $\Gamma \in h(a)$. But $b\overline{C}c$ implies that $b \notin \Gamma$ (otherwise we will get bCc). Conditions $\Gamma \in h(a)$ and $\Gamma \notin h(b)$ show that $h(a) \not\subseteq h(b)$. Thus, by Corollary 5.1, h is a dually dense embedding of $\underline{D}$ into the Boolean algebra $RC(X(\underline{D}))$. It remains to show that h preserves the relations $C, \widehat{C}$ and $\ll$. The proof is almost the same as in the corresponding proof of Lemma 6.3. The only new thing is when we construct a certain clan from prime filters satisfying the relation UR^cV in the form $U \cup V$, then we extend it into a maximal clan. Note also that Claim 6.3 remains true. We demonstrate this by considering only the preservation of $\ll$. We have to show:

$a \not\ll b$ iff $h(a) \cap Cl(-h(b) \neq \varnothing$

($\Rightarrow$) Suppose $a \not\ll b$. Then by Lemma 2.6 $(\exists U, V \in PF(D))(a \in U$ and $b \notin V$ and $UR^cV)$. Define $\Gamma_0 = U \cup V$. Γ_0 is a clan containing U and V. Extend Γ_0 into a maximal clan Γ. Then Γ contains a, so $\Gamma \in h(a)$. We have also that $b \notin V \subseteq \Gamma$, so by the Claim 6.3 $\Gamma \in Cl(-h(b))$.

($\Leftarrow$) The proof is identical to the corresponding proof from Lemma 6.3. □

LEMMA 6.7. *The space $X(D)$ satisfies the following conditions:*
(i) $X(D)$ is T_1,
(ii) $X(D)$ is compact,
(iii) h is C-separable embedding.

PROOF. (i) Let Γ be an arbitrary maximal clan. The space $X(D)$ is T_1 iff the singleton set $\{\Gamma\}$ is closed, i.e. $Cl(\{\Gamma\}) = \{\Gamma\}$. This follows by the maximality of Γ as follows. Let Δ be a maximal clan. Then:
$\Delta \in Cl(\{\Gamma\})$ iff $(\forall c \in D)(\{\Gamma\} \subseteq h(c) \rightarrow \Delta \in h(c))$ iff $(\forall c \in D)(\Gamma \in h(c) \rightarrow \Delta \in h(c))$ iff $(\forall c \in D)(c \in \Gamma \rightarrow c \in \Delta)$ iff $\Gamma \subseteq \Delta$ iff $\Gamma = \Delta$ iff $\Delta \in \{\Gamma\}$.

This chain shows that indeed $Cl(\{\Gamma\}) = \{\Gamma\}$.

(ii) The proof is similar to the proof of Lemma 6.4 (ii)

(iii) follows from (ii) as in the proof of Lemma 6.5. □

THEOREM 6.5. **Topological representation theorem for C-extensional U-rich EDC-lattices** *Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be a C-extensional U-rich EDC-lattice. Then there exists a compact weakly regular T_1-space X and a dually dense and C-separable embedding h of $\underline{D}$ into the Boolean contact algebra $RC(X)$ of the regular closed sets of X. Moreover:*

(i) $\underline{D}$ satisfies (Con C) iff $RC(X)$ satisfies (Con C); in this case X is connected.

(ii) $\underline{D}$ satisfies (Nor 1) iff $RC(X)$ satisfies (Nor 1); in this case X is κ-normal.

PROOF. The proof follows from Lemma 6.6, Lemma 6.7 and Lemma 5.1.

□

6.3. Representations in T_2 spaces. In the previous section we proved representability in T_1 spaces of U-rich EDC-lattices satisfying the axiom of C-extensionality (Ext C). The T_1 property of the topological space was guaranteed by the fact that abstract points are maximal clans. In this section we will show that adding the axiom (Nor 1) we can obtain representability in compact T_2-spaces. The reason for this is that the axiom (Nor 1) makes possible to use new abstract points - the so called clusters, which are maximal clans satisfying some additional properties yielding T_2 separability of the topological space. Clusters have been used in the compactification theory of proximity spaces (see more about their origin in [**37**]). They have been adapted in algebraic form in the representation theory of contact algebras in [**8, 42**]. In [**12**] their definition and some constructions are modified for the distributive case. We remind below the corresponding definition.

DEFINITION 6.2. *Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be an EDC-lattice. A clan Γ in $\underline{D}$ is called a cluster if it satisfies the following condition:*

(Cluster) If for all $b \in \Gamma$ we have aCb, then $a \in \Gamma$.

We denote the set of clusters in $\underline{D}$ by $CLUSTER(D)$.

Let us note that not in all EDC-lattices there are clusters. The following lemma shows that the axiom (Nor 1) guarantees existence of clusters and some important properties needed for the representation theorem.

LEMMA 6.8. [**12**] *Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be an EDC-lattice. Then:*

(i) Every cluster is a maximal clan.

(ii) If $\underline{D}$ satisfies (Nor 1) then every maximal clan is a cluster.

(iii) If Γ and Δ are clusters such that $\Gamma \neq \Delta$, then there are $a \notin \Gamma$ and $b \notin \Delta$ such that $a + b = 1$.

To build the canonical space $X(D)$ we assume in this section that $\underline{D} = (D, C, \widehat{C}, \ll)$ is an U-rich EDC-lattice satisfying the axioms (Ext C) and (Nor 1). We define $X(D) = CLUSTER(D)$, $h(a) = \{\Gamma \in CLUSTER(D) : a \in \Gamma\}$ and define the topology in $X(D)$ considering the set $\mathbf{CB}(X) = \{h(a) : a \in D\}$ as a basis for closed sets in $X(D)$. Since the points of $X(D)$ are maximal clans, just as in Section 6.2 we can prove the following lemma.

LEMMA 6.9. *The space $X(D)$ is a semiregular and h is a dually dense embedding of D into the contact Boolean algebra $RC(X(D))$.*

LEMMA 6.10. *(i) $X(D)$ is T_2,*
(ii) $X(D)$ is compact,
(iii) h is C-separable embedding.

PROOF. (i) To show that the space $X(D)$ is T_2 suppose that Γ, Δ are two different clusters. We have to find two disjoint open sets A, B such that $\Gamma \in A$ and $\Delta \in B$. By Lemma 6.8 (iii) there are $a, b \in D$ such that $a \notin \Gamma$ and $b \notin \Delta$ such that $a + b = 1$. Then by Lemma 6.9 we get $\Gamma \notin h(a)$, $\Delta \notin h(b)$ and $h(a) \cup h(b) = X(D)$, hence $-h(a) \cap -h(b) = \varnothing$. Define $A = -h(a)$, $B = -h(b)$. Since $h(a)$ and $h(b)$ are closed sets, then A and B are open sets which separate the abstract points Γ and Δ.

The proof of (ii) and (iii) is the same as the proof of (ii) and (iii) in Lemma 6.7. □

THEOREM 6.6. **Topological representation theorem for U-rich EDC-lattices satisfying (Ext C) and (Nor 1).** *Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be an U-rich EDC-lattice satisfying (Ext C) and (Nor 1). Then there exists a compact T_2-space X and a dually dense and C-separable embedding h of $\underline{D}$ into the Boolean contact algebra $RC(X)$ of the regular closed sets of X. Moreover $\underline{D}$ satisfies (Con C) iff $RC(X)$ satisfies (Con C) and in this case X is connected.*

PROOF. The proof follows from Lemma 6.9, Lemma 6.10 and 5.1.

□

Let us note that this theorem generalizes to the case of EDC-lattices several representation theorems for contact algebras from [**8, 40, 43, 42**].

7. Topological representation theory of O-rich EDC-lattices

This section is devoted to the theory of dense representations for O-rich EDC-latices (see Definition 5.1). According to Theorem 5.3 we will look for dense representations with regular open sets (see 5.4 (ii)). This case is completely dual to the corresponding theory developed in Section 6. For this reason we will only sketch the main representation scheme and the definitions of abstract points for the T_0, T_1 and T_2 representations.

The representation scheme is dual to the scheme presented in Section 6:

- Define a set $X(D)$ of "abstract points" of $\underline{D}$,
- define a topology in $X(D)$ by the set $\mathbf{OB}(X(D)) = \{h(a) : a \in D\}$, considered as an open base of the topology, where h is the intended embedding of Stone type: $h(a) = \{\Gamma : \Gamma \text{ is "abstract point" and } a \in \Gamma\}$. $X(D)$ is called the *canonical topological space of $\underline{D}$* and h is called *canonical embedding*,
- establish that h is a dense embedding of the lattice $\underline{D}$ into the Boolean algebra $RO(X(D))$ of regular open sets of the space $X(D)$.

For the case of T_0 dense representation we consider a notion of abstract point which is dual to the notion of clan. This is the so called E-filter (Efremovich filter). E-filters were used in the theory of proximity spaces (see [**37**]). In the context of contact algebras they were introduced for the first time in [**8**]. The definition adapted for the language of EDC-lattices is the following.

DEFINITION 7.1. *Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be an EDC-lattice. A subset $\Gamma \subseteq D$ is called an* E-filter *if it satisfies the following properties:*

(E-fil 1) Γ is a proper filter in $\underline{D}$, i.e. $0 \notin \Gamma$,

(E-fil 2) If $a \notin \Gamma$ and $b \notin \Gamma$, then $a\widehat{C}b$.

Γ is a minimal E-filter if it is minimal in the set of all E-filters of $\underline{D}$ with respect to set inclusion.

This definition comes as an abstraction from the following natural example. Let X be a topological space, $x \in X$ and $RO(X)$ be the set of all regular-open sets of X. Then the set $\Gamma_x = \{a \in RO(X) : x \in a\}$ is an E-filter in the contact algebra $RO(X)$. Note that the definition of E-filter is based not on the relation of contact C, but on the dual contact $\widehat{C}$.

A general construction of E-filters can be obtained dualizing the construction of clans from Section 6.1. Just to show how this dual construction works and how the O-rich axioms works, we will repeat some steps of the construction omitting the corresponding proofs.

First we will introduce a new canonical relation between prime filters.

DEFINITION 7.2. *Let U, V be prime ideals. Define a new canonical relation $\widehat{R}_{\widehat{C}}$ ($\widehat{R}_{\widehat{C}}$-*canonical relation*) between prime ideals as follows:*

$U\widehat{R}_{\widehat{C}}V \leftrightarrow_{def} (\forall a \in U)(\forall b \in V)(a\widehat{C}b)$.

If U, V are prime filters then we define $UR_{\widehat{C}}V \leftrightarrow_{def} \overline{U}\widehat{R}_{\widehat{C}}\overline{V}$.

Let us note that the relation $\widehat{R}_{\widehat{C}}$ depends only on $\widehat{C}$ and can be defined also for ideals. It is different from the canonical relation $\widehat{R}^c$ between prime ideals defined in Section 2.3, but the presence of O-rich axioms makes it equivalent to $\widehat{R}^c$ as it is stated in the following lemma.

LEMMA 7.1. *(i) $\widehat{R}_{\widehat{C}}$ is a reflexive and symmetric relation.*

(ii) If $\underline{D}$ satisfies the axioms (O-rich $\ll$) and (O-rich $\widehat{C}$), then $\widehat{R}_{\widehat{C}} = \widehat{R}^c$.

The following statement lists some facts about the relation R_C.

FACTS 7.1. *(1) Let F, G be ideals and $F\widehat{R}_{\widehat{C}}G$ then there are prime ideals U, V such that $F \subseteq U$, $G \subseteq V$ and $U\widehat{R}_{\widehat{C}}V$.*

(2) For all $a, b \in D$: $a\widehat{C}b$ iff there exist prime ideals U, V such that $U\widehat{R}_{\widehat{C}}V$, $a \in U$ and $b \in V$.

(3) For all $a, b \in D$: $a\widehat{C}b$ iff there exist prime filters U, V such that $UR_{\widehat{C}}V$, $a \notin U$ and $b \notin V$.

In the following lemma we list some facts about E-filters.

FACTS 7.2. *(1) Every prime filter is an E-filter.*

(2) If Γ is an E-filter and $a \notin \Gamma$, then there exists a prime filter U such that $\Gamma \subseteq U$ and $a \notin U$.

(3) *Every E-filter* Γ *is the intersection of all prime filters containing* Γ.
(4) *Every E-filter contains a minimal E-filter.*
(5) *Let* Σ *be a nonempty set of prime filters such that for every* $U, V \in \Sigma$ *we have* $UR_{\widehat{C}}V$ *and let* Γ *be the intersection of the elements of* Σ. *Then* Γ *is an E-filter and every E-filter can be obtained in this way.*
(6) *Let* U, V *be prime filters,* Γ *be an E-filter,* $\Gamma \subseteq U$ *and* $\Gamma \subseteq V$. *Then* $UR_{\widehat{C}}V$ *and* UR^cV.

Using the above facts one can prove the following representation theorem.

THEOREM 7.3. **Representation theorem for O-rich EDC-lattices.** *Let* $\underline{D} = (D, C, \widehat{C}, \ll)$ *be an O-rich EDC-lattice. Then there exists a compact semi-regular space* X *and a dense and* $\widehat{C}$*-separable embedding* h *from* $\underline{D}$ *into the contact algebra* $RO(X)$ *of regular-open sets of* X. *Moreover:*
(i) If $\underline{D}$ *satisfies (Ext* $\widehat{C}$*), then* X *is weakly regular,*
(ii) If $\underline{D}$ *satisfies (Con* $\widehat{C}$*), then* X *is a connected space,*
(iii) If $\underline{D}$ *satisfies (Nor 2), then* X *is* κ*-normal.*

Abstract points for dense representations in T_1 spaces are minimal E-filters and abstract points for dense representations in T_2 spaces are duals of clusters introduced in [**8**] under the name *co-clusters*. We adapt this notion for the language of EDC-lattices as follows:

DEFINITION 7.3. *An E-filter* Γ *is a co-cluster if it satisfies the following condition:*

(Co-cluster) If $(\forall b \notin \Gamma)(a\widehat{C}b)$, *then* $a \notin \Gamma$. *(or, equivalently, if* $a \in \Gamma$, *then* $(\exists b \notin \Gamma)(a\overline{\widehat{C}}b)$*).*

Let us show, for instance, the following statement for co-clusters, which is dual to the corresponding property for clusters as maximal clans:

LEMMA 7.2. *Every co-cluster is a minimal E-filter.*

PROOF. Suppose that Γ is a co-cluster which is not a minimal E-filter. Then there exists an E-filter Δ such that $\Delta \subset \Gamma$, so $a \in \Gamma$ and $a \notin \Delta$ for some a. Then there exists $b \notin \Gamma$ such that $a\overline{\widehat{C}}b$. From here we get $b \in \Delta$. Consequently $b \in \Gamma$ - a contradiction.

□

We leave to the reader to prove the dual analogues of Theorem 6.5 and Theorem 6.6 which we formulate below.

THEOREM 7.4. **Topological representation theorem for** $\widehat{C}$**-extensional** O**-rich EDC-lattices.** *Let* $\underline{D} = (D, C, \widehat{C}, \ll)$ *be a* $\widehat{C}$*-extensional O-rich EDC-lattice. Then there exists a compact weakly regular* T_1*-space* X *and a dense and* $\widehat{C}$*-separable embedding* h *of* $\underline{D}$ *into the Boolean contact algebra* $RO(X)$ *of the regular open sets of* X. *Moreover:*

(i) $\underline{D}$ *satisfies (Con* $\widehat{C}$*) iff* $RO(X)$ *satisfies (Con* $\widehat{C}$*); in this case* X *is connected.*

(ii) $\underline{D}$ *satisfies (Nor 2) iff* $RO(X)$ *satisfies (Nor 2); in this case* X *is* κ*-normal.*

THEOREM 7.5. **Topological representation theorem for** O**-rich EDC-lattices satisfying (Ext** $\widehat{C}$**) and (Nor 2).** *Let* $\underline{D} = (D, C, \widehat{C}, \ll)$ *be an O-rich*

EDC-lattice satisfying (Ext $\widehat{C}$*) and (Nor 2). Then there exists a compact* T_2*-space* X *and a dense and* $\widehat{C}$*-separable embedding* h *of* $\underline{D}$ *into the Boolean contact algebra* $RO(X)$ *of the regular open sets of* X*. Moreover* $\underline{D}$ *satisfies (Con* $\widehat{C}$*) iff* $RO(X)$ *satisfies (Con* $\widehat{C}$*) and in this case* X *is connected.*

CHAPTER 2

Extended contact algebras and internal connectedness

In [**38**] is presented a complete quantifier-free axiomatization of several logics on region-based theory of space, based on contact relation and connectedness predicates c and $c^{\leq n}$, and completeness theorems for the logics in question are proved. It was shown in [**38**] that c and $c^{\leq n}$ are definable in contact algebras by the contact C. The predicates c and $c^{\leq n}$ were studied for the first time in [**30, 31**] (see also [**40**]). The expressiveness and complexity of spatial logics containing c and $c^{\leq n}$ has been investigated in [**23, 24, 25, 26, 27**]. In this chapter we consider the predicate c^o - internal connectedness. Let X be a topological space and $x \in RC(X)$. Let $c^o(x)$ means that $Int(x)$ is a connected topological space in the subspace topology. We prove that the predicate internal connectedness cannot be defined in the language of contact algebras. Because of this we add to the language a new ternary predicate symbol $\vdash$ which has the following sense: in the contact algebra of regular closed sets of some topological space $a, b \vdash c$ iff $a \cap b \subseteq c$. It turns out that the predicate c^o can be defined in the new language. We define *extended contact algebras* - Boolean algebras with added relations $\vdash$, C and c^o, satisfying some axioms, and prove that every extended contact algebra can be isomorphically embedded in the contact algebra of the regular closed subsets of some compact, semiregular, T_0 topological space with added relations $\vdash$ and c^o. So extended contact algebra can be considered an axiomatization of the theory, consisting of the universal formulas true in all topological contact algebras with added relations $\vdash$ and c^o.

1. Undefinability of internal connectedness in the language of contact algebras

Let X be a topological space and $x \in RC(X)$. Let $c^o(x)$ means that $Int(x)$ is a connected topological space in the subspace topology.

PROPOSITION 1.1. *There does not exist a formula $A(x)$ in the language of contact algebras such that: for arbitrary topological space, for every regular closed subset x of this topological space, $c^o(x)$ iff $A(x)$ is valid in the algebra of regular closed subsets of the topological space.*

PROOF. Suppose for the sake of contradiction that there exists a formula $A(x)$ in the language of contact algebras such that: for any topological space, for every regular closed subset x of this topological space, $c^o(x)$ iff $A(x)$ is valid in the algebra of regular closed subsets of the topological space.

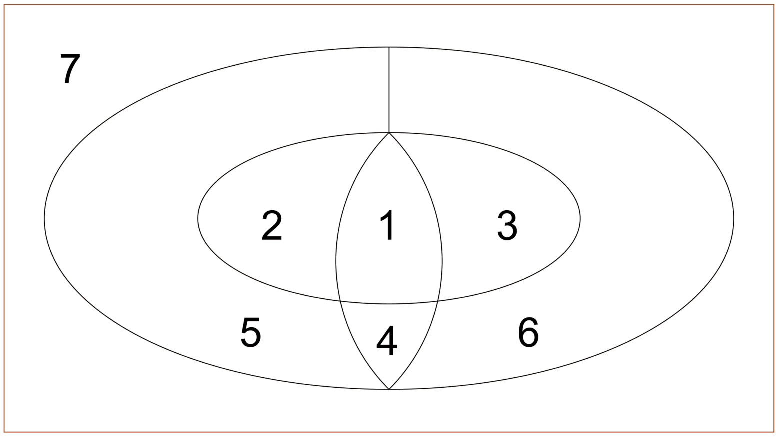

FIGURE 1. The topological space (X, O)

We consider the topological space (X, O), where $X = \{1, 2, 3, 4, 5, 6, 7\}$ and the topology is defined by an open basis: $\{\{1, 2, 3\},\ \{7\},\ \{2, 5, 7\},\ \{3, 6, 7\},\ \{2\},\ \{3\},\ X,\ \emptyset\}$ (see Figure 1).

It can be easily verified that the open sets are $\{1, 2, 3\},\ \{7\},\ \{2, 5, 7\},\ \{3, 6, 7\},\ \{2\},\ \{3\},\ \{2, 3, 5, 6, 7\},\ \{1, 2, 3, 5, 6, 7\},\ \{1, 2, 3, 7\},\ \{2, 7\},\ \{3, 7\},\ \{2, 3\},\ \{2, 3, 7\},\ \{1, 2, 3, 5, 7\},\ \{1, 2, 3, 6, 7\},\ \{2, 3, 5, 7\},\ \{2, 3, 6, 7\},\ X,\ \emptyset$. It can be easily verified that the regular closed sets are $\{4, 5, 6, 7\},\ \{1, 2, 3, 4, 5, 6\},\ \{1, 2, 4, 5\},\ \{1, 3, 4, 6\},\ \{1, 2, 4, 5, 6, 7\},\ \{1, 3, 4, 5, 6, 7\},\ X,\ \emptyset$.

We consider the subspace of X, $Y = X \setminus \{1\}$. It can be easily proved that:

$$Int_Y(c \setminus \{1\}) = Int_X c \setminus \{1\} \qquad \text{for every } c \text{ - closed subset of } X \tag{1.1}$$

Using (1.1) and the fact that for every t, $Cl_Y t = Cl_X t \cap Y = Cl_X t \setminus \{1\}$, we prove that $RC(Y) = \{x \setminus \{1\} : \ x \in RC(X)\}$.

We define a function f from $RC(X)$ to $RC(Y)$ in the following way:

$$f(t) = \begin{cases} t & \text{if } 1 \notin t \\ t \setminus \{1\} & \text{if } 1 \in t \end{cases}$$

It can be easily proved that f is an isomorphism from $(RC(X), \leq, \emptyset, X, \cdot, +, *, C)$ to $(RC(Y), \leq, \emptyset, Y, \cdot, +, *, C)$.

Let $a = \{1, 2, 3, 4, 5, 6\}$. We will prove that a is internally connected. $Int_X a = \{1, 2, 3\}$. The closed sets in $Int(a)$ are: $\{1, 2, 3\},\ \emptyset,\ \{1, 2\},\ \{1, 3\},\ \{1\}$. $Int(a)$ cannot be represented as the union of two non-empty disjoint closed sets and hence $Int(a)$ is connected. Consequently a is internally connected.

Let $b = \{2, 3, 4, 5, 6\}$. $Int_Y b = \{2, 3\}$. We will prove that b is not internally connected. We will prove that $\{2, 3\}$ is not connected. Since $\{2, 3\} = \{2\} \cup \{3\}$, it suffices to prove that $\{2\}$ and $\{3\}$ are closed in $\{2, 3\}$. $\{2, 4, 5\}$ is closed in Y and hence $\{2\} = \{2, 4, 5\} \cap \{2, 3\}$ is closed in $\{2, 3\}$. $\{3, 4, 6\}$ is closed in Y and hence $\{3\} = \{3, 4, 6\} \cap \{2, 3\}$ is closed in $\{2, 3\}$. Consequently $\{2, 3\}$ is not connected, i.e. b is not internally connected.

We have $a \in RC(X)$, $c^o(a)$. Consequently $A(a)$. Now consider the topological space Y. Using $b \in RC(Y)$ and $\neg c^o(b)$, we have $\neg A(b)$. We also have $b = f(a)$. $(RC(X), \leq, \emptyset, X, \cdot, +, *, C)$ and $(RC(Y), \leq, \emptyset, Y, \cdot, +, *, C)$ are isomorphic structures for the language of contact algebras, A is a formula in the same language. Consequently: $A(a)$ is true in $(RC(X)\ldots)$ iff $A(f(a))$ i.e. $A(b)$ is true in $(RC(Y)\ldots)$. We have proven that $A(a)$ is true in $(RC(X)\ldots)$; so $A(b)$ is true in $(RC(Y)\ldots)$ - a contradiction. $\square$

2. Definability of internal connectedness in an extended language

Let X be a topological space. We define the relation $\vdash$ in $RC(X)$ in the following way: $a, b \vdash c$ iff $a \cap b \subseteq c$.

PROPOSITION 2.1. *Let X be a topological space. For every a in $RC(X)$, $c^o(a)$ iff $\forall b \forall c(b \neq 0 \wedge c \neq 0 \wedge a = b + c \rightarrow b, c \nvdash a^*)$.*

PROOF. $\rightarrow$) Let $c^o(a)$. Let $b, c \in RC(X)$, $b \neq 0$, $c \neq 0$, $a = b + c$. We will prove that $b, c \nvdash a^*$. We have $a^* = Cl_X - a = -Int_X a$. Suppose for the sake of contradiction that $b, c \vdash -Int_X a$. It follows that $b \cap c \subseteq -Int_X a$ (1). Suppose for the sake of contradiction that $b \cap Int_X a = \emptyset$. We also have $a = b \cup c$ and consequently $Int_X a \subseteq c$. We will prove that $Int_X b = \emptyset$. Suppose for the sake of contradiction that $Int_X b \neq \emptyset$. Using $Int_X a \subseteq c$ and (1), we have that $Int_X b \cap Int_X a = \emptyset$, but $Int_X b \neq \emptyset$, so $Int_X a \neq Int_X a \cup Int_X b$ (2). We have $a = b \cup c$. Consequently $Int_X a \cup Int_X b \subseteq a$, but $Int_X a \cup Int_X b$ is an open set, so $Int_X a \cup Int_X b \subseteq Int_X a$, i.e. $Int_X a \cup Int_X b = Int_X a$ - a contradiction. Consequently $Int_X b = \emptyset$. We have $b \in RC(X)$, so $b = Cl_X Int_X b = Cl_X \emptyset = \emptyset$ - a contradiction. Consequently $b \cap Int_X a \neq \emptyset$. Similarly $c \cap Int_X a \neq \emptyset$. Let $b_1 = b \cap Int_X a$, $c_1 = c \cap Int_X a$. We have $b_1 \cup c_1 = Int_X a \cap (b \cup c) = Int_X a \cap a = Int_X a$. From $a = b \cup c$ and (1) we get $b_1 \cap c_1 = \emptyset$. We have $Int_X a = b_1 \cup c_1$, $b_1 \neq \emptyset$, $c_1 \neq \emptyset$, $b_1 \cap c_1 = \emptyset$, b_1 and c_1 are closed in $Int_X a$ and therefore $Int_X a$ is not connected, i.e. a is not internally connected - a contradiction.

$\leftarrow$) Let $\forall b, c \in RC(X)(b \neq 0 \wedge c \neq 0 \wedge a = b + c \rightarrow b, c \nvdash a^*)$. We will prove that $Int_X a$ is connected. Suppose for the sake of contradiction that $Int_X a$ is not connected. Consequently there are b_1, c_1 - closed in $Int_X a$, such that $Int_X a = b_1 \cup c_1$ (1), $b_1 \neq \emptyset$, $c_1 \neq \emptyset$, $b_1 \cap c_1 = \emptyset$. We have $b_1 = b \cap Int_X a$, $c_1 = c \cap Int_X a$, where b and c are closed in X because b_1 and c_1 are closed in $Int_X a$. Let $b' = Cl_X b_1$, $c' = Cl_X c_1$. a and b are closed sets in X, $b_1 \subseteq b$, $b_1 \subseteq a$ and therefore $b' \subseteq b$, $b' \subseteq a$. Similarly $c' \subseteq c$, $c' \subseteq a$. Suppose for the sake of contradiction that $a \nsubseteq b' \cup c'$. b' and c' are closed in X and consequently $b' \cup c'$ is closed in X. From $b_1 \subseteq b'$, $c_1 \subseteq c'$, (1) we obtain that $Int_X a \subseteq b' \cup c'$, but $b' \cup c'$ is closed in X and consequently $Cl_X Int_X a \subseteq b' \cup c'$. We have $b' \cup c' \subseteq a$, $b' \cup c' \neq a$. Consequently $Cl_X Int_X a \neq a$ - a contradiction with $a \in RC(X)$. Consequently $a \subseteq b' \cup c'$ and thus $a = b' \cup c'$ (3). We have $c_1 = c \cap Int_X a$, $Int_X a = b_1 \cup c_1$, $b_1 \cap c_1 = \emptyset$ and therefore $b_1 = -c \cap Int_X a$. c is closed in X and hence $-c$ is open in X; $Int_X a$ is open in X; so b_1 is open in X, but $b_1 \subseteq b'$, so $b_1 \subseteq Int_X b'$. Suppose for the sake of contradiction that $Int_X b' \neq b_1$. From (3) we get $Int_X b' \subseteq Int_X a$ (4). From $Int_X a = b_1 \cup c_1$, $b_1 \subseteq Int_X b'$, $b_1 \neq Int_X b'$, (4) we obtain $c_1 \cap Int_X b' \neq \emptyset$, but $Int_X b' \subseteq b' \subseteq b$, so $c_1 \cap b \neq \emptyset$. Consequently $b \cap Int_X a \cap c_1 \neq \emptyset$, but $b \cap Int_X a = b_1$, so $b_1 \cap c_1 \neq \emptyset$ - a contradiction. Consequently $Int_X b' = b_1$. $b' = Cl_X b_1 = Cl_X Int_X b'$, so $b' \in RC(X)$. Similarly $c' \in RC(X)$. We also have $b', c' \neq \emptyset$, $a = b' + c'$, so

$b' \cap c' \nsubseteq a^* = -Int_X a$. Consequently $b' \cap c' \cap Int_X a \neq \emptyset$, but $b' \subseteq b$, $c' \subseteq c$, so $b \cap c \cap Int_X a \neq \emptyset$, i.e. $(b \cap Int_X a) \cap (c \cap Int_X a) \neq \emptyset$. Consequently $b_1 \cap c_1 \neq \emptyset$ - a contradiction. Consequently $Int_X a$ is connected, i.e. $c^o(a)$. □

3. Extended contact algebras

In this section we give an axiomatization of the relation $a, b \vdash c$ used in the characterization of the predicate $c^0(a)$ of internal connectedness given in Section 2.

DEFINITION 3.1. *Extended contact algebra (ECA, for short) is a system* $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$, *where* $(B, \leq, 0, 1, \cdot, +, *)$ *is a nondegenerate Boolean algebra,* $\vdash$ *is a ternary relation in* B *such that the following axioms are true:*
(1) $a, b \vdash c \rightarrow b, a \vdash c$,
(2) $a \leq b \rightarrow a, a \vdash b$,
(3) $a, b \vdash a$,
(4) $a, b \vdash x,\ a, b \vdash y,\ x, y \vdash c \rightarrow a, b \vdash c$,
(5) $a, b \vdash c \rightarrow a \cdot b \leq c$,
(6) $a, b \vdash c \rightarrow a + x, b \vdash c + x$,
C *is a binary relation in* B *such that for all* a, $b \in B$: $aCb \leftrightarrow a, b \nvdash 0$. c^o *is a unary predicate in* B *such that for all* $a \in B$: $c^o(a) \leftrightarrow \forall b \forall c (b \neq 0 \wedge c \neq 0 \wedge a = b + c \rightarrow b, c \nvdash a^*)$.

LEMMA 3.1. *If* $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$ *is an ECA, then* C *is a contact relation in* B *and hence* (B, C) *is a contact algebra.*

PROOF. Routine verification that the axioms of contact C_1 - C_5 are true. □

The above lemma shows that the notion of ECA is a generalization of contact algebra.

The next lemma shows the standard topological example of ECA.

LEMMA 3.2. *Let* X *be a topological space and* $RC(X)$ *be the Boolean algebra of regular closed subsets of* X. *Let for* $a, b, c \in RC(X)$:
aCb *iff* $a \cap b \neq \emptyset$,
$a, b \vdash c$ *iff* $a \cap b \subseteq c$
$c^0(a)$ *iff* $Int(a)$ *is a connected subspace of* X.
Then the Boolean algebra $RC(X)$ *with just defined relations is an ECA, called topological ECA over the space* X.

PROOF. It can be easily verified that the axioms (1)-(6) of ECA are true and for all a, $b \in RC(X)$: $aCb \leftrightarrow a, b \nvdash 0$. Using proposition 2.1, we get that for every $a \in RC(X)$ we have $c^o(a) \leftrightarrow \forall b \forall c (b \neq 0 \wedge c \neq 0 \wedge a = b + c \rightarrow b, c \nvdash a^*)$. □

Our aim is to prove that every ECA can be isomorphically embedded into a topological ECA over a certain topological space X, which will be done in the next section. This will show that the chosen axioms for ECA are right.

REMARK 3.1. *Using axioms (2) and (5), we see that in an ECA* $\underline{B}$ $a \leq b \leftrightarrow a, a \vdash b$ *for every* $a, b \in B$, *i.e. the predicate symbol* $\leq$ *can be removed from the language. Although this we leave it in the language.*

4. Topological representation theory of ECA

DEFINITION 4.1. *Let* $(B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$ *be an ECA and* $S \subseteq B$.
$S \vDash_0 x \overset{def}{\leftrightarrow} x \in S$
$S \vDash_{n+1} x \overset{def}{\leftrightarrow} \exists x_1, x_2 : \; x_1, x_2 \vdash x, \; S \vDash_{k_1} x_1, \; S \vDash_{k_2} x_2$, *where* $k_1, k_2 \leq n$
$S \vDash x \overset{def}{\leftrightarrow} \exists n : \; S \vDash_n x$

For to prove a representation theorem of EC-algebras we will need several lemmas.

LEMMA 4.1. *If* $S \vDash_n y$ *and* $S \subseteq S'$, *then* $S' \vDash_n y$.

PROOF. An induction on n.
Case 1: $n = 0$
Let $S \vDash_0 y$ and $S \subseteq S'$. We have $y \in S$ and consequently $y \in S'$, i.e. $S' \vDash_0 y$.
Case 2: $n > 0$
Let $S \vDash_n y$ and $S \subseteq S'$. We will prove that $S' \vDash_n y$. From $S \vDash_n y$, $n > 0$ we get that there are x_1, x_2 such that $x_1, x_2 \vdash y$, $S \vDash_{k_1} x_1$, $S \vDash_{k_2} x_2$, where $k_1, k_2 < n$. Using $S \vDash_{k_1} x_1$, $S \vDash_{k_2} x_2$, $S \subseteq S'$ and the induction hypothesis, we have $S' \vDash_{k_1} x_1$, $S' \vDash_{k_2} x_2$. Consequently $S' \vDash_n y$. □

LEMMA 4.2. *If* $S \vDash_n y$ *and* $n \leq n'$, *then* $S \vDash_{n'} y$.

PROOF. Let $S \vDash_n y$ and $n \leq n'$. We will prove that $S \vDash_{n'} y$.
Case 1: $n = 0$
By induction on n' we will prove that $\forall n' \forall S \forall y (S \vDash_0 y$ and $0 \leq n' \to S \vDash_{n'} y)$.
Case 1.1: $n' = 0$
Obviously $\forall S \forall y (S \vDash_0 y$ and $0 \leq 0 \to S \vDash_0 y)$.
Case 1.2: $n' > 0$
Let $S \subseteq B$, $y \in B$, $S \vDash_0 y$ and $0 \leq n'$. We will prove that $S \vDash_{n'} y$. From $n' > 0$ we have $0 \leq n' - 1$. By the induction hypothesis we obtain that $\forall S \forall y (S \vDash_0 y$ and $0 \leq n' - 1 \to S \vDash_{n'-1} y)$. Consequently $S \vDash_{n'-1} y$. We also have $y, y \vdash y$ (from axiom (2)). Consequently $S \vDash_{n'} y$.

So we proved that $\forall n' \forall S \forall y (S \vDash_0 y$ and $0 \leq n' \to S \vDash_{n'} y)$. We also have $S \vDash_0 y$ and $0 \leq n'$. Consequently $S \vDash_{n'} y$.
Case 2: $n > 0$
From $S \vDash_n y$, $n > 0$ we get that there are x_1, x_2 such that $x_1, x_2 \vdash y$, $S \vDash_{k_1} x_1$, $S \vDash_{k_2} x_2$, where $k_1, k_2 < n$. But we have $n \leq n'$, so $k_1, k_2 < n'$. Consequently $S \vDash_{n'} y$. □

LEMMA 4.3. *If* $S \vDash x$ *and* $x \leq y$, *then* $S \vDash y$.

PROOF. Let $S \vDash x$ and $x \leq y$. We will prove that $S \vDash y$. From $x \leq y$ and axiom (2) we have that $x, x \vdash y$ (1). From $S \vDash x$ we obtain that: $S \vDash_n x$ for some n (2). From (1) and (2) we have $S \vDash_{n+1} y$, i.e. $S \vDash y$. □

LEMMA 4.4. *If* $\{x\} \cup S \vDash y$, $\{y\} \cup S \vDash z$, *then* $\{x\} \cup S \vDash z$.

PROOF. Let $\{x\} \cup S \vDash y$, $\{y\} \cup S \vDash z$. We will prove that $\{x\} \cup S \vDash z$. We have $\{y\} \cup S \vDash_{n_0} z$ for some n_0. By induction on n we will prove that $\forall n \forall t (\{x\} \cup S \vDash y$, $\{y\} \cup S \vDash_n t \to \{x\} \cup S \vDash t)$. Let n be a natural number and $\forall n' < n \forall t (\{x\} \cup S \vDash y$, $\{y\} \cup S \vDash_{n'} t \to \{x\} \cup S \vDash t)$. We will prove that $\forall t (\{x\} \cup S \vDash y$, $\{y\} \cup S \vDash_n t \to \{x\} \cup S \vDash t)$. Let $t \in B$, $\{x\} \cup S \vDash y$, $\{y\} \cup S \vDash_n t$. We will prove that $\{x\} \cup S \vDash t$.

Case 1: $n = 0$
Case 1.1: $t = y$
Obviously $\{x\} \cup S \vDash t$.
Case 1.2: $t \neq y$
We have $\{y\} \cup S \vDash_0 t$. Consequently $t \in \{y\} \cup S$, but $t \neq y$, so $t \in S$. Consequently $\{x\} \cup S \vDash_0 t$.
Case 2: $n > 0$
We have $\{y\} \cup S \vDash_n t$, $n > 0$. Consequently there are t_1, t_2 such that $t_1, t_2 \vdash t$, $\{y\} \cup S \vDash_{k_1} t_1$, $\{y\} \cup S \vDash_{k_2} t_2$, where $k_1, k_2 < n$. By the induction hypothesis for k_1, k_2, we get $\{x\} \cup S \vDash t_1$, $\{x\} \cup S \vDash t_2$. Consequently $\{x\} \cup S \vDash_{l_1} t_1$, $\{x\} \cup S \vDash_{l_2} t_2$ for some integers l_1, l_2. Let l be the greater among l_1 and l_2. We have $\{x\} \cup S \vDash_l t_1$, $\{x\} \cup S \vDash_l t_2$ by lemma 4.2; $t_1, t_2 \vdash t$; consequently $\{x\} \cup S \vDash_{l+1} t$, i.e. $\{x\} \cup S \vDash t$.
We proved that $\forall n \forall t(\{x\} \cup S \vDash y,\ \{y\} \cup S \vDash_n t \to \{x\} \cup S \vDash t)$. We also have $\{x\} \cup S \vDash y$, $\{y\} \cup S \vDash_{n_0} z$. Consequently $\{x\} \cup S \vDash z$. □

Lemma 4.5. *If* $\{x_1\} \cup S \vDash y$, $\{x_2\} \cup S \vDash y$, *then* $\{x_1 + x_2\} \cup S \vDash y$.

Proof. Let $\{x_1\} \cup S \vDash y$, $\{x_2\} \cup S \vDash y$. We will prove that $\{x_1 + x_2\} \cup S \vDash y$. There is a n_0 such that $\{x_1\} \cup S \vDash_{n_0} y$, $\{x_2\} \cup S \vDash_{n_0} y$. We will prove by induction on n that:
$(*)$ $\forall n \forall u \forall v \forall w(\{u\} \cup S \vDash_n v \to \{u + w\} \cup S \vDash v + w)$
Let n be a natural number and $\forall t < n \forall u \forall v \forall w(\{u\} \cup S \vDash_t v \to \{u + w\} \cup S \vDash v + w)$. We will prove that $\forall u \forall v \forall w(\{u\} \cup S \vDash_n v \to \{u + w\} \cup S \vDash v + w)$. Let $u, v, w \in B$ and $\{u\} \cup S \vDash_n v$. We will prove that $\{u + w\} \cup S \vDash v + w$.
Case 1: $n = 0$
Case 1.1: $v \in S$
We have $\{u + w\} \cup S \vDash_0 v$ and by lemma 4.3, we obtain that $\{u + w\} \cup S \vDash v + w$.
Case 1.2: $v \notin S$
We have $\{u\} \cup S \vDash_0 v$, $v \notin S$. Consequently $v = u$. It is sufficient to prove that $\{v + w\} \cup S \vDash v + w$ which obviously is true.
Case 2: $n > 0$
We have $\{u\} \cup S \vDash_n v$, $n > 0$. Consequently there are v_1, v_2 such that $v_1, v_2 \vdash v$, $\{u\} \cup S \vDash_{k_1} v_1$, $\{u\} \cup S \vDash_{k_2} v_2$, where $k_1, k_2 < n$. From the induction hypothesis for k_1 and k_2 we get that $\{u + w\} \cup S \vDash v_1 + w$ (1) and $\{u + w\} \cup S \vDash v_2 + w$ (2). From $v_1, v_2 \vdash v$ and axiom (6) we obtain $v_1 + w, v_2 \vdash v + w$; so $v_2, v_1 + w \vdash v + w$ (by axiom (1)); so $v_2 + w, v_1 + w \vdash v + w + w$ (by axiom (6)); consequently $v_1 + w, v_2 + w \vdash v + w$ (3) (by axiom (1)). Using (1),(2) and (3) we have $\{u + w\} \cup S \vDash v + w$.

We proved that $(*)$ is true. From $(*)$ and $\{x_1\} \cup S \vDash_{n_0} y$ we get that $\{x_1 + x_2\} \cup S \vDash y + x_2$ (4). From $(*)$ and $\{x_2\} \cup S \vDash_{n_0} y$, we obtain that $\{x_2 + y\} \cup S \vDash y + y$, i.e. $\{y + x_2\} \cup S \vDash y$ (5). Using (4), (5) and lemma 4.4, we have $\{x_1 + x_2\} \cup S \vDash y$. □

Lemma 4.6. *Let* $S \vDash x$. *Then there is a finite nonempty subset of* S, S_0, *such that* $S_0 \vDash x$.

Proof. We will prove by induction on n that $\forall n \forall x(S \vDash_n x \to \exists$ finite nonempty subset S_0 of S such that $S_0 \vDash_n x)$.
Case 1: $n = 0$
Let $S \vDash_0 x$. Consequently $x \in S$. Thus $\{x\}$ is a finite nonempty subset of S and $\{x\} \vDash_0 x$.
Case 2: $n \neq 0$
Let $S \vDash_n x$. Consequently there are x_1, x_2 such that $x_1, x_2 \vdash x$, $S \vDash_{k_1} x_1$, $S \vDash_{k_2} x_2$,

where $k_1, k_2 < n$. Using the induction hypothesis, we have that there exist finite nonempty subsets of S, S_1 and S_2, such that $S_1 \vDash_{k_1} x_1$, $S_2 \vDash_{k_2} x_2$. By lemma 4.1, we get $S_1 \cup S_2 \vDash_{k_1} x_1$, $S_1 \cup S_2 \vDash_{k_2} x_2$. Thus $S_1 \cup S_2 \vDash_n x$, $S_1 \cup S_2 \neq \emptyset$, $S_1 \cup S_2$ is finite, $S_1 \cup S_2 \subseteq S$. □

LEMMA 4.7. *Let $S = \{a_1, \dots, a_n\} \cup \{b_1, \dots, b_k\}$ for some $n, k > 0$ and $S \vDash x$. Let $a = a_1 \cdot \dots \cdot a_n$, $b = b_1 \cdot \dots \cdot b_k$. Then $a, b \vdash x$.*

PROOF. By induction on n we will prove that $\forall n \forall x (S \vDash_n x \rightarrow a, b \vdash x)$.
Case 1: $n = 0$
Let $x \in B$, $S \vDash_0 x$. We will prove that $a, b \vdash x$. We have $x \in S$. Without loss of generality $x = a_1$. From $a \leq a_1$ by axiom (2), we obtain that $a, a \vdash a_1$. From axiom (3) we get $a, b \vdash a$. From here and $a, a \vdash a_1$ by axiom (4), we get that $a, b \vdash a_1$.
Case 2: $n \neq 0$
Let $x \in B$ and $S \vDash_n x$. We will prove that $a, b \vdash x$. There are x_1, x_2 such that $x_1, x_2 \vdash x$, $S \vDash_{k_1} x_1$, $S \vDash_{k_2} x_2$, where $k_1, k_2 < n$. Using the induction hypothesis, we get $a, b \vdash x_1$, $a, b \vdash x_2$. But $x_1, x_2 \vdash x$, so by axiom (4), we obtain $a, b \vdash x$. □

DEFINITION 4.2. *Let $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$ be an ECA. A subset of B, Γ, is an abstract point if the following conditions are satisfied:*
1) $1 \in \Gamma$
2) $0 \notin \Gamma$
3) $a + b \in \Gamma \rightarrow a \in \Gamma$ *or* $b \in \Gamma$
4) $a, b \in \Gamma,\ a, b \vdash c \rightarrow c \in \Gamma$

Note that ultrafilters are abstract points.

LEMMA 4.8. *Let X be a topological space. For every n and for any $b_1, \dots, b_n \in RC(X)$, we have $ClInt(b_1 \cap \dots \cap b_n) = b_1 \cdot \dots \cdot b_n$.*

PROOF. An induction on n.
• $n = 1$
$ClIntb_1 = b_1$ because $b_1 \in RC(X)$.
• $n \rightarrow n + 1$
We will prove that $ClInt(b_1 \cap \dots \cap b_{n+1}) = b_1 \cdot \dots \cdot b_{n+1}$. Let $b = b_2 \cap \dots \cap b_{n+1}$. We will prove that $Int(b_1 \cap b) = Int(b_1 \cap ClIntb)$. We have $Int(b_1 \cap b) \subseteq Intb$ and hence $Int(b_1 \cap b) \subseteq ClIntb$. We also have $Int(b_1 \cap b) \subseteq b_1 \cap b \subseteq b_1$. Consequently $Int(b_1 \cap b) \subseteq b_1 \cap ClIntb$. Consequently $Int(b_1 \cap b) \subseteq Int(b_1 \cap ClIntb)$. Since $b_2, \dots, b_{n+1} \in RC(X)$ and $b = b_2 \cap \dots \cap b_{n+1}$, we have that b is closed. We also have $Intb \subseteq b$, so $ClIntb \subseteq b$. Consequently $b_1 \cap ClIntb \subseteq b_1 \cap b$ and hence $Int(b_1 \cap ClIntb) \subseteq Int(b_1 \cap b)$. Thus $Int(b_1 \cap b) = Int(b_1 \cap ClIntb)$. We have $ClInt(b_1 \cap b) = ClInt(b_1 \cap ClInt(b_2 \cap \dots \cap b_{n+1})) = ClInt(b_1 \cap (b_2 \cdot \dots \cdot b_{n+1})) = b_1 \cdot (b_2 \cdot \dots \cdot b_{n+1})$. □

LEMMA 4.9. *Let $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$ be an ECA. Let $A \neq \emptyset$, $A \subseteq B$, $a \in B$, $A \nvDash a$. Then there exists an abstract point Γ such that $A \subseteq \Gamma$ and $a \notin \Gamma$.*

PROOF. We consider the set $(M, \subseteq)$, where $M = \{P \subseteq B :\ A \subseteq P;\ a \notin P;\ x, y \in P,\ x, y \vdash z \rightarrow z \in P\}$. We will prove that $(M, \subseteq)$ has a maximal element Γ and Γ is an abstract point. Let $P_0 = \{t :\ A \vDash t\}$. We will prove that $P_0 \in M$. Obviously $A \subseteq P_0$ and $a \notin P_0$. Let $x, y \in P_0$, $x, y \vdash z$. We will prove that $z \in P_0$. We have $A \vDash_n x$ and $A \vDash_n y$ for some n. Consequently $A \vDash_{n+1} z$. Consequently $z \in P_0$. Thus $P_0 \in M$. We will prove that $(M, \subseteq)$ has a maximal element. Let L

be a chain in M.
Case 1: $L = \emptyset$
P_0 is an upper bound of L.
Case 2: $L \neq \emptyset$
We will prove that $\bigcup L \in M$. Obviously $\bigcup L \subseteq B$, $A \subseteq \bigcup L$, $a \notin \bigcup L$. Let $x, y \in \bigcup L$, $x, y \vdash z$. We will prove that $z \in \bigcup L$. We have $x \in P_1$, $y \in P_2$, where $P_1, P_2 \in L$. Without loss of generality $P_1 \subseteq P_2$. Thus $x, y \in P_2$, $x, y \vdash z$, $P_2 \in M$. Consequently $z \in P_2$ and hence $z \in \bigcup L$. Consequently $\bigcup L \in M$. Obviously $\bigcup L$ is an upper bound of L.

Thus $(M, \subseteq)$ satisfies the Zorn condition. Consequently $(M, \subseteq)$ has a maximal element Γ. We will prove that Γ is an abstract point. $A \neq \emptyset$ and hence $a_1 \in A$ for some a_1. $\Gamma \in M$ and therefore $A \subseteq \Gamma$, so $a_1 \in \Gamma$. From $a_1 \leq 1$ by axiom (2), we get that $a_1, a_1 \vdash 1$. We also have $\Gamma \in M$, $a_1 \in \Gamma$, so $1 \in \Gamma$.

Suppose for the sake of contradiction that $0 \in \Gamma$. From $0 \leq a$ by axiom (2), we obtain $0, 0 \vdash a$. Consequently $a \in \Gamma$ - a contradiction with $\Gamma \in M$. Consequently $0 \notin \Gamma$.

Condition 4) from the definition of abstract point is satisfied for Γ because $\Gamma \in M$.

Let $x + y \in \Gamma$. We will prove that $x \in \Gamma$ or $y \in \Gamma$. For the sake of contradiction suppose that $\{x\} \cup \Gamma \vDash a$, $\{y\} \cup \Gamma \vDash a$. From lemma 4.5 we have $\{x + y\} \cup \Gamma \vDash a$, but $\{x + y\} \cup \Gamma = \Gamma$, so $\Gamma \vDash a$. Consequently there is a n_0 such that $\Gamma \vDash_{n_0} a$. By induction on n we will prove that
$\forall n \forall x (\Gamma \vDash_n x \rightarrow x \in \Gamma)$ (1)
Case 1: $n = 0$
Let $x \in B$ and $\Gamma \vDash_0 x$. Obviously $x \in \Gamma$.
Case 2: $n > 0$
Let $x \in B$ and $\Gamma \vDash_n x$. We will prove that $x \in \Gamma$. There are x_1, x_2 such that $x_1, x_2 \vdash x$, $\Gamma \vDash_{k_1} x_1$, $\Gamma \vDash_{k_2} x_2$, where k_1, $k_2 < n$. By the induction hypothesis and $\Gamma \vDash_{k_1} x_1$, $\Gamma \vDash_{k_2} x_2$, we get that x_1, $x_2 \in \Gamma$. We also have $\Gamma \in M$, $x_1, x_2 \vdash x$, so $x \in \Gamma$.
Consequently (1) is true. We also have $\Gamma \vDash_{n_0} a$. Consequently $a \in \Gamma$ - a contradiction with $\Gamma \in M$. Consequently $\{x\} \cup \Gamma \nvDash a$ or $\{y\} \cup \Gamma \nvDash a$. Without loss of generality, $\{x\} \cup \Gamma \nvDash a$. Let $\Gamma' = \{z : \{x\} \cup \Gamma \vDash z\}$. We will prove that $\Gamma' \in M$. Obviously $\Gamma' \subseteq B$, $A \subseteq \Gamma \subseteq \Gamma'$. Since $\{x\} \cup \Gamma \nvDash a$, $a \notin \Gamma'$. Let x_1, $x_2 \in \Gamma'$, $x_1, x_2 \vdash x_3$. We will prove that $x_3 \in \Gamma'$. We have $\{x\} \cup \Gamma \vDash_n x_1$, $\{x\} \cup \Gamma \vDash_n x_2$ for some n. Consequently $\{x\} \cup \Gamma \vDash_{n+1} x_3$. Consequently $x_3 \in \Gamma'$. Thus $\Gamma' \in M$. We have $\Gamma \subseteq \Gamma'$, Γ is a maximal element of $(M, \subseteq)$, $\Gamma' \in M$, so $\Gamma = \Gamma'$ and hence $x \in \Gamma$. Consequently Γ is an abstract point. □

THEOREM 4.1. *(Representation theorem) Let* $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$ *be an ECA. Then there is a compact, semiregular,* T_0 *topological space* X *and an embedding* h *of* $\underline{B}$ *into* $RC(X)$.

PROOF. Let X be the set of all abstract points of $\underline{B}$ and for $a \in B$, suppose $h(a) = \{\Gamma \in X : a \in \Gamma\}$. The set $\{h(a) : a \in B\}$ can be taken as a closed basis for a topology of X. From the definition of abstract point we obtain $h(0) = \emptyset$, $h(1) = X$.

Let a, $b \in B$. We will prove that $h(a + b) = h(a) + h(b)$. $h(a + b) = \{\Gamma \in X : a + b \in \Gamma\}$, $h(a) + h(b) = \{\Gamma \in X : a \in \Gamma\} \cup \{\Gamma \in X : b \in \Gamma\}$. Obviously $h(a + b) \subseteq h(a) + h(b)$. Let $\Gamma \in h(a) \cup h(b)$. Without loss of generality $\Gamma \in h(a)$, i.e. $\Gamma \in X$ and $a \in \Gamma$. From $a \leq a+b$ and axiom (2) we get $a, a \vdash a+b$. But $a \in \Gamma$ and Γ is an abstract point, so $a+b \in \Gamma$, so $\Gamma \in h(a+b)$.

Let a, b, $c \in B$. Obviously $a, b \vdash c$ implies $h(a), h(b) \vdash h(c)$. Suppose that $h(a), h(b) \vdash h(c)$. We will prove that $a, b \vdash c$. Suppose for the sake of contradiction that $\{a, b\} \nvDash c$. By lemma 4.9, we get that there is an abstract point Γ such that $a, b \in \Gamma$, $c \notin \Gamma$. We have $\Gamma \in h(a) \cap h(b)$; $h(a) \cap h(b) \subseteq h(c)$ (since $h(a), h(b) \vdash h(c)$); so $\Gamma \in h(c)$; so $c \in \Gamma$ - a contradiction. Consequently $\{a, b\} \vDash c$. By lemma 4.7, $a, b \vdash c$.

Let a, $b \in B$. We have $a \leq b \leftrightarrow a, a \vdash b \leftrightarrow h(a), h(a) \vdash h(b) \leftrightarrow h(a) \cap h(a) \subseteq h(b) \leftrightarrow h(a) \subseteq h(b)$.

In a similar way as in [**40**] (Proposition 2.3.4 (1),(2)) we prove that $h(a^*) = Cl(-h(a))$, $h(a)$ is a regular closed set for every $a \in B$. Consequently X is semiregular.

We have $h(a.b) = h((a^* + b^*)^*) = (h(a)^* + h(b)^*)^* = h(a).h(b)$ for all a, $b \in B$.

Let a, $b \in B$. Obviously aCb iff $h(a)Ch(b)$.

Let $a \in B$. Clearly $c^o(h(a))$ implies $c^o(a)$. Let $c^o(a)$. Suppose for the sake of contradiction that $\neg c^o(h(a))$. Consequently there are b, $c \in RC(X)$ such that $b \neq \emptyset$, $c \neq \emptyset$, $h(a) = b \cup c$ and $b \cap c \subseteq h(a)^*$ (proposition 2.1). b and c are closed, so $b = \bigcap_{i \in I} h(b_i)$, $c = \bigcap_{j \in J} h(c_j)$ for some sets I and J. Let $A = \{b_i :\ i \in I\} \cup \{c_j :\ j \in J\}$. Suppose for the sake of contradiction that $A \vDash a^*$. Thus by lemma 4.6, we get that there is a finite nonempty subset of A, A', such that $A' \vDash a^*$. Let $b_{i_1} \in \{b_i :\ i \in I\}$, $c_{j_1} \in \{c_j :\ j \in J\}$. Let $A' = \{b_{i_2}, b_{i_3}, \ldots, b_{i_k}\} \cup \{c_{j_2}, c_{j_3}, \ldots, c_{j_l}\}$ for some k, $l \geq 1$. Let $b' = b_{i_1} \cdot b_{i_2} \cdot \ldots \cdot b_{i_k}$, $c' = c_{j_1} \cdot c_{j_2} \cdot \ldots \cdot c_{j_l}$. From $A' \vDash a^*$ and lemma 4.1 we get that $\{b_{i_1}, b_{i_2}, \ldots, b_{i_k}\} \cup \{c_{j_1}, c_{j_2}, \ldots, c_{j_l}\} \vDash a^*$. Using this fact, the definitions of b' and c' and lemma 4.7, we obtain $b', c' \vdash a^*$. Suppose for the sake of contradiction that $b' \cdot a = 0$. Consequently $h(b_{i_1}) \cdot h(b_{i_2}) \cdot \ldots \cdot h(b_{i_k}) \cdot h(a) = h(0) = \emptyset$. Thus by lemma 4.8, we have $ClInt(h(b_{i_1}) \cap h(b_{i_2}) \cap \ldots \cap h(b_{i_k}) \cap h(a)) = \emptyset$, so $Int(h(b_{i_1}) \cap h(b_{i_2}) \cap \ldots \cap h(b_{i_k}) \cap h(a)) = \emptyset$. We have $h(a) = b \cup c$ and therefore $b = b \cap h(a) \subseteq h(b_{i_1}) \cap h(b_{i_2}) \cap \ldots \cap h(b_{i_k}) \cap h(a)$. Consequently $Intb \subseteq Int(h(b_{i_1}) \cap h(b_{i_2}) \cap \ldots \cap h(b_{i_k}) \cap h(a)) = \emptyset$, i.e. $Intb = \emptyset$. We have $b \in RC(X)$ and from here $b = ClIntb = Cl\emptyset = \emptyset$ - a contradiction. Consequently $b' \cdot a \neq 0$ (1). Similarly $c' \cdot a \neq 0$ (2). We have $h(a) = b \cup c \subseteq h(b_{i_m}) \cup h(c_{j_n})$ for all $m = 1, \ldots, k$, $n = 1, \ldots, l$. Consequently $a \leq b_{i_m} + c_{j_n}$ for all $m = 1, \ldots, k$, $n = 1, \ldots, l$. We also have $b' + c' = (b_{i_1} \cdot \ldots \cdot b_{i_k}) + (c_{j_1} \cdot \ldots \cdot c_{j_l}) = (b_{i_1} + c_{j_1}) \cdot \ldots \cdot (b_{i_k} + c_{j_1}) \cdot \ldots \cdot (b_{i_1} + c_{j_l}) \cdot \ldots \cdot (b_{i_k} + c_{j_l})$. Consequently $a \leq b' + c'$. Thus $a = (b' + c') \cdot a = b' \cdot a + c' \cdot a$ (3). From $b' \cdot a \leq b'$ by axiom (2), we have $b' \cdot a, b' \cdot a \vdash b'$ (4). From axiom (3) we get $b' \cdot a, c' \cdot a \vdash b' \cdot a$ (5). From (5) and (4) by axiom (4), we obtain $b' \cdot a, c' \cdot a \vdash b'$ (6). Similarly $c' \cdot a, b' \cdot a \vdash c'$ and from here by axiom (1), we have $b' \cdot a, c' \cdot a \vdash c'$ (7). From (6), (7) and $b', c' \vdash a^*$ we get, by axiom (4), that $b' \cdot a, c' \cdot a \vdash a^*$ (8). From $c^o(a)$, (1), (2) and (3) we obtain $b' \cdot a, c' \cdot a \nvdash a^*$ - a contradiction with (8). Consequently $A \nvDash a^*$. Thus by lemma 4.9, we get that there is an abstract point Γ_1 such that $A \subseteq \Gamma_1$, $a^* \notin \Gamma_1$. Since $A \subseteq \Gamma_1$, we have $b_i \in \Gamma_1$ for every $i \in I$ and $c_j \in \Gamma_1$ for every $j \in J$. We also have that Γ_1 is an abstract point, so $\Gamma_1 \in h(b_i)$ for every $i \in I$ and $\Gamma_1 \in h(c_j)$ for every $j \in J$. Consequently $\Gamma_1 \in b$, $\Gamma_1 \in c$. We have $a^* \notin \Gamma_1$, so $\Gamma_1 \notin h(a^*)$. Thus $b \cap c \nsubseteq h(a^*)$, i.e. $b \cap c \nsubseteq h(a)^*$ - a contradiction. Consequently $c^o(h(a))$. Consequently h is an embedding.

As in [**40**] (Lemma 2.3.6), replacing the notion clan with abstract point, we prove that X is a compact, T_0 space. □

5. Concluding remarks

One of the motivations to introduce ECA is that its language is more rich and makes possible to express the predicate of internal connectedness of a region. Here we mention without proof some other things which can be expressed in its language and also some things which are not expressible and need further extension. It is known that the intersection of regular closed sets is not in general a regular closed set. Let X be a topological space and for the elements of $RC(X)$ consider the relation: $RC \cap (a, b) \leftrightarrow a \cap b$ is a regular closed set. Very probably this relation is not expressible in contact algebras, but it is expressible in ECA as follows: $RC{\cap}(a, b) \leftrightarrow (\exists c)(a, b \vdash c$ and $c \leq a$ and $c \leq b)$. Another interesting property which is expressible in ECA is related to the existence or not existence of holes in a region like for instance the hole of a region with the form of torus. Then the complement $-a$ is an open set which is not connected. So connectedness of $-a$ expresses that a has no holes. This is expressible in ECA by $c^o(a^*)$. If the internal part of a region is not connected then we cannot express the number of its components. For that purpose we need a more general relation between finite number of regions, which topological meaning is expressible in $RC(X)$ by the relation: $a_1, \ldots, a_n \vdash b$ iff $a_1 \cap \ldots \cap a_n \subseteq b$. Such relations for all n are studied in the paper [**39**].

By this relation one can express also n-ary contact by $C_n(a_1, \ldots, a_n)$ iff $a_1, \ldots, a_n$ $\nvdash 0$, which is not expressible neither in contact algebras nor in ECA.

CHAPTER 3

Quantifier-free logics, related to EDC-lattices and EC-algebras

In this chapter we consider a first-order language without quantifiers corresponding to EDCL. We give completeness theorems with respect to both algebraic and topological semantics for several logics for this language. It turns out that all these logics are decidable. We also consider a quantifier-free first-order language corresponding to ECA and a logic for ECA which is decidable.

1. Preliminaries

Here we have constructions almost the same as in [**4**] (pages 57-59).

Let $\mathcal{L}$ be a quantifier-free countable first-order language with equality. Let δ be a formula in $\mathcal{L}$. We define $\bot = \delta\wedge\neg\delta$, $\top = \delta\vee\neg\delta$. Let I be an arbitrary set; for every $i \in I$ β_i be a formula for $\mathcal{L}$ with variables among $p_{i_1}, \dots, p_{i_{n_i}}, q_{i_1}, \dots, q_{i_{m_i}}$; for every $i \in I$ γ_i be a formula for $\mathcal{L}$ with variables among $q_{i_1}, \dots, q_{i_{m_i}}$. ($p_{i_1}, \dots, p_{i_{n_i}}, q_{i_1}, \dots, q_{i_{m_i}}$ are different variables.)

Let $\mathbb{L}$ be a logic for $\mathcal{L}$, containing all axioms of the classical propositional logic, whose rules are MP and all rules of the type:

$$\frac{\varphi \to \beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\, q_{i_1}\dots q_{i_{m_i}}}{r_{i_1}\dots r_{i_{n_i}}\, a_{i_1}\dots a_{i_{m_i}}}\right] \text{ for all sequences of variables } r_{i_1}\dots r_{i_{n_i}}}{\varphi \to \gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]}, \qquad (1.1)$$

where $i \in I$, φ is a formula for $\mathcal{L}$, $a_{i_1}\dots a_{i_{m_i}}$ are terms for $\mathcal{L}$. Let also if α is an axiom of $\mathbb{L}$ with variables $p_1, \dots, p_n$ and $a_1, \dots, a_n$ are terms in $\mathcal{L}$, then $\alpha\left[\frac{p_1,\dots,p_n}{a_1,\dots,a_n}\right]$ is also an axiom of $\mathbb{L}$. (Here $[\dots]$ means a simultaneous substitution.)

We call the following axiom corresponding to the rule 1.1:
$\neg\gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right] \to \exists x_{i_1}\dots\exists x_{i_{n_i}}\neg\beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\, q_{i_1}\dots q_{i_{m_i}}}{x_{i_1}\dots x_{i_{n_i}}\, a_{i_1}\dots a_{i_{m_i}}}\right]$, where $x_{i_1}, \dots, x_{i_{n_i}}$ are some variables, not occurring in $a_{i_1}, \dots, a_{i_{m_i}}$, different from $p_{i_1}, \dots, p_{i_{n_i}}, q_{i_1}\dots, q_{i_{m_i}}$.

Remark 1.1. *Another approach is to be considered rules of the kind:* $\dfrac{\varphi\to\beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\, q_{i_1}\dots q_{i_{m_i}}}{r_{i_1}\dots r_{i_{n_i}}\, a_{i_1}\dots a_{i_{m_i}}}\right]}{\varphi\to\gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]}$*, where $r_{i_1}\dots r_{i_{n_i}}$ are variables not occuring in $a_{i_1}, \dots, a_{i_{m_i}}$ and φ (see [**4**]).*

Definition 1.1. *A set of formulas for $\mathcal{L}$ Γ is a $\mathbb{L}$-theory, if satisfies the following conditions:*
(i) Γ contains all theorems of $\mathbb{L}$;

(ii) If α, $\alpha \to \beta \in \Gamma$, *then* $\beta \in \Gamma$;
(iii) For every rule of the type above, we have: if $\varphi \to \beta_i \left[\frac{p_{i_1} \dots p_{i_{n_i}} q_{i_1} \dots q_{i_{m_i}}}{r_{i_1} \dots r_{i_{n_i}} a_{i_1} \dots a_{i_{m_i}}}\right] \in \Gamma$ *for all sequences of variables* $r_{i_1}, \dots, r_{i_{n_i}}$, *then* $\varphi \to \gamma_i \left[\frac{q_{i_1} \dots q_{i_{m_i}}}{a_{i_1} \dots a_{i_{m_i}}}\right] \in \Gamma$.

A $\mathbb{L}$*-theory* Γ *is consistent, if* $\perp \notin \Gamma$.

Γ *is a maximal* $\mathbb{L}$*-theory, if it is a consistent* $\mathbb{L}$*-theory and for every consistent* $\mathbb{L}$*-theory* Δ, *if* $\Gamma \subseteq \Delta$, *then* $\Gamma = \Delta$.

LEMMA 1.1 (Extension lemma). *Let* Γ *be a* $\mathbb{L}$*-theory and* α *be a formula. Let* $\Delta = \Gamma + \alpha \stackrel{def}{=} \{\beta : \ \alpha \to \beta \in \Gamma\}$. *Then:*
(i) Δ *is the smallest* $\mathbb{L}$*-theory, containing* Γ *and* α;
(ii) Δ *is inconsistent* $\leftrightarrow \neg\alpha \in \Gamma$;
(iii) For any $i \in I$, φ *- a formula for* $\mathcal{L}$, $a_{i_1}, \dots, a_{i_{m_i}}$ *- terms for* $\mathcal{L}$, *we have: if* $\Gamma + \neg\left(\varphi \to \gamma_i \left[\frac{q_{i_1} \dots q_{i_{m_i}}}{a_{i_1} \dots a_{i_{m_i}}}\right]\right)$ *is consistent, then there are variables* $r_{i_1}, \dots, r_{i_{n_i}}$ *such that* $\left(\Gamma + \neg\left(\varphi \to \gamma_i \left[\frac{q_{i_1} \dots q_{i_{m_i}}}{a_{i_1} \dots a_{i_{m_i}}}\right]\right)\right) + \neg\left(\varphi \to \beta_i \left[\frac{p_{i_1} \dots p_{i_{n_i}} q_{i_1} \dots q_{i_{m_i}}}{r_{i_1} \dots r_{i_{n_i}} a_{i_1} \dots a_{i_{m_i}}}\right]\right)$ *is a consistent* $\mathbb{L}$*-theory.*

PROOF. (i) We will prove that $\Gamma \subseteq \Delta$. Let $\gamma \in \Gamma$. We will prove that $\gamma \in \Delta$. It suffices to prove that $\alpha \to \gamma \in \Gamma$. The formula $\gamma \to (\alpha \to \gamma)$ is a theorem of the classical propositional logic, Γ is a $\mathbb{L}$-theory, so $\gamma \to (\alpha \to \gamma) \in \Gamma$. We also have $\gamma \in \Gamma$, Γ is closed under MP, so $\alpha \to \gamma \in \Gamma$.

We will prove that $\alpha \in \Delta$. It suffices to prove that $\alpha \to \alpha \in \Gamma$. But this is true because $\alpha \to \alpha$ is a theorem of $\mathbb{L}$.

We will prove that Δ is a $\mathbb{L}$-theory. Γ contains all theorems of $\mathbb{L}$ and $\Gamma \subseteq \Delta$, consequently Δ contains all theorems of $\mathbb{L}$. Let γ_1, $\gamma_1 \to \gamma_2 \in \Delta$. We will prove that $\gamma_2 \in \Delta$. We have $\alpha \to \gamma_1 \in \Gamma$, $\alpha \to (\gamma_1 \to \gamma_2) \in \Gamma(1)$. The formula $(\alpha \to \gamma_1) \to ((\alpha \to (\gamma_1 \to \gamma_2)) \to (\alpha \to \gamma_2))$ is a theorem of the classical propositional logic and consequently is in Γ. Using this fact, (1) and the closeness of Γ under MP, we get $\alpha \to \gamma_2 \in \Gamma$, so $\gamma_2 \in \Delta$. Let $i \in I$, φ is a formula for $\mathcal{L}$, $a_{i_1}, \dots, a_{i_{m_i}}$ are terms for $\mathcal{L}$. Let $\varphi \to \beta_i \left[\frac{p_{i_1} \dots p_{i_{n_i}} q_{i_1} \dots q_{i_{m_i}}}{r_{i_1} \dots r_{i_{n_i}} a_{i_1} \dots a_{i_{m_i}}}\right] \in \Delta$ for all sequences of variables $r_{i_1}, \dots, r_{i_{n_i}}$. Let $\gamma_i' = \gamma_i \left[\frac{q_{i_1} \dots q_{i_{m_i}}}{a_{i_1} \dots a_{i_{m_i}}}\right]$. We will prove that $\varphi \to \gamma_i' \in \Delta$. We have $\alpha \to \left(\varphi \to \beta_i \left[\frac{p_{i_1} \dots p_{i_{n_i}} q_{i_1} \dots q_{i_{m_i}}}{r_{i_1} \dots r_{i_{n_i}} a_{i_1} \dots a_{i_{m_i}}}\right]\right) \in \Gamma$ for all sequences of variables $r_{i_1}, \dots, r_{i_{n_i}}$, so $(\alpha \wedge \varphi) \to \beta_i \left[\frac{p_{i_1} \dots p_{i_{n_i}} q_{i_1} \dots q_{i_{m_i}}}{r_{i_1} \dots r_{i_{n_i}} a_{i_1} \dots a_{i_{m_i}}}\right] \in \Gamma$ for all sequences of variables $r_{i_1}, \dots, r_{i_{n_i}}$. From here and the fact that Γ is a $\mathbb{L}$-theory, we obtain $(\alpha \wedge \varphi) \to \gamma_i' \in \Gamma$, so $\alpha \to (\varphi \to \gamma_i') \in \Gamma$, so $\varphi \to \gamma_i' \in \Delta$. Consequently Δ is a $\mathbb{L}$-theory.

Let Δ' is a $\mathbb{L}$-theory, containing Γ and α. We will prove that $\Delta \subseteq \Delta'$. Let $\gamma \in \Delta$. We will prove that $\gamma \in \Delta'$. We have $\alpha \to \gamma \in \Gamma$, $\Gamma \subseteq \Delta'$, so $\alpha \to \gamma \in \Delta'$. But $\alpha \in \Delta'$ and Δ' is closed under MP, so $\gamma \in \Delta'$. Consequently Δ is the smallest $\mathbb{L}$-theory, containing Γ and α.
(ii) Let Δ is inconsistent. We will prove that $\neg\alpha \in \Gamma$. $\perp \in \Delta$ and hence $\alpha \to \perp \in \Gamma$. $(\alpha \to \perp) \to \neg\alpha$ is a theorem of the classical propositional logic and therefore is in Γ. Consequently $\neg\alpha \in \Gamma$.

Let $\neg\alpha \in \Gamma$. We will prove that Δ is inconsistent, i.e. that $\perp \in \Delta$. The formula $\neg\alpha \to (\alpha \to \perp) \in \Gamma$, $\neg\alpha \in \Gamma$, so $\alpha \to \perp \in \Gamma$, i.e. $\perp \in \Delta$.
(iii) Let $i \in I$, φ be a formula for $\mathcal{L}$, $a_{i_1}, \dots, a_{i_{m_i}}$ be terms for $\mathcal{L}$, $\Gamma + \neg\Big(\varphi \to$

$\gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]\Big)$ is consistent. Suppose for the sake of contradiction that for all sequences of variables $r_{i_1},\dots,r_{i_{n_i}}$ $\Big(\Gamma+\neg\Big(\varphi\to\gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]\Big)\Big)+\neg\Big(\varphi\to\beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\,q_{i_1}\dots q_{i_{m_i}}}{r_{i_1}\dots r_{i_{n_i}}\,a_{i_1}\dots a_{i_{m_i}}}\right]\Big)$ is inconsistent. Consequently $\Big(\varphi\to\beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\,q_{i_1}\dots q_{i_{m_i}}}{r_{i_1}\dots r_{i_{n_i}}\,a_{i_1}\dots a_{i_{m_i}}}\right]\Big)\in\Gamma+\neg\Big(\varphi\to\gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]\Big)$ for all sequences of variables $r_{i_1},\dots,r_{i_{n_i}}$. Thus we get $\varphi\to\gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]\in\Gamma+\neg\Big(\varphi\to\gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]\Big)$. We also have $\neg\Big(\varphi\to\gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]\Big)\in\Gamma+\neg\Big(\varphi\to\gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]\Big)$, so $\bot\in\Gamma+\neg\Big(\varphi\to\gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]\Big)$ - a contradiction. Consequently there is a sequence of variables $r_{i_1},\dots,r_{i_{n_i}}$ such that $\Big(\Gamma+\neg\Big(\varphi\to\gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]\Big)\Big)+\neg\Big(\varphi\to\beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\,q_{i_1}\dots q_{i_{m_i}}}{r_{i_1}\dots r_{i_{n_i}}\,a_{i_1}\dots a_{i_{m_i}}}\right]\Big)$ is consistent. □

LEMMA 1.2 (Lindenbaum lemma for $\mathbb{L}$-theories). *Every consistent $\mathbb{L}$-theory Γ can be extended to a maximal $\mathbb{L}$-theory Δ.*

PROOF. Let Γ be a consistent $\mathbb{L}$-theory and the formulas of $\mathcal{L}$ be $\alpha_1,\dots,\alpha_n,\dots$, $n<\omega$. Let an enumeration of the finite sequences of variables be fixed. We define a sequence of consistent $\mathbb{L}$-theories $\Gamma_1\subseteq\Gamma_2\subseteq\dots$ by induction in the following way: $\Gamma_1=\Gamma$ and let $\Gamma_1,\dots,\Gamma_n$ be defined. We define Γ_{n+1} in the following way:
Case 1: $\Gamma_n+\alpha_n$ is consistent
Case 1.1: α_n is not of the kind $\neg\Big(\varphi\to\gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]\Big)$, where φ is a formula for $\mathcal{L}$, $i\in I$, $a_{i_1},\dots,a_{i_{m_i}}$ are terms for $\mathcal{L}$.
In this case $\Gamma_{n+1}\overset{def}{=}\Gamma_n+\alpha_n$.
Case 1.2: α_n is of the kind $\neg\Big(\varphi\to\gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]\Big)$, where φ is a formula for $\mathcal{L}$, $i\in I$, $a_{i_1},\dots,a_{i_{m_i}}$ are terms for $\mathcal{L}$. By the Extension lemma, we get that there are variables $r_{i_1},\dots,r_{i_{n_i}}$ such that $\Big(\Gamma_n+\neg\Big(\varphi\to\gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]\Big)\Big)+\neg\Big(\varphi\to\beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\,q_{i_1}\dots q_{i_{m_i}}}{r_{i_1}\dots r_{i_{n_i}}\,a_{i_1}\dots a_{i_{m_i}}}\right]\Big)$ is a consistent $\mathbb{L}$-theory and let $r_{i_1},\dots,r_{i_{n_i}}$ be the first in the enumeration sequence of variables with this property. In this case $\Gamma_{n+1}\overset{def}{=}\Big(\Gamma_n+\neg\Big(\varphi\to\gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]\Big)\Big)+\neg\Big(\varphi\to\beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\,q_{i_1}\dots q_{i_{m_i}}}{r_{i_1}\dots r_{i_{n_i}}\,a_{i_1}\dots a_{i_{m_i}}}\right]\Big)$.
Case 2: $\Gamma_n+\alpha_n$ is not consistent
In this case $\Gamma_{n+1}\overset{def}{=}\Gamma_n$.

Let $\Delta=\bigcup_{n=1}^{\infty}\Gamma_n$. Obviously $\Gamma\subseteq\Delta$. We will prove that Δ is a maximal $\mathbb{L}$-theory. Obviously Δ contains all theorems of $\mathbb{L}$. Let α, $\alpha\to\beta\in\Delta$. We will prove that $\beta\in\Delta$. There is an n such that α, $\alpha\to\beta\in\Gamma_n$; Γ_n is a $\mathbb{L}$-theory; so $\beta\in\Gamma_n$, i.e. $\beta\in\Delta$. Let $i\in I$, φ be a formula for $\mathcal{L}$, $a_{i_1},\dots,a_{i_{m_i}}$ be terms for $\mathcal{L}$, $\varphi\to\beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\,q_{i_1}\dots q_{i_{m_i}}}{r_{i_1}\dots r_{i_{n_i}}\,a_{i_1}\dots a_{i_{m_i}}}\right]\in\Delta$ for all sequences of variables $r_{i_1},\dots,r_{i_{n_i}}$ (1). For the sake of contradiction suppose that $\varphi\to\gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]\notin\Delta$(2). $\neg\Big(\varphi\to\gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]\Big)$ is α_m for some m. By the Extension lemma (ii) and (2), we obtain that $\Gamma_m+\alpha_m$ is consistent. $\Gamma_{m+1}=(\Gamma_m+\alpha_m)+\neg\Big(\varphi\to\beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\,q_{i_1}\dots q_{i_{m_i}}}{r'_{i_1}\dots r'_{i_{n_i}}\,a_{i_1}\dots a_{i_{m_i}}}\right]\Big)$ for some sequence of variables $r'_{i_1},\dots,r'_{i_{n_i}}$(3). From (1) we get that $\varphi\to\beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\,q_{i_1}\dots q_{i_{m_i}}}{r'_{i_1}\dots r'_{i_{n_i}}\,a_{i_1}\dots a_{i_{m_i}}}\right]\in\Gamma_l$ for some l. From here and (3) we obtain that there is a k such that $\varphi\to$

$\beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\,q_{i_1}\dots q_{i_{m_i}}}{r'_{i_1}\dots r'_{i_{n_i}}\,a_{i_1}\dots a_{i_{m_i}}}\right]$, $\neg\left(\varphi \to \beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\,q_{i_1}\dots q_{i_{m_i}}}{r'_{i_1}\dots r'_{i_{n_i}}\,a_{i_1}\dots a_{i_{m_i}}}\right]\right) \in \Gamma_k$. Consequently $\bot \in \Gamma_k$, i.e. Γ_k is not consistent - a contradiction. Consequently Δ is a $\mathbb{L}$-theory.

For every n, Γ_n is consistent and hence $\bot \notin \Gamma_n$ for every n. Consequently $\bot \notin \Delta$, i.e. Δ is consistent.

Let Δ' be a consistent $\mathbb{L}$-theory and $\Delta \subseteq \Delta'$. We will prove that $\Delta' \subseteq \Delta$. Let $\alpha_n \in \Delta'$. We will prove that $\alpha_n \in \Delta$. For the sake of contradiction suppose that $\neg\alpha_n \in \Gamma_n$. Consequently $\neg\alpha_n \in \Delta$ and $\neg\alpha_n \in \Delta'$. We also have $\alpha_n \in \Delta'$, so $\bot \in \Delta'$ - a contradiction. Consequently $\neg\alpha_n \notin \Gamma_n$. From here and the Extension lemma (ii) we get that $\Gamma_n + \alpha_n$ is consistent. Consequently $\alpha_n \in \Delta$. Consequently Δ is a maximal $\mathbb{L}$-theory. □

LEMMA 1.3. *Let S be a maximal $\mathbb{L}$-theory. Then:*
(i) for every formula α, $\alpha \in S$ or $\neg\alpha \in S$;
(ii) for all formulas α and β:
1) $\neg\alpha \in S \leftrightarrow \alpha \notin S$;
2) $\alpha \wedge \beta \in S \leftrightarrow \alpha \in S$ and $\beta \in S$;
3) $\alpha \vee \beta \in S \leftrightarrow \alpha \in S$ or $\beta \in S$.

PROOF. (i) Let α be a formula for $\mathcal{L}$. For the sake of contradiction suppose that $S' = S + \neg\alpha$ and $S'' = S + \alpha$ are inconsistent. Consequently $\neg\alpha \to \bot \in S$ and $\alpha \to \bot \in S$. The formula $(\neg\alpha \to \bot) \to ((\alpha \to \bot) \to \bot)$ is a theorem of the classical propositional logic and consequently is in S. Thus using that S is closed under MP, we get that $\bot \in S$ - a contradiction. Consequently S' is consistent or S'' is consistent, so $S' = S$ or $S'' = S$, i.e. $\neg\alpha \in S$ or $\alpha \in S$.
(ii) Let α and β be formulas for $\mathcal{L}$.
1) If $\neg\alpha \in S$, then $\alpha \notin S$ because otherwise S is inconsistent. If $\alpha \notin S$, then $\neg\alpha \in S$ because (i) is true.
2) Let $\alpha \wedge \beta \in S$. The formula $(\alpha \wedge \beta) \to \alpha$ is in S. Consequently $\alpha \in S$. Similarly $\beta \in S$. Let α, $\beta \in S$. The formula $\alpha \to (\beta \to \alpha \wedge \beta)$ is in S. Consequently $\alpha \wedge \beta \in S$.
3) Let $\alpha \vee \beta \in S$. Suppose for the sake of contradiction that $\alpha \notin S$, $\beta \notin S$. From (i) we get $\neg\alpha \in S$ and $\neg\beta \in S$. We have $\neg\alpha \to (\neg\beta \to \neg(\alpha \vee \beta)) \in S$. Thus $\neg(\alpha \vee \beta) \in S$. Consequently S is inconsistent - a contradiction.

Let $\alpha \in S$ or $\beta \in S$. The formulas $\alpha \to (\alpha \vee \beta)$ and $\beta \to (\alpha \vee \beta)$ are in S. Consequently $\alpha \vee \beta \in S$. □

Let S be a maximal $\mathbb{L}$-theory. We define the relation $\equiv$ in the set of all terms of $\mathcal{L}$ in the following way: $a \equiv b \Leftrightarrow a = b \in S$. $\equiv$ is an equivalence relation. Let $B_s = \{|a| : a \text{ is a term}\}$. We define the structure $\mathcal{B}_s$ with universe B_s in the following way:
- for every constant c: $c^{\mathcal{B}_s} = |c|$;
- for every n-ary function symbol f: $f^{\mathcal{B}_s}(|a_1|, \dots, |a_n|) = |f(a_1, \dots, a_n)|$;
- for every n-ary predicate symbol p: $p^{\mathcal{B}_s}(|a_1|, \dots, |a_n|) \leftrightarrow p(a_1, \dots, a_n) \in S$.

We define a valuation in $\mathcal{B}_s$ in the following way: $v_s(p) = |p|$ for every variable p. It can be easily verified that $v_s(a) = |a|$ for every term a. We call $(\mathcal{B}_s, v_s)$ canonical model, corresponding to S.

The semantics of $\mathcal{L}$ is the standard one.

LEMMA 1.4. *For every formula α: $(\mathcal{B}_s, v_s) \models \alpha \Leftrightarrow \alpha \in S$.*

PROOF. Induction on the complexity of α. $\square$

PROPOSITION 1.1. *All theorems of* $\mathbb{L}$ *are true in* $(\mathcal{B}_s, v_s)$. *For every* $i \in I$ *and for any* $a_{i_1}, \ldots, a_{i_{m_i}}$ *- terms we have: if* $(\mathcal{B}_s, v_s) \nvDash \gamma_i\left[\frac{q_{i_1} \ldots q_{i_{m_i}}}{a_{i_1} \ldots a_{i_{m_i}}}\right]$, *then there are terms* $p'_{i_1}, \ldots, p'_{i_{n_i}}$ *such that* $(\mathcal{B}_s, v_s) \nvDash \beta_i\left[\frac{p_{i_1} \ldots p_{i_{n_i}} q_{i_1} \ldots q_{i_{m_i}}}{p'_{i_1} \ldots p'_{i_{n_i}} a_{i_1} \ldots a_{i_{m_i}}}\right]$.

PROOF. Since S contains all theorem of $\mathbb{L}$ by lemma 1.4, we get that all theorems of $\mathbb{L}$ are true in $(\mathcal{B}_s, v_s)$.

Let $i \in I$, $a_{i_1}, \ldots, a_{i_{m_i}}$ be terms and $(\mathcal{B}_s, v_s) \nvDash \gamma_i\left[\frac{q_{i_1} \ldots q_{i_{m_i}}}{a_{i_1} \ldots a_{i_{m_i}}}\right]$. Consequently $\gamma_i\left[\frac{q_{i_1} \ldots q_{i_{m_i}}}{a_{i_1} \ldots a_{i_{m_i}}}\right] \notin S$. For simplicity for any k and any terms $\tau_1, \ldots, \tau_k$ we will denote $\tau_1, \ldots, \tau_k$ by $\overline{\tau}$. Thus $\gamma_i\left[\frac{\overline{q}}{\overline{a}}\right] \notin S$. For the sake of contradiction suppose that for any terms $\overline{p'}$, $\beta_i\left[\frac{\overline{p}, \overline{q}}{\overline{p'}, \overline{a}}\right] \in S$. For any terms $\overline{p'}$, $\beta_i\left[\frac{\overline{p}, \overline{q}}{\overline{p'}, \overline{a}}\right] \rightarrow \left(\top \rightarrow \beta_i\left[\frac{\overline{p}, \overline{q}}{\overline{p'}, \overline{a}}\right]\right)$ is a theorem of $\mathbb{L}$ and hence is in S. Consequently $\top \rightarrow \beta_i\left[\frac{\overline{p}, \overline{q}}{\overline{p'}, \overline{a}}\right] \in S$ for any terms $\overline{p'}$. By condition (iii) from the definition of $\mathbb{L}$-theory, $\top \rightarrow \gamma_i\left[\frac{\overline{q}}{\overline{a}}\right] \in S$. Consequently $\gamma_i\left[\frac{\overline{q}}{\overline{a}}\right] \in S$ - a contradiction. $\square$

PROPOSITION 1.2. *Let S be a maximal* $\mathbb{L}$*-theory. Then the canonical structure, corresponding to S,* $\mathcal{B}_s$ *satisfies all axioms of* $\mathbb{L}$ *and the axioms, corresponding to the rules of* $\mathbb{L}$.

PROOF. Let α be an axiom of $\mathbb{L}$ with variables among $p_1, \ldots, p_n$, where $n \geq 0$. Let v be a valuation in $\mathcal{B}_s$. We will prove that $(\mathcal{B}_s, v) \vDash \alpha$, i.e. $\alpha\left[\frac{p_1, \ldots, p_n}{v(p_1), \ldots, v(p_n)}\right]$ is true. There are terms $a_1, \ldots, a_n$ such that $v(p_1) = |a_1|, \ldots, v(p_n) = |a_n|$. (Here we use the definition of the canonical structure $\mathcal{B}_s$, corresponding to S - $B_s = \{|a| : a$ is a term$\}$.) $\alpha\left[\frac{p_1, \ldots, p_n}{a_1, \ldots, a_n}\right]$ is also an axiom of $\mathbb{L}$ and hence by lemma 1.4, $(\mathcal{B}_s, v_s) \vDash \alpha\left[\frac{p_1, \ldots, p_n}{a_1, \ldots, a_n}\right]$. Consequently $\alpha\left[\frac{p_1, \ldots, p_n}{|a_1|, \ldots, |a_n|}\right]$ is true.

If $\mathbb{L}$ includes rules, different from MP, we prove that their corresponding axioms are true in $\mathcal{B}_s$, using proposition 1.1, in the following way: For simplicity for any k and any terms $\tau_1, \ldots, \tau_k$ we denote $\tau_1, \ldots, \tau_k$ by $\overline{\tau}$, $|\tau_1|, \ldots, |\tau_k|$ by $\overline{|\tau|}$ and $v(\tau_1), \ldots, v(\tau_k)$ by $\overline{v(\tau)}$, where v is some valuation. Let $i \in I$ and $\overline{a}$ be terms. Let v be a valuation in $\mathcal{B}_s$ and $(\mathcal{B}_s, v) \vDash \neg\gamma_i\left[\frac{\overline{q}}{\overline{a}}\right]$. We will prove that $(\mathcal{B}_s, v) \vDash \exists x_{i_1} \ldots \exists x_{i_{n_i}} \neg\beta_i\left[\frac{\overline{p}, \overline{q}}{\overline{x}, \overline{a}}\right]$, where $\overline{x}$ are some variables, not occurring in $\overline{a}$, different from $\overline{p}$, $\overline{q}$. Let $v(a_{i_1}) = |b_{i_1}|, \ldots, v(a_{i_{m_i}}) = |b_{i_{m_i}}|$. We have $\neg\gamma_i\left[\frac{\overline{q}}{\overline{v(a)}}\right]$, i.e. $\neg\gamma_i\left[\frac{\overline{q}}{\overline{v_s(b)}}\right]$, i.e. $(\mathcal{B}_s, v_s) \vDash \neg\gamma_i\left[\frac{\overline{q}}{\overline{b}}\right]$. By proposition 1.1, we obtain that there are terms $\overline{p'}$ such that $(\mathcal{B}_s, v_s) \nvDash \beta_i\left[\frac{\overline{p}, \overline{q}}{\overline{p'}, \overline{b}}\right]$, i.e. $\neg\beta_i\left[\frac{\overline{p}, \overline{q}}{\overline{|p'|}, \overline{v(a)}}\right]$. Consequently $\left(\mathcal{B}_s, v\left[\frac{\overline{x}}{\overline{|p'|}}\right]\right) \vDash \neg\beta_i\left[\frac{\overline{p}, \overline{q}}{\overline{x}, \overline{a}}\right]$ and hence $(\mathcal{B}_s, v) \vDash \exists x_{i_1} \ldots \exists x_{i_{n_i}} \neg\beta_i\left[\frac{\overline{p}, \overline{q}}{\overline{x}, \overline{a}}\right]$. $\square$

THEOREM 1.2 (Completeness theorem). *The following conditions are equivalent for every formula* α:

(i) α *is a theorem of* $\mathbb{L}$;

(ii) α is true in all structures for $\mathcal{L}$ in which the axioms of $\mathbb{L}$ and the corresponding to the rules of $\mathbb{L}$ axioms are true.

PROOF. (i)→(ii) It suffices to prove that for every $i \in I$, φ - a formula, $a_{i_1}, \dots, a_{i_{m_i}}$ - terms:

(1) if for arbitrary variables $r_{i_1}, \dots, r_{i_{n_i}}$ $\varphi \to \beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\, q_{i_1}\dots q_{i_{m_i}}}{r_{i_1}\dots r_{i_{n_i}}\, a_{i_1}\dots a_{i_{m_i}}}\right]$ is true in all structures for $\mathcal{L}$ in which the axioms of $\mathbb{L}$ and the corresponding to the rules of $\mathbb{L}$ axioms are true, then $\varphi \to \gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]$ is true in all structures for $\mathcal{L}$ in which the axioms of $\mathbb{L}$ and the corresponding to the rules of $\mathbb{L}$ axioms are true.

Let $i \in I$, φ be a formula, $a_{i_1}, \dots, a_{i_{m_i}}$ be terms and the premise of (1) be true. Let $\mathcal{B}$ be a structure for $\mathcal{L}$ in which the axioms of $\mathbb{L}$ and the corresponding to the rules of $\mathbb{L}$ axioms are true, and v be a valuation in $\mathcal{B}$. We will prove that $(\mathcal{B}, v) \models \varphi \to \gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]$. Suppose for the sake of contradiction the contrary. Consequently $(\mathcal{B}, v) \models \varphi$ and $(\mathcal{B}, v) \not\models \gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right]$. But in $\mathcal{B}$ is true the corresponding to the considered rule axiom: $\neg\gamma_i\left[\frac{q_{i_1}\dots q_{i_{m_i}}}{a_{i_1}\dots a_{i_{m_i}}}\right] \to \exists x_{i_1} \dots \exists x_{i_{n_i}} \neg\beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\, q_{i_1}\dots q_{i_{m_i}}}{x_{i_1}\dots x_{i_{n_i}}\, a_{i_1}\dots a_{i_{m_i}}}\right]$, where $x_{i_1}, \dots, x_{i_{n_i}}$ are some variables, not occurring in $a_{i_1}, \dots, a_{i_{m_i}}$, different from $p_{i_1}, \dots, p_{i_{n_i}}, q_{i_1} \dots, q_{i_{m_i}}$. Consequently $(\mathcal{B}, v) \models \exists x_{i_1} \dots \exists x_{i_{n_i}} \neg\beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\, q_{i_1}\dots q_{i_{m_i}}}{x_{i_1}\dots x_{i_{n_i}}\, a_{i_1}\dots a_{i_{m_i}}}\right]$ and hence there are $r'_{i_1}, \dots, r'_{i_{n_i}} \in B$ such that $\left(\mathcal{B}, v\left[\frac{r_{i_1}\dots r_{i_{n_i}}}{r'_{i_1}\dots r'_{i_{n_i}}}\right]\right) \not\models \beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\, q_{i_1}\dots q_{i_{m_i}}}{r_{i_1}\dots r_{i_{n_i}}\, a_{i_1}\dots a_{i_{m_i}}}\right]$, where $r_{i_1}, \dots, r_{i_{n_i}}$ are some variables, not occurring in $a_{i_1}, \dots, a_{i_{m_i}}$ and φ. We have $\left(\mathcal{B}, v\left[\frac{r_{i_1}\dots r_{i_{n_i}}}{r'_{i_1}\dots r'_{i_{n_i}}}\right]\right) \models \varphi$ and $\left(\mathcal{B}, v\left[\frac{r_{i_1}\dots r_{i_{n_i}}}{r'_{i_1}\dots r'_{i_{n_i}}}\right]\right) \models \varphi \to \beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\, q_{i_1}\dots q_{i_{m_i}}}{r_{i_1}\dots r_{i_{n_i}}\, a_{i_1}\dots a_{i_{m_i}}}\right]$. Consequently $\left(\mathcal{B}, v\left[\frac{r_{i_1}\dots r_{i_{n_i}}}{r'_{i_1}\dots r'_{i_{n_i}}}\right]\right) \models \beta_i\left[\frac{p_{i_1}\dots p_{i_{n_i}}\, q_{i_1}\dots q_{i_{m_i}}}{r_{i_1}\dots r_{i_{n_i}}\, a_{i_1}\dots a_{i_{m_i}}}\right]$ - a contradiction.

(ii)→(i) Let α be true in all structures for $\mathcal{L}$ in which the axioms of $\mathbb{L}$ and the corresponding to the rules of $\mathbb{L}$ axioms are true. Suppose for the sake of contradiction that α is not a theorem of $\mathbb{L}$. Let T be the set of all theorems of $\mathbb{L}$. T is a $\mathbb{L}$-theory. Let $T' = T + \{\neg\alpha\}$. We have $\neg\neg\alpha \notin T$, so using the extension lemma (ii), we get that T' is a consistent $\mathbb{L}$-theory. From the Lindenbaum lemma it follows that T' can be extended to a maximal $\mathbb{L}$-theory S. From proposition 1.2 we obtain that the canonical structure $\mathcal{B}_s$ satisfies all axioms of $\mathbb{L}$ and the axioms, corresponding to the rules of $\mathbb{L}$. Consequently $(\mathcal{B}_s, v_s) \models \alpha$. By lemma 1.4, we get that $\alpha \in S$. But $\neg\alpha$ is also in S. Consequently S is inconsistent - a contradiction. Consequently α is a theorem of $\mathbb{L}$. □

2. Quantifier-free logics for extended distributive contact lattices

We consider the quantifier-free first-order language with equality $\mathcal{L}$ which includes:

- constants: 0, 1;
- function symbols: $+$, $\cdot$;
- predicate symbols: $\leq$, C, $\widehat{C}$, $\ll$.

Let $\bot \stackrel{def}{=} (0 \leq 0) \wedge \neg(0 \leq 0)$, $\top \stackrel{def}{=} (0 \leq 0) \vee \neg(0 \leq 0)$. Every EDCL is a structure for $\mathcal{L}$.

We consider the logic L with rule MP and the following axioms:

- the axioms of the classical propositional logic;

• the axiom schemes of distributive lattice;
• the axioms for C, $\widehat{C}$, $\ll$ and the mixed axioms of EDCL - considered as axiom schemes.

We consider the following additional rules and an axiom scheme:

(R Ext $\widehat{O}$) $\frac{\alpha\rightarrow(a+p\neq 1\vee b+p=1) \text{ for all variables } p}{\alpha\rightarrow(a\leq b)}$, where α is a formula, a, b are terms

(R U-rich $\ll$) $\frac{\alpha\rightarrow(b+p\neq 1\vee aCp) \text{ for all variables } p}{\alpha\rightarrow(a\ll b)}$, where α is a formula, a, b are terms

(R U-rich $\widehat{C}$) $\frac{\alpha\rightarrow(a+p\neq 1\vee b+q\neq 1\vee pCq) \text{ for all variables } p,\ q}{\alpha\rightarrow a\widehat{C}b}$, where α is a formula, a, b are terms

(R Ext C) $\frac{\alpha\rightarrow(p\neq 0\rightarrow aCp) \text{ for all variables } p}{\alpha\rightarrow(a=1)}$, where α is a formula, a is a term

(R Nor1) $\frac{\alpha\rightarrow(p+q\neq 1\vee aCp\vee bCq) \text{ for all variables } p,\ q}{\alpha\rightarrow aCb}$, where α is a formula, a, b are terms

(Con C) $p\neq 0\wedge q\neq 0\wedge p+q=1\rightarrow pCq$

The corresponding to these rules axioms are equivalent respectively to the axioms (Ext $\widehat{O}$), (U-rich $\ll$), (U-rich $\widehat{C}$), (Ext C), (Nor1).

Let L' be for example the extension of L with the rule (R Ext $\widehat{O}$) and the axiom scheme (Con C). Then we denote L' by $L_{ConC,Ext\widehat{O}}$ and call the axioms (Con C) and (Ext $\widehat{O}$) corresponding to L' additional axioms. In a similar way we denote any extension of L with some of the considered additional rules and axiom scheme and in a similar way we define its corresponding additional axioms.

Using theorem 1.2, we obtain:

THEOREM 2.1 (Completeness theorem with respect to algebraic semantics). *Let L' be some extension of L with 0 or more of the considered additional rules and axiom scheme. The following conditions are equivalent for any formula α:*
(i) α is a theorem of L';
(ii) α is true in all EDCL, satisfying the corresponding to L' additional axioms.

We consider the following logics, corresponding to the EDC-lattices, considered in chapter 1:
1) L;
2) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C}}$;
3) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC}$;
4) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ConC}$;
5) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},Nor1}$;
6) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,ConC}$;
7) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},Nor1,ConC}$;
8) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,Nor1}$;
9) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,ConC,Nor1}$.

To every of these logics we juxtapose a class of topological spaces:
1) the class of all T_0, semiregular, compact topological spaces;
2) the class of all T_0, semiregular, compact topological spaces;
3) the class of all T_0, compact, weakly regular topological spaces;
4) the class of all T_0, semiregular, compact, connected topological spaces;

5) the class of all T_0, semiregular, compact, κ - normal topological spaces;
6) the class of all T_0, compact, weakly regular, connected topological spaces;
7) the class of all T_0, semiregular, compact, κ - normal, connected topological spaces;
8) the class of all T_0, compact, weakly regular, κ - normal topological spaces;
9) the class of all T_0, compact, weakly regular, connected, κ - normal topological spaces.

Later we will prove that some of the rules of these logics can be eliminated and these logics are reducible to other logics. Because of this some other logics also will be considered.

PROPOSITION 2.1. *For every EDCL $\underline{B}$, satisfying the corresponding of some of the considered above logics additional axioms, there exists a topological space X from the corresponding class and an embedding of $\underline{B}$ in $RC(X)$.*

PROOF. In [**40**] (Theorem 2.3.9) it is proved that: if $\underline{B}$ is a contact algebra, then there is a compact, semiregular, T_0 topological space X and an embedding of $\underline{B}$ in $RC(X)$. From here and corollary 2.1 in chapter 1 it follows that: if $\underline{B}$ is an EDCL (i.e. EDCL, satisfying the corresponding to L zero additional axioms), then there is a compact, semiregular, T_0 topological space X and an embedding of $\underline{B}$ in $RC(X)$.

For the other eight logics the proposition follows from theorem 6.4 in chapter 1. $\square$

THEOREM 2.2 (Completeness theorem with respect to topological semantics). *Let L' be any of the considered logics. The following conditions are equivalent for any formula α:*
(i) α is a theorem of L';
(ii) α is true in all contact algebras over a topological space from the corresponding to L' class.

PROOF. From the previous completeness theorem we have: (i)$\leftrightarrow$

$$(ii')\alpha \text{ is true in all EDCL, satisfying the corresponding to } L' \text{ additional axioms}$$

We will prove that (ii')$\leftrightarrow$ (ii).
(ii')$\rightarrow$(ii) Let α be true in all EDCL, satisfying the corresponding to L' additional axioms. Let X be a topological space from the corresponding to L' class. From lemma 5.1 in chapter 1 it follows that $RC(X)$ satisfies the corresponding to L' additional axioms. Consequently α is true in $RC(X)$.
(ii)$\rightarrow$(ii') Let α be true in all contact algebras over a topological space from the corresponding to L' class. Let $\underline{B}$ be an EDCL, satisfying the corresponding to L' additional axioms, and v be a valuation in $\underline{B}$. We will prove that $(\underline{B}, v) \models \alpha$. By proposition 2.1, we get that there is a topological space X from the corresponding to L' class and an isomorphic embedding h of $\underline{B}$ in $RC(X)$. We define a valuation v' in $RC(X)$ in the following way: $v'(p) = h(v(p))$ for every variable p. By $\underline{B}'$ we denote the sublattice of $RC(X)$ to which $\underline{B}$ is isomorphic. We have $(RC(X), v') \models \alpha$, so $(\underline{B}', v') \models \alpha$, so $(\underline{B}, v) \models \alpha$. $\square$

PROPOSITION 2.2. *L and $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C}}$ have the same theorems.*

PROOF. The proposition follows from the completeness theorem with respect to topological semantics because to L and $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C}}$ corresponds the same class of topological spaces. □

3. Decidability via finite models and admissibility of some rules of inference

In this section we discuss admissibility of some rules of inference and decidability via finite models of the logics introduced in Section 2. We will not discuss in this dissertation the complexity of the corresponding logics.

3.1. Decidability of the logic L.

PROPOSITION 3.1. *The following conditions are equivalent for any formula α:*
(i) α is true in all EDCL;
(ii) α is true in all finite EDCL with a number of the elements less or equal to $2^{2^n-1}+1$, where n is the number of the variables of α.

PROOF. Obviously (i) implies (ii). Let (ii) be true. We will prove (i). Let $\underline{B}$ be an EDCL, v be a valuation in $\underline{B}$. We will prove that $(\underline{B}, v) \models \alpha$. Let the variables of α be $p_1, \dots, p_n$, where $n \geq 0$. It is a well known fact that $v(p_1), \dots, v(p_n)$ generate a distributive sublattice $\underline{B'}$ of $\underline{B}$ with a number of the elements less or equal to $2^{2^n-1}+1$. $\underline{B'}$ is an EDCL. We define a valuation v' in $\underline{B'}$ in the following way:

$$v'(p) = \begin{cases} v(p) & \text{if } p = p_1 \text{ or } p = p_2 \text{ or } \dots \text{ or } p = p_n \\ 0 & \text{otherwise} \end{cases}$$

It suffices to prove that $(\underline{B'}, v') \models \alpha$. But this is true because (ii) is fulfilled. □

COROLLARY 3.1. *L is decidable.*

3.2. Admissibility of the rule (R Ext C). As in [**4**] we define a p-morphism and prove a lemma for it. Let (W, R) and (W', R') be relational structures and f be a surjection from W to W'. We call f p-morphism from (W, R) to (W', R'), if the following conditions are fulfilled for any x, $y \in W$ and any x', $y' \in W'$:
(p1) If xRy, then $f(x)R'f(y)$;
(p2) If $x'R'y'$, then $(\exists x, y \in W)(x' = f(x), y' = f(y), xRy)$.

Let $\underline{B}$ be the contact algebra over (W, R), $\underline{B'}$ be the contact algebra over (W', R'), v and v' be valuations respectively in $\underline{B}$ and $\underline{B'}$. We say that f is a p-morphism from $(\underline{B}, v)$ to $(\underline{B'}, v')$, if for every variable p and every $x \in W$: $x \in v(p) \leftrightarrow f(x) \in v'(p)$. It can be easily proved that for every term a and every $x \in W$: $x \in v(a) \leftrightarrow f(x) \in v'(a)$.

LEMMA 3.1. [**4**] *Let f be a p-morphism from $(\underline{B}, v)$ to $(\underline{B'}, v')$. Then for any formula for $\mathcal{L}$, φ we have: $(\underline{B}, v) \models \varphi \leftrightarrow (\underline{B'}, v') \models \varphi$.*

PROOF. Induction on the complexity of φ. □

PROPOSITION 3.2. *The rule (R Ext C) is admissible in $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C}}$ and $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ConC}$.*

PROOF. The construction is almost the same as in [**4**] (Lemma 6.1). Let L' be any of these logics. Let α be a formula, a be a term. Let $\alpha \to (p \neq 0 \to aCp)$ be a theorem of L' for every variable p. We will prove that $\alpha \to (a = 1)$ is a theorem of L'. Suppose for the sake of contradiction the contrary. There is an EDCL $\underline{B}$, satisfying the corresponding to L' additional axioms, and a valuation in it v such that $(\underline{B}, v) \not\models \alpha \to (a = 1)$. Consequently $(\underline{B}, v) \models \alpha$ and $(\underline{B}, v) \not\models a = 1$. $\underline{B}$ is an U-rich EDCL and by theorem 6.4 in chapter 1, we get that there is a topological space X and an embedding h of $\underline{B}$ in $RC(X)$. Moreover if $\underline{B}$ satisfies (Con C), then $RC(X)$ also satisfies (Con C). We define a valuation v' in $RC(X)$ in the following way: $v'(p) = h(v(p))$ for every variable p. We have $(RC(X), v') \models \alpha$ and $(RC(X), v') \not\models a = 1$.

Let Q be the set of all variables, occurring in α and a. $v'(Q)$ is a finite subset of $RC(X)$. The subalgebra $\underline{B_1}$ of $RC(X)$, generated by $v'(Q)$, is a finite Boolean contact algebra. If $RC(X)$ satisfies (Con C), then $\underline{B_1}$ also satisfies the axiom (Con C). We define a valuation v_1 in $\underline{B_1}$ in the following way:

$$v_1(p) = \begin{cases} v'(p) & \text{if } p \in Q \\ 0 & \text{otherwise} \end{cases}$$

We have $(\underline{B_1}, v_1) \models \alpha$ and $(\underline{B_1}, v_1) \not\models a = 1$. There is a relational structure (W_2, R_2) and an isomorphism h_1 from $\underline{B_1}$ to the contact algebra $\underline{B_2}$ over (W_2, R_2). We define a valuation v_2 in $\underline{B_2}$ in the following way $v_2(p) = h_1(v_1(p))$ for every variable p. $(\underline{B_2}, v_2) \models \alpha$ and $(\underline{B_2}, v_2) \not\models a = 1$. Consequently $v_2(a) \neq W_2$. Let $w_1 \in W_2 - v_2(a)$, $w_0 \notin W_2$. We define $W_3 = W_2 \cup \{w_0\}$, $R_3 = R_2 \cup \{(w_0, w_0), (w_0, w_1), (w_1, w_0)\}$. We define $f : W_3 \to W_2$ in the following way:

$$f(w) = \begin{cases} w & \text{if } w \neq w_0 \\ w_1 & \text{if } w = w_0 \end{cases}$$

Let $\underline{B_3}$ be the contact algebra over (W_3, R_3). We define a valuation v_3 in $\underline{B_3}$ in the following way: $v_3(p) = f^{-1}(v_2(p))$ for every variable p. It can be easily verified that f is a p-morphism from $(\underline{B_3}, v_3)$ to $(\underline{B_2}, v_2)$. Consequently $(\underline{B_3}, v_3) \models \alpha$ and $(\underline{B_3}, v_3) \not\models a = 1$. If $\underline{B}$ satisfies the axiom (Con C), then $\underline{B_1}$ also satisfies (Con C) and since $\underline{B_1}$ is isomorphic to $\underline{B_2}$, we have that $\underline{B_2}$ also satisfies (Con C). From here and the definition of R_3 we get that if $\underline{B}$ satisfies (Con C), then $\underline{B_3}$ also satisfies the axiom (Con C)(1). Since $\underline{B_3}$ is a contact algebra, we have that $\underline{B_3}$ satisfies (Ext $\widehat{O}$), (U-rich $\ll$) and (U-rich $\widehat{C}$)(2). Let p be a variable, not occurring in a and α. We have $(\underline{B_3}, v_3\left[\frac{p}{\{w_0\}}\right]) \models \alpha$ and $v_3\left[\frac{p}{\{w_0\}}\right](a) = v_3(a) = f^{-1}(v_2(a)) = v_2(a)$; $v_3\left[\frac{p}{\{w_0\}}\right](p) \neq \emptyset$; $v_3\left[\frac{p}{\{w_0\}}\right](a)\overline{C_{R_3}}v_3\left[\frac{p}{\{w_0\}}\right](p)$. Consequently $(\underline{B_3}, v_3\left[\frac{p}{\{w_0\}}\right]) \not\models \alpha \to (p \neq 0 \to aCp)$. Also from (1) and (2) it follows that $\underline{B_3}$ satisfies the corresponding to L' additional axioms. But $\alpha \to (p \neq 0 \to aCp)$ is a theorem of L', so $(\underline{B_3}, v_3\left[\frac{p}{\{w_0\}}\right]) \models \alpha \to (p \neq 0 \to aCp)$ - a contradiction. □

3.3. Admissibility of the rule (R Nor1).

PROPOSITION 3.3. *The rule (R Nor1) is admissible in the logics* $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C}}$ *and* $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ConC}$.

PROOF. The construction is almost the same as in [**4**] (Lemma 6.2). Let L' be any of these logics. Let α be a formula, a and b be terms. Let $\alpha \to (p + q \neq$

$1 \vee aCp \vee bCq)$ be a theorem of L' for all variables p and q. We will prove that $\alpha \to aCb$ is a theorem of L'. Suppose for the sake of contradiction the contrary. The same way as in the proof of the previous proposition we obtain that there is a contact algebra $\underline{B}$ over some relational structure (W, R) and a valuation in it v such that $(\underline{B}, v) \models \alpha$ and $(\underline{B}, v) \models a\overline{C}b$. Moreover if L' is the second logic, then $\underline{B}$ satisfies (Con C). Let $A \subseteq W$. We define $\langle R\rangle A = \{x \in W : (\exists y \in A)(yRx)\}$. Let $W_{defects} = \langle R\rangle(v(a)) \cap \langle R\rangle(v(b))$. We define
$W_1 = W \times \{1, 2\}$,
$(x,i)R_1(y,j) \leftrightarrow xRy$ and $((j = 1 \wedge x \in v(a) \wedge y \in W_{defects})$
or $(i = 1 \wedge y \in v(a) \wedge x \in W_{defects})$
or $(j = 2 \wedge x \in v(b) \wedge y \in W_{defects})$
or $(i = 2 \wedge y \in v(b) \wedge x \in W_{defects})$
or $(x \notin v(a) \cup v(b) \cup W_{defects} \wedge y \in W_{defects})$
or $(y \notin v(a) \cup v(b) \cup W_{defects} \wedge x \in W_{defects})$
or $(x \in W_{defects} \wedge y \in W_{defects})$
or $(x \notin W_{defects} \wedge y \notin W_{defects}))$,
$v_1(q) = v(q) \times \{1, 2\}$,
$f((x, i)) = x$.
Let $\underline{B_1}$ be the contact algebra over (W_1, R_1). It can be easily verified that f is a p-morphism from $(\underline{B_1}, v_1)$ to $(\underline{B}, v)$. Consequently $(\underline{B_1}, v_1) \models \alpha$ and $(\underline{B_1}, v_1) \models a\overline{C}b$. It can be easily verified that if L' is the second logic, then $\underline{B_1}$ satisfies (Con C). Let p, q be variables which do not occur in a, b and φ. We define a valuation v_1' in $\underline{B_1}$ eventually different from v_1 only in p and q: $v_1'(p) = \langle R_1\rangle(v_1(b))$, $v_1'(q) = \overline{\langle R_1\rangle(v_1(b))}$. Obviously $v_1'(p) + v_1'(q) = 1$. Suppose for the sake of contradiction that $v_1'(a)Cv_1'(p)$. Consequently $v_1(a)C_{R_1}\langle R_1\rangle v_1(b)$. From here we obtain that there are $(x,i) \in v_1(a)$, $(y,j) \in \langle R_1\rangle v_1(b)$ such that $(x,i)R_1(y,j)$. From $(y,j) \in \langle R_1\rangle(v_1(b))$ we obtain that there is $(z,k) \in v_1(b)$ such that $(z,k)R_1(y,j)$. Consequently $z \in v(b)$ and yRz and hence $y \in \langle R\rangle(v(b))$ (1). From $(x,i) \in v_1(a)$ we obtain $x \in v(a)$ (2). From $(x,i)R_1(y,j)$ we get xRy (3). Using (2) and (3), we get $y \in \langle R\rangle(v(a))$ (4). From (4) and (1) we get $y \in W_{defects}$. From $(x,i)R_1(y,j)$, $y \in W_{defects}$, $x \in v(a)$, $(\underline{B}, v) \models a\overline{C}b$ and the definition of R_1 we get $j = 1$. Using $(z,k)R_1(y,j)$, $y \in W_{defects}$, $z \in v(b)$, $(\underline{B}, v) \models a\overline{C}b$ and the definition of R_1, we get $j = 2$ - a contradiction. Consequently $v_1'(a)\overline{C}v_1'(p)$. From the definition of $v_1'(q)$ we obtain that $v_1'(b)\overline{C}v_1'(q)$. Thus $(\underline{B_1}, v_1') \not\models p + q \neq 1 \vee aCp \vee bCq$ and $(\underline{B_1}, v_1') \models \alpha$; $\underline{B_1}$ satisfies the corresponding to L' additional axioms - a contradiction. □

3.4. The rule (R U-rich $\ll$) is not admissible in L_{ConC}.

LEMMA 3.2. *Let $\underline{B} = (B, \ldots)$ be an EDCL, satisfying (U-rich $\ll$) and (Con C). Then for every $a \in B$, different from 0 and 1, we have $a\overline{\ll}a$.*

PROOF. Let $a \in B$, $a \neq 0$, $a \neq 1$. Suppose for the sake of contradiction that $a \ll a$. Since $\underline{B}$ satisfies (U-rich $\ll$), there is a $c \in B$ such that $c + a = 1$ and $a\overline{C}c$. We have that $\underline{B}$ satisfies (Con C) and $c + a = 1$, $a \neq 0$, $c \neq 0$ (because $a \neq 1$), so aCc - a contradiction. Consequently $a\overline{\ll}a$. □

PROPOSITION 3.4. *The rule (R U-rich $\ll$) is not admissible in L_{ConC}.*

PROOF. We will prove that there is a theorem of $L_{ConC,U-rich\ll}$ which is not a theorem of L_{ConC}. We consider the formula $\alpha :\ p \neq 0 \wedge p \neq 1 \to p\overline{\ll}p$. Using lemma 3.2, we obtain that α is true in every EDCL, satisfying (Con C) and (U-rich $\ll$). Consequently α is a theorem of $L_{ConC,U-rich\ll}$.

We consider the relational structure (W, R), where $W = \{x, y\}$, $R = \{(x, x), (y, y)\}$. Let $\underline{B}$ be the contact algebra over (W, R). Let $B' = \{\emptyset, W, \{x\}\}$. It can be easily verified that B' is closed under $\cup$ and $\cap$. Consequently $\underline{B'} = (B', \subseteq, \emptyset, W, \cap, \cup)$ is a distributive lattice. We can consider $\underline{B'}$ as a substructure $(B', \subseteq, \emptyset, W, \cap, \cup, C_R, \widehat{C_R}, \ll_R)$ of $\underline{B}$. $\underline{B}$ is an EDCL and the axioms of EDCL are quantifier-free and therefore $\underline{B'}$ is an EDCL. We have $\{x\} \neq \emptyset$, $\{x\} \neq W$ and $\{x\} \ll \{x\}$, so α is not true in $\underline{B'}$. It can be easily verified that $\underline{B'}$ satisfies (Con C). Consequently α is not a theorem of L_{ConC}. □

3.5. A technical lemma with applications to admissibility of some rules of inference and decidability of some logics.

LEMMA 3.3. *Let $\underline{B}$ be an EDCL, satisfying (Con C) and (U-rich $\ll$) and v be a valuation in it. Let α be a formula in $\mathcal{L}$. Then there is a finite connected Boolean contact algebra $\underline{B^*}$ and a valuation in it v^* such that: $(\underline{B^*}, v^*) \models \alpha$ iff $(\underline{B}, v) \models \alpha$. The number of the elements of $\underline{B^*}$ is $\leq 2^{\frac{(2^{2^n-1}+1)2^{2^n-1}}{2}}$, where n is the number of the variables of α.*

PROOF. Let $\underline{B}$ be an EDCL, satisfying (Con C) and (U-rich $\ll$), and v be a valuation in it. Let α be a formula in $\mathcal{L}$. From the relational representation theorem of EDC-lattices (Theorem 2.3 in Chapter 1) it follows that there is a relational structure (W', R') with R' reflexive and symmetric and an isomorphic embedding h of $\underline{B}$ in the contact algebra $\underline{B'}$ over (W', R'). $\underline{B}$ is isomorphic of some substructure of $\underline{B'}$, $\underline{B_1}$, which is an EDCL, satisfying (Con C) and (U-rich $\ll$). We define a valuation v_1 in $\underline{B_1}$ in the following way: for every variable p $v_1(p) \stackrel{def}{=} h(v(p))$. It can be easily proved that $(\underline{B}, v) \models \alpha$ iff $(\underline{B_1}, v_1) \models \alpha$. Let the variables of α be $p_1, \ldots, p_n$, where $n \geq 0$. $v_1(p_1), \ldots, v_1(p_n)$ generate a finite sublattice $\underline{B_2} = (B_2, \subseteq, \emptyset, W', \cap, \cup, C_{R'}, \widehat{C_{R'}}, \ll_{R'})$ of $\underline{B_1}$ which is an EDCL, satisfying (Con C), and has number of the elements $\leq 2^{2^n-1} + 1$. We define a valuation v_2 in $\underline{B_2}$ in the following way:

$$v_2(p) = \begin{cases} v_1(p) & \text{if } p = p_1 \text{ or } p = p_2 \text{ or } \ldots \text{ or } p = p_n \\ \emptyset & \text{otherwise} \end{cases}$$

We have $(\underline{B_1}, v_1) \models \alpha$ iff $(\underline{B_1}, v_2) \models \alpha$ iff $(\underline{B_2}, v_2) \models \alpha$.

For every $A \in W'$ we define, using that B_2 is finite, $s_A \stackrel{def}{=} \bigcap\{a \in B_2 : A \in a\}$, i.e. s_A is the smallest element of B_2 which contains A.

We will define *special sets* and with their help we will obtain a Boolean algebra $\underline{B_3}$. Let $A \in W'$ and $b \in B_2$, $b \subseteq s_A$, $A \notin b$, $\forall c(c \neq \emptyset, c \in B_2, c \subseteq s_A, A \notin c \rightarrow b \cap c \neq \emptyset)$. Then $s_A - b$ we call a special set, determined by the ordered pair (s_A, b).

Let (a, b) be an ordered pair of elements of B_2. We have:

1) if $b \subseteq a$, $a \neq b$, then (a, b) determines at most one special set;
2) if b is not a proper subset of a, then (a, b) does not determine a special set;

Using this fact, we get that the number of the special sets is $\leq$ half of the ordered pairs of different elements of B_2. Let C be the set of all special sets, N be the number of the elements of B_2. We have $|C| \leq \frac{N(N-1)}{2} \leq \frac{(2^{2^n-1}+1)2^{2^n-1}}{2}$. Let D be the set of all finite unions of special sets. We have that $|D| \leq$ the number of the nonempty subsets of C, i.e. $|D| \leq 2^{|C|} - 1 \leq 2^{\frac{(2^{2^n-1}+1)2^{2^n-1}}{2}} - 1$. Let $B_3 \stackrel{def}{=} D \cup \{\emptyset\}$. We have $|B_3| \leq 2^{\frac{(2^{2^n-1}+1)2^{2^n-1}}{2}}$; $B_3 \subseteq B'$.

We will prove:

CLAIM 3.1. *If $a, b \in B_2$, then $a - b \in B_3$.*

PROOF. **Case 1:** $a - b = \emptyset$
We have $a - b \in B_3$.
Case 2: $a - b \neq \emptyset$
Let $A \in a - b$. By t_A we denote the largest element of B_2 which is a subset of s_A and does not contain A ($t_A = \cup\{e \in B_2 : \; e \subseteq s_A, A \notin e\}$). s_A is the smallest element of B_2 which contains A; a is an element of B_2 which contains A; so $s_A \subseteq a(2)$. We have $s_A \cap b \in B_2$, $s_A \cap b \subseteq s_A$, $A \notin s_A \cap b$; so $s_A \cap b \subseteq t_A(3)$. From (2) and (3) we get that $s_A - t_A \subseteq a - b$. Thus we juxtapose to every point A of $a - b$ an ordered pair elements of B_2 (s_A, t_A) such that $A \in s_A - t_A \subseteq a - b$. Let the obtained this way ordered pairs be $(s_1, t_1), \ldots, (s_k, t_k)$, where $k > 0$. Obviously $a - b \subseteq (s_1 - t_1) \cup \ldots \cup (s_k - t_k) \subseteq a - b$, i.e. $a - b = (s_1 - t_1) \cup \ldots \cup (s_k - t_k)$. Let $i \in \{1, \ldots, k\}$. Using the definition of t_i, we get that $s_i - t_i$ is a special set, determined by (s_i, t_i). Consequently $a - b$ is a finite union of special sets. Consequently $a - b \in B_3$. □

CLAIM 3.2. *$\underline{B_3} = (B_3, \subseteq, \emptyset, W', \cup, \cap)$ is a Boolean algebra and $B_2 \subseteq B_3$. (We do not use $*$ in the notation of $\underline{B_3}$ because we do not want to change the language.)*

PROOF. Let $a \in B_2$. We have $a = a - \emptyset$. Using claim 3.1, we obtain that $a \in B_3$. Consequently $B_2 \subseteq B_3$. Consequently $W' \in B_3$. We will prove that B_3 is closed under $\cup$ and $\cap$. Obviously B_3 is closed under $\cup$. Let a_1, $a_2 \in B_3$. We will prove that $a_1 \cap a_2 \in B_3$. If $a_1 = \emptyset$ or $a_2 = \emptyset$, then obviously $a_1 \cap a_2 \in B_3$. Let a_1, $a_2 \neq \emptyset$. We have $a_1 \cap a_2 = (a_{11} \cup \ldots \cup a_{1l}) \cap (a_{21} \cup \ldots \cup a_{2m})$, where l, $m > 0$, $a_{11}, \ldots, a_{2m}$ are special sets. $a_1 \cap a_2 = (a_{11} \cap a_{21}) \cup \ldots \cup (a_{11} \cap a_{2m}) \ldots (a_{1l} \cap a_{21}) \cup \ldots \cup (a_{1l} \cap a_{2m})$. It is sufficient to prove that the intersection of two special sets is $\emptyset$ or a finite union of special sets. Let $s_{A_1} - b_1$ and $s_{A_2} - b_2$ be special sets. It can be easily verified that $(s_{A_1} - b_1) \cap (s_{A_2} - b_2) = (s_{A_1} \cap s_{A_2}) - ((b_1 \cap s_{A_2}) \cup (b_2 \cap s_{A_1}))$. Using this fact and claim 3.1, we obtain that $(s_{A_1} - b_1) \cap (s_{A_2} - b_2) \in B_3$. Consequently $a_1 \cap a_2 \in B_3$. Thus $\underline{B_3} = (B_3, \subseteq, \emptyset, W', \cup, \cap)$ is a distributive lattice of sets.

We will prove that for every $a \in B_3$, we have $\overline{a} \in B_3$. Let $a \in B_3$. If $a = \emptyset$, then $\overline{a} \in B_3$. Let $a = (s_{A_1} - b_1) \cup \ldots \cup (s_{A_l} - b_l)$, where $l > 0$, $s_{A_1} - b_1, \ldots, s_{A_l} - b_l$ are special sets, determined respectively by $(s_{A_1}, b_1), \ldots, (s_{A_l}, b_l)$. $\overline{a} = \overline{s_{A_1} - b_1} \cap \ldots \cap \overline{s_{A_l} - b_l}$. Let $i \in \{1, \ldots, l\}$. We will prove that $\overline{s_{A_i} - b_i} \in B_3$. $\overline{s_{A_i} - b_i} = \overline{s_{A_i}} \cup b_i = (W' - s_{A_i}) \cup b_i$. Using $W' \in B_2$, $s_{A_i} \in B_2$ and claim 3.1, we get that $W' - s_{A_i} \in B_3(4)$. (s_{A_i}, b_i) determines a special set and therefore $b_i \in B_2$; but $B_2 \subseteq B_3$, so $b_i \in B_3(5)$. Using (4), (5) and the fact that B_3 is closed under $\cup$, we get that $\overline{s_{A_i} - b_i} \in B_3$ for every $i \in \{1, \ldots, l\}$. But B_3 is closed under $\cap$, so $\overline{a} \in B_3$. Consequently $\underline{B_3} = (B_3, \subseteq, \emptyset, W', \cup, \cap)$ is a Boolean algebra. □

We will call the elements of W' points. Let T, $U \in W'$ and suppose there is an $a \in B_2$ such that $T \in a$, $U \notin a$. We define $S_{T,U} = \bigcup\{a \in B_2 : \; T \in a, U \notin a\}$, i.e. $S_{T,U}$ is the largest element of B_2, containing T and not-containing U.

Let T, $U \in W'$ and suppose there is an $a \in B_2$ such that $T \in a$, $U \notin a$. We call U corresponding to T, if $(\forall a \in B_2)(U \in a \to T \in a)$ and $s_T \overline{\ll_{R'}} S_{T,U}$.

We define a binary relation R in W' in the following way: TRU iff $TR'U$ or U is corresponding to T or T is corresponding to U. Obviously R is reflexive and symmetric. We consider the Boolean contact algebra $\underline{B_4} = (B_3, \subseteq, \emptyset, W', \cup, \cap, C_R, \widehat{C_R}, \ll_R)$. (Here $C_R, \widehat{C_R}, \ll_R$ are defined in the following way: $aC_Rb \leftrightarrow$ there are $F_1 \in a$, $F_2 \in b$ such that F_1RF_2, $a\widehat{C_R}b \leftrightarrow$ there are $F_1 \in \overline{a}$, $F_2 \in \overline{b}$ such that F_1RF_2, $a \ll_R b \leftrightarrow (\forall F_1 \in a)(\forall F_2 \in \overline{b})(F_1\overline{R}F_2)$). We consider the following substructure of $\underline{B_4}$: $\underline{B_5} = (B_2, \subseteq, \emptyset, W', \cup, \cap, C_R, \widehat{C_R}, \ll_R)$. We will prove:

Claim 3.3. *$\underline{B_5}$ is isomorphic to* $\underline{B_2} = (B_2, \subseteq, \emptyset, W', \cup, \cap, C_{R'}, \widehat{C_{R'}}, \ll_{R'})$.

Proof. The isomorphism will be the mapping $id: B_2 \to B_2$ ($id(a) \stackrel{def}{=} a$ for every $a \in B_2$).
•) We will prove that for all a_1, $a_2 \in B_2$ we have: $a_1C_{R'}a_2$ iff $a_1C_Ra_2$. Obviously $a_1C_{R'}a_2$ implies $a_1C_Ra_2$. Let $a_1C_Ra_2$. Consequently there are $F_1 \in a_1$, $F_2 \in a_2$ such that F_1RF_2.
Case 1: $F_1R'F_2$
Obviously $a_1C_{R'}a_2$.
Case 2: $F_1\overline{R'}F_2$
F_2 is corresponding to F_1 or F_1 is corresponding to F_2. Without loss of generality F_2 is corresponding to F_1. Consequently every element of B_2 which contains F_2, also contains F_1; $F_2 \in a_2$; $a_2 \in B_2$; so $F_1 \in a_2$. We also have $F_1 \in a_1$, so $a_1C_{R'}a_2$.
•) We will prove that for all a_1, $a_2 \in B_2$ we have: $a_1\widehat{C_{R'}}a_2$ iff $a_1\widehat{C_R}a_2$. Obviously $a_1\widehat{C_{R'}}a_2$ implies $a_1\widehat{C_R}a_2$. Let $a_1\widehat{C_R}a_2(6)$. Suppose for the sake of contradiction that $a_1\overline{\widehat{C_{R'}}}a_2(7)$. Consequently $\overline{a_1} \cap \overline{a_2} = \emptyset(8)$. From here and (6) we get that there are $F_1 \in a_1$, $F_2 \in a_2$ such that F_1RF_2. There is an element of B_2 (a_2) which contains F_2 but does not contain F_1; there is an element of B_2 (a_1) which contains F_1 but does not contain F_2; so F_2 is not corresponding to F_1 and F_1 is not corresponding to F_2. From (7) we get that $F_1\overline{R'}F_2$. Consequently $F_1\overline{R}F_2$ - a contradiction. Consequently $a_1\widehat{C_{R'}}a_2$.
•) We will prove that for all a_1, $a_2 \in B_2$ we have: $a_1 \ll_{R'} a_2$ iff $a_1 \ll_R a_2$. Obviously $a_1 \ll_R a_2$ implies $a_1 \ll_{R'} a_2$. Let $a_1 \ll_{R'} a_2(9)$. Suppose for the sake of contradiction that $a_1\overline{\ll_R}a_2$. Consequently there are $F_1 \in a_1$, $F_2 \notin a_2$ such that F_1RF_2. From (9) we obtain that $F_1\overline{R'}F_2$. We have $F_1 \in a_2$, $a_2 \in B_2$, $F_2 \notin a_2$, so F_1 is not corresponding to F_2. Consequently F_2 is corresponding to F_1. Consequently $s_{F_1}\overline{\ll_{R'}}S_{F_1,F_2}(10)$. We have $a_2 \in B_2$, $F_1 \in a_2$, $F_2 \notin a_2$, so $a_2 \subseteq S_{F_1,F_2}(11)$. We have $F_1 \in a_1$, $a_1 \in B_2$, so $s_{F_1} \subseteq a_1(12)$. From (9) we get that $a_1 \subseteq a_2(13)$. From (10), (12), (13) and (11) we obtain $a_1\overline{\ll_{R'}}a_2$ - a contradiction with (9). Consequently $a_1 \ll_R a_2$.

Consequently $\underline{B_5}$ is isomorphic to $\underline{B_2}$. □

From this claim we get $(\underline{B_2}, v_2) \models \alpha$ iff $(\underline{B_5}, v_2) \models \alpha$. $\underline{B_5}$ is a substructure of $\underline{B_4}$, α is quantifier-free, so $(\underline{B_5}, v_2) \models \alpha$ iff $(\underline{B_4}, v_2) \models \alpha$.

Claim 3.4. *$\underline{B_4}$ satisfies (Con C).*

Proof. It suffices to prove that for every non-empty and different from W' $a \in B_3$, there are $F_1 \in a$ and $F_2 \notin a$ such that F_1RF_2. Let $a \in B_3$, $a \neq \emptyset$ and $a \neq W'$. We have $a = (s_{A_1} - b_1) \cup \ldots \cup (s_{A_k} - b_k)$, where $k > 0$; $s_{A_1} - b_1, \ldots, s_{A_k} - b_k$ are special sets, determined respectively by $(s_{A_1}, b_1), \ldots, (s_{A_k}, b_k)$.

Case 1: $(\exists i \in \{1,\dots,k\})(\exists T \in b_i - a)(\exists U \in s_{A_i} - b_i)(U$ is corresponding to $T)$
We have $U \in a$, $T \notin a$ and URT.
Case 2: $(\forall i \in \{1,\dots,k\})(\forall T \in b_i - a)(\forall U \in s_{A_i} - b_i)(U$ is not corresponding to $T)$
We will prove that for every $i \in \{1,\dots,k\}$ there is a $c_i \in B_1$ such that $s_{A_i} - b_i \subseteq c_i \subseteq a$. Let $i \in \{1,\dots,k\}$.
Case 2.1: $b_i \subseteq a$
$s_{A_i} - b_i \subseteq s_{A_i} \subseteq a$. We have $s_{A_i} \in B_2 \subseteq B_1$, i.e. $s_{A_i} \in B_1$.
Case 2.2: $b_i \nsubseteq a$
The idea of finding c_i is shortly the following: Let $T \in b_i - a$. For T we divide the points from $s_{A_i} - b_i$ into two kinds:
1 kind) all U such that $(\forall b \in B_2)(U \in b \to T \in b)$
2 kind) all U such that $(\exists b \in B_2)(U \in b,\ T \notin b)$
We will prove that there is an element of B_2 t_T such that $s_T \ll_{R'} t_T$ and every point of the first kind is not in t_T. Since B_2 is finite, we can obtain finitely many such pairs (s_T, t_T). For every pair (s_T, t_T), using $s_T \ll_{R'} t_T$, we get that there is a q_r such that q_r does not intersect s_T and q_r contains all points of the first kind. Thus every point T from $b_i - a$ which determines the pair in question (s_T, t_T), is not in q_r, the points for T of the first kind are in q_r. We will find a set q'_r such that $s_{A_i} - b_i \subseteq q_r \cup q'_r$, every point T which determines the pair (s_T, t_T), is not in q'_r. Thus for every pair (s_T, t_T) we get a set $q_r \cup q'_r$ which includes $s_{A_i} - b_i$ and does not contain any point T, determining the pair (s_T, t_T). We consider the intersection q of all sets of the kind $q_r \cup q'_r$. We have $s_{A_i} - b_i \subseteq q$. Every point T from $b_i - a$ is not in some $q_r \cup q'_r$ and therefore is not in q. As a c_i we can take $q \cap s_{A_i}$.

Now we will give the proof in details. Let $T \in b_i - a$. We consider arbitrary $U \in s_{A_i} - b_i$ such that $(\forall b \in B_2)(U \in b \to T \in b)$. U is not corresponding to T. Consequently $s_T \ll_{R'} S_{T,U}$. We have $b_i \subseteq S_{T,U}$. Let $P_T \stackrel{def}{=} \{S_{T,U} :\ U \in s_{A_i} - b_i$ and $(\forall b \in B_2)(U \in b \to T \in b)\}$. B_2 is finite and therefore P_T is finite and let $P_T = \{t_1,\dots,t_l\}$, where $l > 0$. Let $t_T \stackrel{def}{=} t_1 \cap \ldots \cap t_l$. We have $t_T \in B_2$. We have $\forall U$(If $U \in s_{A_i} - b_i$ and $(\forall b \in B_2)(U \in b \to T \in b)$, then $U \notin t_T$)(14); $b_i \subseteq t_T$(15). For every $j \in \{1,\dots,l\}$ $s_T \ll_{R'} t_j$, so $s_T \ll_{R'} t_T$(16).

Let $Q \stackrel{def}{=} \{(s_T, t_T) :\ T \in b_i - a\}$. Since B_2 is finite, we have that Q is finite and let $Q = \{(p_{11}, p_{12}),\dots,(p_{m1}, p_{m2})\}$, where $m > 0$.

Let $r \in \{1,\dots,m\}$. We consider (p_{r1}, p_{r2}). Using (16), we get $p_{r1} \ll_{R'} p_{r2}$. We also have p_{r1}, $p_{r2} \in B_2 \subseteq B_1$; $\underline{B_1}$ satisfies (U-rich $\ll$); so there is a $q_r \in B_1$ such that $p_{r2} \cup q_r = W'$, $q_r \overline{C_{R'}} p_{r1}$. Consequently $q_r \cap p_{r1} = \emptyset$. Let $V_r = \{T \in b_i - a :\ (s_T, t_T) = (p_{r1}, p_{r2})\}$. We have:
(17) If $T \in V_r$, then $T \in p_{r1}$ and $T \notin q_r$.
Using (14) and $p_{r2} \cup q_r = W'$, we obtain that:
(18) If $T \in V_r$, then
$\forall U$(If $U \in s_{A_i} - b_i$ and $(\forall b \in B_2)(U \in b \to T \in b)$, then $U \notin p_{r2}$ and $U \in q_r$).
Let $q'_r \stackrel{def}{=} \bigcup\{s_U :\ U \in s_{A_i} - b_i, (\forall T \in V_r)(\exists b \in B_2)(U \in b$ and $T \notin b)\}$. We will prove that $s_{A_i} - b_i \subseteq q_r \cup q'_r$. Let $U \in s_{A_i} - b_i$.
Case 1: $(\exists T \in V_r)(\forall b \in B_2)(U \in b \to T \in b)$
Using (18), we get $U \in q_r$.
Case 2: $(\forall T \in V_r)(\exists b \in B_2)(U \in b$ and $T \notin b)$. From the definition of q'_r we obtain that $s_U \subseteq q'_r$. $U \in s_U$, so $U \in q'_r$.

We proved that:
(19) $s_{A_i} - b_i \subseteq q_r \cup q'_r$.

We will prove that: if $T \in V_r$, then $T \notin q'_r$. Let $T \in V_r$. Suppose for the sake of contradiction that $T \in q'_r$. Consequently $T \in s_U$ for some U such that $U \in s_{A_i} - b_i$, $(\forall T \in V_r)(\exists b \in B_2)(U \in b$ and $T \notin b)$. Consequently $(\exists b \in B_2)(U \in b$ and $T \notin b)$ and hence $T \notin s_U$ (we have $s_U \subseteq b$) - a contradiction with $T \in s_U$. Consequently $T \notin q'_r$. We proved that:
(20) if $T \in V_r$, then $T \notin q'_r$.
From (17) and (20) we get that:
(21) if $T \in V_r$, then $T \notin q_r \cup q'_r$.

Let $q \stackrel{def}{=} (q_1 \cup q'_1) \cap \ldots \cap (q_m \cup q'_m)$. For every point T of $b_i - a$, there is a $r \in \{1, \ldots, m\}$ such that $(s_T, t_T) = (p_{r1}, p_{r2})$. We have $T \in V_r$ and by (21), we obtain $T \notin q_r \cup q'_r$. We proved that for every point T of $b_i - a$, there is a $r \in \{1, \ldots, m\}$ such that $T \notin q_r \cup q'_r$. Consequently
(22) $(\forall T \in b_i - a)(T \notin q)$
We have proved ((19)) that
(23) $(\forall r \in \{1, \ldots, m\})(s_{A_i} - b_i \subseteq q_r \cup q'_r)$
Consequently $s_{A_i} - b_i \subseteq q$ (24)
We have that for every $r \in \{1, \ldots, m\}$: $q_r \in B_1$, $q'_r \in B_2 \subseteq B_1$, so $q \in B_1$ (25).
Let $c_i \stackrel{def}{=} q \cap s_{A_i}$. From here and (25) we obtain $c_i \in B_1$ (26). From (22) we get:
(27) $(\forall T \in b_i - a)(T \notin c_i)$
From (24) we get:
(28) $s_{A_i} - b_i \subseteq c_i$
We will prove that $c_i \subseteq a$. Let $F \in c_i$. Consequently $F \in s_{A_i}$.
Case 1: $F \in s_{A_i} - b_i$
We have $F \in a$.
Case 2: $F \in b_i$
Suppose for the sake of contradiction that $F \notin a$. Consequently $F \in b_i - a$. From (27) we obtain $F \notin c_i$ - a contradiction. Consequently $F \in a$.
We proved that $c_i \subseteq a$ (29)
From (26), (28) and (29) we get that there is a $c_i \in B_1$ such that $s_{A_i} - b_i \subseteq c_i \subseteq a$.

We proved that for every $i \in \{1, \ldots, k\}$, there is a $c_i \in B_1$ such that $s_{A_i} - b_i \subseteq c_i \subseteq a$. Consequently $a = (s_{A_1} - b_1) \cup \ldots \cup (s_{A_k} - b_k) \subseteq c_1 \cup \ldots \cup c_k \subseteq a$. Consequently $a = c_1 \cup \ldots \cup c_k$. Consequently $a \in B_1$. We have $a \neq \emptyset$, $a \neq W'$, $\underline{B_1}$ is an EDCL, satisfying (Con C) and (U-rich $\ll$), so by lemma 3.2, we get that $\overline{a \ll_{R'} a}$. Consequently there are $F_1 \in a$, $F_2 \notin a$ such that $F_1 R' F_2$. Consequently $F_1 R F_2$.

We proved that for every non-empty and different from W' $a \in B_3$, there are $F_1 \in a$ and $F_2 \notin a$ such that $F_1 R F_2$. Thus we proved that $\underline{B_4}$ satisfies (Con C). □

Thus $\underline{B_4}$ is a finite connected Boolean contact algebra and v_2 is a valuation in it; $(\underline{B_4}, v_2) \vDash \alpha$ iff $(\underline{B}, v) \vDash \alpha$; the number of the elements of $\underline{B_4}$ is $\leq 2^{\frac{(2^{2^n-1}+1)2^{2^n-1}}{2}}$, where n is the number of the variables of α. □

PROPOSITION 3.5. *The rule (R U-rich $\widehat{C}$) is admissible in* $L_{ConC,U-rich\ll}$.

PROOF. It suffices to show that every theorem of $L_{ConC,U-rich\ll,U-rich\widehat{C}}$ is a theorem of $L_{ConC,U-rich\ll}$. Let α be a theorem of $L_{ConC,U-rich\ll,U-rich\widehat{C}}$ (1).

We will prove that α is a theorem of $L_{ConC,U-rich\ll}$. It suffices to prove that α is true in all EDCL, satisfying (Con C) and (U-rich $\ll$). Let $\underline{B}$ be an EDCL, satisfying (Con C) and (U-rich $\ll$) and v be a valuation in it. We will prove that $(\underline{B}, v) \models \alpha$. By lemma 3.3, we get that there is a finite connected Boolean contact algebra $\underline{B^*}$ and a valuation in it v^* such that $(\underline{B^*}, v^*) \models \alpha$ iff $(\underline{B}, v) \models \alpha$. $\underline{B^*}$ is a Boolean contact algebra and therefore satisfies (U-rich $\ll$) and (U-rich $\widehat{C}$). Using this fact, the connectedness of $\underline{B^*}$ and (1), we have $(\underline{B^*}, v^*) \models \alpha$. Consequently $(\underline{B}, v) \models \alpha$. □

PROPOSITION 3.6. *The rule (R Ext $\widehat{O}$) is admissible in the logic* $L_{ConC,U-rich\ll,U-rich\widehat{C}}$.

PROOF. The proof is similar to the proof of proposition 3.5. Here we use that in all Boolean contact algebras are true (U-rich $\ll$), (U-rich $\widehat{C}$) and (Ext $\widehat{O}$). □

PROPOSITION 3.7. $L_{ConC,U-rich\ll}$ *is decidable.*

PROOF. It suffices to prove that the following are equivalent for every formula α in $\mathcal{L}$:
(i) α is a theorem of $L_{ConC,U-rich\ll}$;
(ii) α is true in all finite EDCL, satisfying (Con C) and (U-rich $\ll$) with number of the elements $\leq 2^{\frac{(2^{2^n-1}+1)2^{2^n-1}}{2}}$, where n is the number of the variables of α.

Let α be a formula in $\mathcal{L}$. Obviously (i) implies (ii). Let (ii) be true. We will prove (i). Let $\underline{B}$ be an EDCL, satisfying (Con C), (U-rich $\ll$) and v be a valuation in it. It suffices to prove that $(\underline{B}, v) \models \alpha$. By lemma 3.3, we get that there is a finite connected Boolean contact algebra $\underline{B^*}$ and a valuation in it v^* such that $(\underline{B^*}, v^*) \models \alpha$ iff $(\underline{B}, v) \models \alpha$. The number of the elements of $\underline{B^*}$ is $\leq 2^{\frac{(2^{2^n-1}+1)2^{2^n-1}}{2}}$, where n is the number of the variables of α. We have $(\underline{B^*}, v^*) \models \alpha$. Consequently $(\underline{B}, v) \models \alpha$. □

3.6. The main theorem.

COROLLARY 3.2. *(i) The logics* L, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C}}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},Nor1}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,Nor1}$ *have the same theorems and are decidable;*
(ii) The logics $L_{ConC,U-rich\ll}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ConC}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ConC,Nor1}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,ConC}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,ConC,Nor1}$ *have the same theorems and are decidable.*

PROOF. (i) follows from proposition 3.3, proposition 3.2, proposition 2.2 and corollary 3.1.
(ii) follows from proposition 3.3, proposition 3.2, proposition 3.6, proposition 3.5, proposition 3.7. □

In a dual way we can obtain logics for O-rich EDC-lattices.

4. A quantifier-free logic for extended contact algebras

We consider a quantifier-free first-order language $\mathcal{L}'$ with equality which has:
- constants: 0, 1
- functional symbols: $+$, $\cdot$, $*$

• predicate symbols: $\leq$, $\vdash$, C, c^o

Let $\bot \stackrel{def}{=} (0 \leq 0) \wedge \neg(0 \leq 0)$, $\top \stackrel{def}{=} (0 \leq 0) \vee \neg(0 \leq 0)$.
Every ECA is a structure for $\mathcal{L}'$.
We consider the logic L_{c^o} which has the following:

• axioms:
- the axioms of the classical propositional logic
- the axioms of Boolean algebra - as axiom schemes
- the axioms $(1), \ldots, (6)$ of ECA - as axiom schemes
- the axiom scheme:
(Ax C) $pCq \leftrightarrow p, q \not\vdash 0$
- the axiom scheme:
(Ax c^o) $c^o(p) \wedge q \neq 0 \wedge r \neq 0 \wedge p = q + r \rightarrow q, r \not\vdash p^*$

• rules:
- MP
- (Rule c^o) $\frac{\alpha \rightarrow (p \neq 0 \wedge q \neq 0 \wedge a = p + q \rightarrow p, q \not\vdash a^*) \text{ for all variables } p,\ q}{\alpha \rightarrow c^o(a)}$, where α is a formula, a is a term.

Let α be a formula, a be a term. The corresponding to (Rule c^o) axiom is:
$(*)$ $\neg c^o(a) \rightarrow \exists x_1 \exists x_2 \neg(x_1 \neq 0 \wedge x_2 \neq 0 \wedge a = x_1 + x_2 \rightarrow x_1, x_2 \not\vdash a^*)$

We also consider the logic L_{Axc^o} which is obtained from L_{c^o} by removing the rule (Rule c^o).

THEOREM 4.1 (Completeness theorem). *For every formula α in $\mathcal{L}'$ the following conditions are equivalent:*
(i) α is a theorem of L_{c^o};
(ii) α is true in all ECA;
(iii) α is true in all ECA over a compact, T_0, semiregular topological space.

PROOF. (i)$\leftrightarrow$(ii) is obtained by the completeness theorem in section Preliminaries. Obviously (ii) implies (iii). Using the representation theorem of ECA (theorem 4.1 in chapter 2), we get (iii)$\rightarrow$(ii). $\square$

LEMMA 4.1. *For every formula α in $\mathcal{L}'$ the following conditions are equivalent:*
(i) α is a theorem of L_{Axc^o};
(ii) α is true in all Boolean algebras in which the predicates $\vdash$, C and c^o are defined in such a way that the axioms $(1), \ldots, (6)$ of ECA and the axioms (Ax C), (Ax c^o) are true.

PROOF. We use the completeness theorem in section Preliminaries. $\square$

PROPOSITION 4.1. *The rule (Rule c^o) is not admissible in L_{Axc^o}.*

PROOF. L_{Axc^o} is complete in the class of all Boolean algebras in which the predicates $\vdash$, C and c^o are defined in such a way that the axioms $(1), \ldots, (6)$ of ECA and the axioms (Ax C), (Ax c^o) are true. L_{c^o} is complete in the class of all ECA. Consequently it suffices to find a formula in $\mathcal{L}'$ which is true in all ECA, but is not true in all Boolean algebras in which the predicates $\vdash$, C and c^o are defined in such a way that the axioms $(1), \ldots, (6)$ of ECA and the axioms (Ax C), (Ax c^o) are true. We consider the formula α: $\neg c^o(p) \rightarrow p \neq 0$. Let $\underline{B}$ be an ECA and v be a valuation in it. We will prove that $(\underline{B}, v) \models \alpha$. Let $v(p) = a$. We will prove that if $\neg c^o(a)$, then $a \neq 0$. Suppose $\neg c^o(a)$. $\underline{B}$ is an ECA and therefore there are b and c such that $b \neq 0$, $c \neq 0$ and $a = b + c$. Consequently $a \neq 0$.

We consider the Boolean algebra $\underline{B_1} = (\{0,1\}, \leq, 0, 1, \cdot, +, *)$. We define in $\underline{B_1}$ the predicate $\vdash$ in the following way: for arbitrary a, b, $c \in \{0,1\}$, $a, b \vdash c \overset{def}{\leftrightarrow} a = 0$ or $b = 0$ or $a = b = c = 1$. It can be easily verified that $\underline{B_1}$ satisfies the axioms $(1), \ldots, (6)$ of ECA. We define C in $\underline{B_1}$ in the following way: for arbitrary a, $b \in \{0,1\}$, $aCb \leftrightarrow a, b \nvdash 0$. We define c^o in $\underline{B_1}$ in the following way: $\neg c^o(0)$ and $\neg c^o(1)$. Obviously $\underline{B_1}$ satisfies (Ax c^o). We define a valuation v_1 in $\underline{B_1}$ in the following way: $v_1(p) = 0$ for every variable p. We have $\neg c^o(0)$ and $0 = 0$ and consequently $\neg c^o(0) \rightarrow 0 \neq 0$ is not true, i.e. $(\underline{B_1}, v_1) \nvDash \alpha$. □

PROPOSITION 4.2. *L_{c^o} is decidable.*

PROOF. It suffices to prove that for every formula α in $\mathcal{L}'$ the following conditions are equivalent:
(i) α is true in all ECA;
(ii) α is true in all finite ECA with number of the elements less or equal to $2^{(2^{3 \cdot h_n})}$, where $h_n = 2^{(2^n)}$ and n is the number of the variables of α.
Let α be a formula for $\mathcal{L}'$. Obviously (i) implies (ii). Let (ii) be true. We will prove (i). Let $\underline{B}$ be an ECA and v be a valuation in it. We will prove that $(\underline{B}, v) \vDash \alpha$. Let the variables of α be $p_1, \ldots, p_n$, where $n \geq 0$. $v(p_1), \ldots, v(p_n)$ generate a finite substructure $\underline{B_1} = (B, \leq, 0, 1, \cdot, +, *, \vdash, C, c^o)$ of $\underline{B}$ which is a Boolean algebra with number of the elements less or equal to $2^{(2^n)}$. $\underline{B_1}$ satisfies $(1), \ldots, (6)$, (Ax c^o) because they are quantifier-free, true in $\underline{B}$ and $\underline{B_1}$ is a substructure of $\underline{B}$. We define a valuation v_1 in $\underline{B_1}$ in the following way:

$$v_1(p) = \begin{cases} v(p) & \text{if } p = p_1 \text{ or } p = p_2 \text{ or } \ldots \text{ or } p = p_n \\ 0 & \text{otherwise} \end{cases}$$

We have $(\underline{B}, v) \vDash \alpha$ iff $(\underline{B_1}, v_1) \vDash \alpha$. Let $a_1, \ldots, a_k$ be the elements of B_1 for which $\neg c^o$. For every $i = 1, \ldots, k$ we have $\neg c^o(a_i)$ in $\underline{B}$ and $\underline{B}$ is an ECA; so $\exists b_i, c_i \in B$ such that $b_i \neq 0$, $c_i \neq 0$, $a_i = b_i + c_i$, $b_i, c_i \vdash a_i^*$. Let $C = B_1 \cup \{b_i : i \in \{1, \ldots, k\}\} \cup \{c_i : i \in \{1, \ldots, k\}\}$. We have $|C| \leq 3.h_n$. C generates a finite Boolean subalgebra of $\underline{B}$ $\underline{B_2}$ with a number of the elements less or equal to $2^{2^{3.h_n}}$. We define in $\underline{B_2}$ $\vdash$ as a restriction of $\vdash$ in $\underline{B}$. Consequently $\underline{B_2}$ satisfies $(1), \ldots, (6)$. We define c^o in $\underline{B_2}$ in the following way: for every $a \in B_2$, $c^o(a) \overset{def}{\leftrightarrow} c^o(a)$ in $\underline{B}$ or $(\forall b, c \in B_2)(b \neq 0,\ c \neq 0,\ a = b + c \rightarrow b, c \nvdash a^*)$. We define C in $\underline{B_2}$ in the following way: $aCb \leftrightarrow a, b \nvdash 0$ for arbitrary a, $b \in B_2$.

We will prove that $\underline{B_1}$ is a substructure of $\underline{B_2}$. It suffices to prove that for every $a \in B_1$ we have: $c^o(a)$ in $\underline{B_1} \leftrightarrow c^o(a)$ in $\underline{B_2}$.
$\rightarrow$) Let $c^o(a)$ in $\underline{B_1}$. Consequently $c^o(a)$ in $\underline{B}$ because $\underline{B_1}$ is a substructure of $\underline{B}$. Thus $c^o(a)$ in $\underline{B_2}$.
$\leftarrow$) Let $c^o(a)$ in $\underline{B_2}$.
Case 1: $c^o(a)$ in $\underline{B}$
Consequently $c^o(a)$ in $\underline{B_1}$ because $\underline{B_1}$ is a substructure of $\underline{B}$.
Case 2: $\neg c^o(a)$ in $\underline{B}$
Since $c^o(a)$ in $\underline{B_2}$, we have $(\forall b, c \in B_2)(b \neq 0,\ c \neq 0,\ a = b + c \rightarrow b, c \nvdash a^*)$. We have $\neg c^o(a)$ in $\underline{B}$ and $a \in B_1$. Consequently $a = a_i$ for some $i \in \{1, \ldots, k\}$. Consequently $(\exists b_i, c_i \in B_2)(b_i \neq 0,\ c_i \neq 0,\ a = b_i + c_i,\ b_i, c_i \vdash a^*)$. This is a contradiction. Consequently case 2 is impossible.
Consequently $\underline{B_1}$ is a substructure of $\underline{B_2}$; so $(\underline{B_1}, v_1) \vDash \alpha \leftrightarrow (\underline{B_2}, v_1) \vDash \alpha$.

We will prove that $\underline{B_2}$ satisfies (Ax c^o). Let a, b, $c \in B_2$ and $c^o(a)$, $b \neq 0$, $c \neq 0$, $a = b + c$. We will prove that $b, c \nvdash a^*$.

Case 1: $c^o(a)$ in $\underline{B}$

We have $b \neq 0$, $c \neq 0$, $a = b + c$ in $\underline{B}$ because $\underline{B_2}$ is a Boolean subalgebra of $\underline{B}$. Using that $\underline{B}$ is an ECA and $c^o(a)$ in $\underline{B}$, we get that $b, c \nvdash a^*$ in $\underline{B}$. Consequently $b, c \nvdash a^*$ in $\underline{B_2}$.

Case 2: $\neg c^o(a)$ in $\underline{B}$

We have $(\forall b, c \in B_2)(b \neq 0,\ c \neq 0,\ a = b + c \rightarrow b, c \nvdash a^*)$. We have $b, c \nvdash a^*$ in $\underline{B_2}$. Consequently $\underline{B_2}$ satisfies (Ax c^o).

Let $a \in B_2$ and $\neg c^o(a)$ in $\underline{B_2}$. We will prove that there are b, $c \in B_2$ such that $b \neq 0$, $c \neq 0$, $a = b + c$, $b, c \vdash a^*$ in $\underline{B_2}$. From the definition of c^o in $\underline{B_2}$ we get that: $\neg c^o(a)$ in $\underline{B}$ and $(\exists b, c \in B_2)(b \neq 0,\ c \neq 0,\ a = b + c,\ b, c \vdash a^*)$.

Consequently for every $a \in B_2$ we have $c^o(a) \leftrightarrow \forall b, c(b \neq 0,\ c \neq 0,\ a = b + c \rightarrow b, c \nvdash a^*)$.

Consequently $\underline{B_2}$ is an ECA. B_2 has a number of the elements less or equal to $2^{2^{3.h_n}}$. So α is true in $\underline{B_2}$, i.e. $(\underline{B_2}, v_1) \models \alpha$. Consequently $(\underline{B}, v) \models \alpha$. □

CHAPTER 4

Conclusion

In the dissertation have been obtained the following results:
In the first part of the first chapter the language of distributive contact lattices is extended by considering as non-definable primitives the relations of contact, non-tangential inclusion and dual contact. It is obtained an axiomatization of the theory consisting of the universal formulas in the language $\mathcal{L}(0, 1; +, \cdot; \leq, C, \widehat{C}, \ll)$ true in all contact algebras. The structures in $\mathcal{L}$, satisfying the axioms in question, are called extended distributive contact lattices (EDC-lattices). A relational representation theorem is proved, stating that each EDC-lattice can be isomorphically embedded into a contact algebra. The axiomatization and the relational representation theorem have been obtained by T. Ivanova.

In part II of chapter 1 is obtained topological representation theory of EDC-lattices and some of their axiomatic extensions yielding representations in T_1 and T_2 spaces. Special attention is given to dual dense and dense representations (defined in Section 4.1) in contact algebras of regular closed and regular open subsets of topological spaces. These results are common with Prof. D. Vakarelov.

In chapter 2 is considered the predicate c^o - internal connectedness. It is proved that this predicate cannot be defined in the language of contact algebras. Because of this to the language is added a new ternary predicate symbol $\vdash$ which has the following sense: in the contact algebra of regular closed sets of some topological space $a, b \vdash c$ iff $a \cap b \subseteq c$. It turns out that the predicate c^o can be defined in the new language. It is defined *extended contact algebra* - a Boolean algebra with added relations $\vdash$, C and c^o, satisfying some axioms, and is proved that every extended contact algebra can be isomorphically embedded in the contact algebra of the regular closed subsets of some compact, semiregular, T_0 topological space with added relations $\vdash$ and c^o. So extended contact algebra can be considered an axiomatization of the theory, consisting of the universal formulas true in all topological contact algebras with added relations $\vdash$ and c^o. The results in chapter 2 except the idea that c^o can be defined by the relation $\vdash$, have been obtained by T. Ivanova.

In chapter 3 is considered a first-order language without quantifiers corresponding to EDCL. Completeness theorems are given with respect to both algebraic and topological semantics for several logics for this language. It turns out that all these logics are decidable. It is also considered a quantifier-free first-order language corresponding to ECA and a logic for ECA which is decidable. The results in this chapter have been obtained by T. Ivanova.

Publications:

1)T. Ivanova. Extended contact algebras and internal connectedness. Proceedings

of the 10th Panhellenic Logic Symposium, 2015, Samos, Greece
http://samosweb.aegean.gr/pls10/pls10-proceedings.pdf
2)T. Ivanova and D. Vakarelov. Distributive mereotopology: extended distributive contact lattices. *Annals of Mathematics and Artificial Intelligence*, 77(1), 3-41, DOI 10.1007/s10472-016-9499-5

Presentations:

1)T. Ivanova, Extended contact algebras and internal connectedness, 10th Panhellenic Logic Symposium, 11-15 June 2015, Samos, Greece
2)T. Ivanova and D. Vakarelov, Extended distributive contact lattices, Spring scientific session of FMI, 28 March 2015, Sofia, Bulgaria
3)T. Ivanova, Extended contact algebras and internal connectedness, Mathematical logic seminar, 10 December 2015, Sofia, Bulgaria

Declaration for originality: The author declares that the dissertation is an original work. Using of past results is accompanied by suitable citations.

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