THE JUMP OPERATOR ON THE ω-ENUMERATION DEGREES

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Abstract. The jump operator on the ω-enumeration degrees is introduced in [?]. In the present paper we prove a jump inversion theorem which allows us to show that the enumeration degrees are first order definable in the structure $\mathcal{D}'_\omega$ of the ω-enumeration degrees augmented by the jump operator. Further on we show that the groups of the automorphisms of $\mathcal{D}'_\omega$ and of the enumeration degrees are isomorphic.

In the second part of the paper we study the jumps of the ω-enumeration degrees below $\mathbf{0}_\omega{}'$. We define the ideal of the almost zero degrees and obtain a natural characterization of the class H of the ω-enumeration degrees below $\mathbf{0}_\omega{}'$ which are high n for some n and of the class L of the ω-enumeration degrees below $\mathbf{0}_\omega{}'$ which are low n for some n.

1. Introduction

The upper semi-lattice $\mathcal{D}_\omega$ of ω-enumeration degrees is introduced by the first author in [?]. It is an extension of the semi-lattice $\mathcal{D}_e$ of the enumeration degrees and hence of the semi-lattice $\mathcal{D}_T$ of the Turing degrees. In [?] a jump operator on the ω-enumeration degrees is defined and a jump inversion theorem is proved also from which follows that the range of the jump operator is equal to the cone of all ω-enumeration degrees greater than the jump $\mathbf{0}_\omega{}'$ of the least ω-enumeration degree $\mathbf{0}_\omega$, a property true for the Turing jump but not true for the enumeration jump.

It turns out that the jump on the ω-enumeration degrees has an even stronger jump inversion property. Namely, for every ω-enumeration degree $\mathbf{a}$ above $\mathbf{0}_\omega{}'$ there exists a least degree among the degrees whose jump is equal to $\mathbf{a}$. This property is not true neither for the enumeration jump nor for the Turing jump.

Using the existence of least jump inverts we show in the first part of the paper that the set of the enumeration degrees is first order definable in the structure $\mathcal{D}'_\omega$ of the ω-enumeration degrees augmented by the jump operator. This definability result allows us to obtain further that the groups of the automorphisms of $\mathcal{D}_e{}'$ and $\mathcal{D}'_\omega$ are isomorphic. Since the enumeration jump is first order definable in $\mathcal{D}_e$, see [?], it follows that the groups of the automorphisms of $\mathcal{D}_e$ and $\mathcal{D}'_\omega$ are also isomorphic.

Thus we obtain that the structures $\mathcal{D}_e{}'$ and $\mathcal{D}'_\omega$ are closely related but $\mathcal{D}_e{}'$ and $\mathcal{D}'_\omega$ are not elementary equivalent.

In the second part of the paper we study the jumps of the ω-enumeration degrees below $\mathbf{0}_\omega{}'$. Here we consider a monotonically decreasing sequence $\{o_n\}_{n\geq 1}$ of explicitly defined degrees, where o_n is the least degree with n-th jump equal to $\mathbf{0}_\omega{}^{(n+1)}$. We call a degree $\mathbf{a}$ *almost zero* (a.z.) if for all n, $\mathbf{a}$ is below o_n. We prove that the a.z. degrees form a non-trivial ideal. The a.z. degrees are used to obtain a characterization of the classes H and L, where

$$H = \{\mathbf{a} : \mathbf{a} \leq \mathbf{0}_\omega{}' \ \&\ (\exists n)(\mathbf{a}^{(n)} = \mathbf{0}_\omega{}^{(n+1)})\} \text{ and}$$
$$L = \{\mathbf{a} : \mathbf{a} \leq \mathbf{0}_\omega{}' \ \&\ (\exists n)(\mathbf{a}^{(n)} = \mathbf{0}_\omega{}^{(n)})\}$$

Namely, we show that a degree $\mathbf{a} \leq \mathbf{0}_\omega{}'$ belongs to H if and only if $\mathbf{a}$ is above all a.z. degrees and $\mathbf{a} \in L$ if and only if there are no nonzero a.z. degrees below $\mathbf{a}$.

Since the ω-enumeration jump agrees with the enumeration jump and with the Turing jump the characterization of the classes H and L remains the same also for the enumeration and for the Turing degrees.

The last result shows that the study of the ω-enumeration degrees can provide us with tools which are useful for the study of the enumeration degrees and of the Turing degrees. A similar methodological observation about the usefulness of the study of the enumeration degrees for obtaining results about the Turing degrees was recently made by Soskova and Cooper [**?**].

2. Preliminaries

2.1. **The enumeration degrees.** We shall assume that an effective coding of all finite sets of natural numbers is fixed and shall identify the finite sets and their codes. Finite sets will be denoted by the letters D, F and S.

2.1. Definition. Given sets A and B of natural numbers, let

$$A(B) = \{x : (\exists D)(\langle x, D\rangle \in A \ \&\ D \subseteq B\}$$

Let $W_0, \dots, W_a, \dots$ be a Gödel enumeration of the recursively enumerable (r.e.) sets of natural numbers.

The operators $\lambda B.W_a(B)$ are called *enumeration operators*. For $A, B \subseteq \mathbb{N}$, $A \leq_e B$ (*A is enumeration reducible to B*) if there exists an r.e. set W such that $A = W(B)$. Let $A \equiv_e B \iff A \leq_e B \ \&\ B \leq_e A$. The relation $\equiv_e$ is an equivalence relation and the respective equivalence classes are called enumeration degrees. Given a set A of natural numbers, by $d_e(A)$ we shall denote the enumeration degree containing A. Let $d_e(A) \leq_e d_e(B)$ if $A \leq_e B$. Clearly $\leq_e$ is a partial ordering with least element $\mathbf{0}_e$ which is equal to the set of all r.e. sets. The set of all enumeration degree is denoted by $\mathcal{D}_e$. We shall use the same notation and for the structure $(\mathcal{D}_e; \mathbf{0}_e; \leq_e)$. For an introduction to the enumeration degrees the reader might consult [**?**].

For every set A of natural numbers let $A^+ = A \oplus (\mathbb{N} \setminus A)$. Then a set B is r.e. in A if and only if $B \leq_e A^+$ and A is Turing reducible to B if and only if $A^+ \leq_e B^+$.

Denote by $\mathcal{D}_T = (\mathcal{D}_T; \mathbf{0}_T; \leq_T)$ the partial ordering of the Turing degrees. Let $\iota : \mathcal{D}_T \to \mathcal{D}_e$ be defined by $\iota(d_T(A)) = d_e(A^+)$. Then ι is an isomorphic embedding of $\mathcal{D}_T$ into $\mathcal{D}_e$ called the Rogers' embedding. The enumeration degrees which belong to the range of ι are called *total*. Notice that an enumeration degree $\mathbf{a}$ is total if and only if for some $A \subseteq \mathbb{N}$, $A^+ \in \mathbf{a}$.

The enumeration jump operator is defined in [**?**] and further studied in [**?**]. Here we shall use the following definition of the enumeration jump which is m-equivalent to the original one, see [**?**].

2.2. Definition. Given a set A of natural numbers, set $L_A = \{\langle a, x\rangle : x \in W_a(A)\}$ and let the *enumeration jump* of A be the set L_A^+.

One can easily check that for every $A \subseteq \mathbb{N}$, $A \lneq_e L_A^+$ and if $A \leq_e B$ then $L_A^+ \leq_e L_B^+$. So we may define a jump operation on $\mathcal{D}_e$ by letting $d_e(A)' = d_e(L_A^+)$. Clearly the jump of every enumeration degree is a total degree. Since there exist enumeration degrees above $\mathbf{0}_e{}'$ which are not total, not every enumeration degree above $\mathbf{0}_e{}'$ is in the range of the enumeration jump operator. The enumeration jump agrees with the Turing jump under Rogers' embedding i.e.

$$(\forall \mathbf{a} \in \mathcal{D}_T)(\iota(\mathbf{a}') = \iota(\mathbf{a})').$$

To simplify the notation, given $A \subseteq \mathbb{N}$, by A' we shall denote the enumeration jump of A. Let $A^{(0)} = A$ and $A^{(n+1)} = (A^{(n)})'$.

We shall need the following Jump inversion theorem proved in [?].

Given a sequence $\mathcal{B} = \{B_k\}_{k<\omega}$ of sets of natural numbers we define the respective *jump sequence* $\mathcal{P}(\mathcal{B}) = \{\mathcal{P}_k(\mathcal{B})\}_{k<\omega}$ by induction on k:

(i) $\mathcal{P}_0(\mathcal{B}) = B_0$;
(ii) $\mathcal{P}_{k+1}(\mathcal{B}) = \mathcal{P}_k(\mathcal{B})' \oplus B_{k+1}$.

2.3. Theorem. *Let $\mathcal{B} = \{B_k\}_{k<\omega}$ be a sequence of sets of natural numbers. Suppose that for some $X \subseteq \mathbb{N}$ and for some $n \in \mathbb{N}$, $\mathcal{P}_n(\mathcal{B}) \leq_e X^+$. Then there exists $F \subseteq \mathbb{N}$ satisfying the following conditions:*

(1) $(\forall k \leq n)(B_k \leq_e (F^+)^{(k)})$
(2) $(\forall k < n)((F^+)^{(k+1)} \equiv_e (F^+) \oplus \mathcal{P}_k(\mathcal{B})')$.
(3) $(F^+)^{(n)} \equiv_e X^+$.

2.2. **The ω-enumeration degrees.** Denote by $\mathcal{S}$ the set of all sequences $\mathcal{B} = \{B_k\}_{k<\omega}$ of sets of natural numbers. Consider an element $\mathcal{B}$ of $\mathcal{S}$ and let the *jump class* $J_\mathcal{B}$ defined by $\mathcal{B}$ be the set of the Turing degrees of all $X \subseteq \mathbb{N}$ such that $(\forall k)(B_k$ is r.e. in $X^{(k)}$ uniformly in $k)$.

Given two sequences $\mathcal{A}$ and $\mathcal{B}$ let $\mathcal{A} \leq_u \mathcal{B}$ ($\mathcal{A}$ is *uniformly reducible to* $\mathcal{B}$) if $J_\mathcal{B} \subseteq J_\mathcal{A}$ and $\mathcal{A} \equiv_u \mathcal{B}$ if $J_\mathcal{B} = J_\mathcal{A}$. Clearly "$\leq_u$" is a reflexive and transitive relation on $\mathcal{S}$ and "$\equiv_u$" is an equivalence relation on $\mathcal{S}$.

For every sequence $\mathcal{B}$ let $d_\omega(\mathcal{B}) = \{\mathcal{A} : \mathcal{A} \equiv_u \mathcal{B}\}$ and let $\mathcal{D}_\omega = \{d_\omega(\mathcal{B}) : \mathcal{B} \in \mathcal{S}\}$. The elements of $\mathcal{D}_\omega$ are called the *ω-enumeration degrees.*

The ω-enumeration degrees can be ordered in the usual way. Given two elements $\mathbf{a} = d_\omega(\mathcal{A})$ and $\mathbf{b} = d_\omega(\mathcal{B})$ of $\mathcal{D}_\omega$, let $\mathbf{a} \leq_\omega \mathbf{b}$ if $\mathcal{A} \leq_u \mathcal{B}$. Clearly $\mathcal{D}_\omega = (\mathcal{D}_\omega, \leq_\omega)$ is a partial ordering with least element $\mathbf{0}_\omega = d_\omega(\emptyset_\omega)$, where all members of the sequence $\emptyset_\omega$ are equal to $\emptyset$.

Given two sequences $\mathcal{A} = \{A_k\}$ and $\mathcal{B} = \{B_k\}$ of sets of natural numbers let $\mathcal{A} \oplus \mathcal{B} = \{A_k \oplus B_k\}$. Is it easy to see that $J_{\mathcal{A}\oplus\mathcal{B}} = J_\mathcal{A} \cap J_\mathcal{B}$ and hence every two elements $\mathbf{a} = d_\omega(\mathcal{A})$ and $\mathbf{b} = d_\omega(\mathcal{B})$ of $\mathcal{D}_\omega$ have a least upper bound $\mathbf{a} \cup \mathbf{b} = d_\omega(\mathcal{A} \oplus \mathcal{B})$.

Given a set W of natural numbers and $k \in \mathbb{N}$, let $W[k] = \{u : \langle k, u\rangle \in W\}$.

2.4. Definition. For every $W \subseteq \mathbb{N}$ and every sequence $\mathcal{B} = \{B_k\}_{k<\omega}$ of sets of natural numbers, let $W(\mathcal{B}) = \{W[k](B_k)\}_{k<\omega}$.

2.5. Definition. Let $\mathcal{A} = \{A_k\}_{k<\omega}$ and $\mathcal{B} = \{B_k\}_{k<\omega}$ be elements of $\mathcal{S}$. Then $\mathcal{A} \leq_e \mathcal{B}$ (*$\mathcal{A}$ is enumeration reducible to* $\mathcal{B}$) if $\mathcal{A} = W(\mathcal{B})$ for some r.e. set W.

A simple application of the S_n^m-Theorem shows that $\mathcal{A} \leq_e \mathcal{B}$ if and only if there exists a primitive recursive function h such that $(\forall k)(A_k = W_{h(k)}(B_k))$.

Let $\mathcal{A} \equiv_e \mathcal{B}$ if $\mathcal{A} \leq_e \mathcal{B}$ and $\mathcal{B} \leq_e \mathcal{A}$.

The following facts follow easily from the definitions.

2.6. Proposition. *Let $\mathcal{A}, \mathcal{B} \in \mathcal{S}$. Then the following assertions hold:*

(1) $\mathcal{A} \leq_e \mathcal{P}(\mathcal{A})$;
(2) $\mathcal{P}(\mathcal{P}(\mathcal{A})) \leq_e \mathcal{P}(\mathcal{A})$.
(3) $\mathcal{A} \leq_e \mathcal{B} \Rightarrow \mathcal{P}(\mathcal{A}) \leq_e \mathcal{P}(\mathcal{B})$.

The following Theorem from [**?**] gives an explicit characterization of the uniform reducibility.

2.7. Theorem. *For every two sequences $\mathcal{A}$ and $\mathcal{B}$ of sets of natural numbers*

$$\mathcal{A} \leq_u \mathcal{B} \iff \mathcal{A} \leq_e \mathcal{P}(\mathcal{B}).$$

2.8. Corollary.

(1) *For all $\mathcal{A} \in \mathcal{S}$, $\mathcal{A} \equiv_u \mathcal{P}(\mathcal{A})$.*
(2) *For all $\mathcal{A}, \mathcal{B} \in \mathcal{S}$, $\mathcal{A} \leq_e \mathcal{B} \Rightarrow \mathcal{A} \leq_u \mathcal{B}$.*

There is a natural embedding of the enumeration degrees into the ω-enumeration degrees. Given a set A of natural numbers denote by $A \uparrow \omega$ the sequence $\{A_k\}_{k<\omega}$, where $A_0 = A$ and for all $k \geq 1$, $A_k = \emptyset$.

2.9. Proposition. *For every $A, B \subseteq \mathbb{N}$, $A \uparrow \omega \leq_u B \uparrow \omega \iff A \leq_e B$.*

Proof. Suppose that $A \uparrow \omega \leq_u B \uparrow \omega$. Then $J_{B\uparrow\omega} \subseteq J_{A\uparrow\omega}$ and hence for every $X \subseteq \mathbb{N}$, B is r.e. in X implies A is r.e. in X. By the Selman's Theorem [**?**], $A \leq_e B$.

The implication $A \leq_e B \Rightarrow J_{B\uparrow\omega} \subseteq J_{A\uparrow\omega}$ is obvious. □

Let $\mathcal{D}_1 = \{d_\omega(A \uparrow \omega) : A \subseteq \mathbb{N}\}$ and $\mathcal{D}_1 = (\mathcal{D}_1; \mathbf{0}_\omega; \leq_\omega \restriction \mathcal{D}_1)$.

Define the mapping $\kappa : \mathcal{D}_e \to \mathcal{D}_1$ by $\kappa(d_e(A)) = d_\omega(A \uparrow \omega)$. Then κ is an isomorphism from $\mathcal{D}_e$ to $\mathcal{D}_1$ and hence κ is an embedding of $\mathcal{D}_e$ into $\mathcal{D}_\omega$.

Recall the Rogers' embedding ι of the Turing degrees into the enumeration degrees defined by $\iota(d_T(X)) = d_e(X^+)$ and let $\lambda : \mathcal{D}_T \to \mathcal{D}_\omega$ be defined by $\lambda(\mathbf{x}) = \kappa(\iota(\mathbf{x}))$. Clearly λ is an isomorphic embedding of $\mathcal{D}_T$ into $\mathcal{D}_\omega$.

2.10. Proposition. *Let $\mathcal{A} \in \mathcal{S}$. Then $J_\mathcal{A} = \{\mathbf{x} : \mathbf{x} \in \mathcal{D}_T \ \&\ d_\omega(\mathcal{A}) \leq_\omega \lambda(\mathbf{x})\}$.*

Proof. Let $\mathbf{x} \in J_\mathcal{A}$. Fix an element X of $\mathbf{x}$. Then for all k, $A_k \leq_e (X^+)^{(k)}$ uniformly in k. Clearly $\mathcal{P}(X^+ \uparrow \omega) \equiv_e \{(X^+)^{(k)}\}_{k<\omega}$. From here it follows that $\mathcal{A} \leq_u X^+ \uparrow \omega$ and hence $d_\omega(\mathcal{A}) \leq_\omega \lambda(\mathbf{x})$.

Let $d_\omega(\mathcal{A}) \leq_\omega \lambda(\mathbf{x})$. Consider a $X \in \mathbf{x}$. Then $\mathcal{A} \leq_e \mathcal{P}(X^+ \uparrow \omega)$ and hence for all k, $A_k \leq_e (X^+)^{(k)}$ uniformly in k. So, $\mathbf{x} \in J_\mathcal{A}$. □

2.11. Corollary. *Let $\mathbf{a}, \mathbf{b} \in \mathcal{D}_\omega$. Then*

$$\mathbf{a} \leq_\omega \mathbf{b} \iff (\forall \mathbf{x} \in \mathcal{D}_T)(\mathbf{b} \leq_\omega \lambda(\mathbf{x}) \Rightarrow \mathbf{a} \leq_\omega \lambda(\mathbf{x})).$$

For every $\mathcal{A} \in \mathcal{S}$ set $J^e_\mathcal{A} = \{\mathbf{x} : \mathbf{x} \in \mathcal{D}_e \ \&\ d_\omega(\mathcal{A}) \leq_\omega \kappa(\mathbf{x})\}$.

Clearly $J_\mathcal{A} = \{\mathbf{x} : \mathbf{x} \in \mathcal{D}_T \ \&\ \iota(\mathbf{x}) \in J^e_\mathcal{A}\}$. Hence for every two sequences $\mathcal{A}$ and $\mathcal{B}$ we have that

$$\mathcal{A} \leq_u \mathcal{B} \iff J^e_\mathcal{B} \subseteq J^e_\mathcal{A}.$$

2.12. Corollary. *Let $\mathbf{a}, \mathbf{b} \in \mathcal{D}_\omega$. Then*

$$\mathbf{a} \leq_\omega \mathbf{b} \iff (\forall \mathbf{x} \in \mathcal{D}_e)(\mathbf{b} \leq_\omega \kappa(\mathbf{x}) \Rightarrow \mathbf{a} \leq_\omega \kappa(\mathbf{x})).$$

2.13. Proposition. *$\mathcal{D}_1$ is a base of the automorphisms of $\mathcal{D}_\omega$.*

Proof. Suppose that φ is an automorphism of $\mathcal{D}_\omega$ and $\varphi(\mathbf{y}) = \mathbf{y}$ for $\mathbf{y} \in \mathcal{D}_1$. Consider an element $\mathbf{a} \in \mathcal{D}_\omega$. Then for all $\mathbf{x} \in \mathcal{D}_e$,

$$\mathbf{a} \leq_\omega \kappa(\mathbf{x}) \iff \varphi(\mathbf{a}) \leq_\omega \varphi(\kappa(\mathbf{x})) \iff \varphi(\mathbf{a}) \leq_\omega \kappa(\mathbf{x}).$$

Hence $\mathbf{a} = \varphi(\mathbf{a})$. □

3. The jump operator

In this section we shall give the definition of the jump operator on the ω-enumeration degrees and study it's properties.

3.1. Definition. For every $\mathcal{A} \in \mathcal{S}$ let $\mathcal{A}' = \{\mathcal{P}_{k+1}(\mathcal{A})\}_{k<\omega}$.

3.2. Proposition. *Let $\mathcal{A} = \{A_k\}_{k<\omega} \in \mathcal{S}$. Then $J_{\mathcal{A}'} = \{\mathbf{a}' : \mathbf{a} \in J_\mathcal{A}\}$.*

Proof. Let $\mathbf{a} \in J_\mathcal{A}$. Since $\mathcal{P}(\mathcal{A}) \equiv_u \mathcal{A}$, $\mathbf{a} \in J_{\mathcal{P}(\mathcal{A})}$ and hence for some $X \in \mathbf{a}$ we have that for all k, $\mathcal{P}_k(\mathcal{A}) \leq_e (X^+)^{(k)}$ uniformly in k. From here it follows that for all k, $\mathcal{P}_{k+1}(\mathcal{A}) \leq_e ((X^+)')^{(k)}$ uniformly in k. Thus $\mathbf{a}' \in J_{\mathcal{A}'}$.

Suppose now that $\mathbf{b} \in J_{\mathcal{A}'}$. Then for some $X \in \mathbf{b}$ and for all k, $\mathcal{P}_{k+1}(\mathcal{A}) \leq_e (X^+)^{(k)}$ uniformly in k. In particular $\mathcal{P}_1(\mathcal{A}) \leq_e X^+$. By Theorem 2.3 there exists $F \subseteq \mathbb{N}$ such that $A_0 \leq_e F^+$ and $(F^+)' \equiv_e X^+$. Let $\mathbf{a} = d_T(F)$. Then $\mathbf{a} \in J_\mathcal{A}$ and $\mathbf{a}' = \mathbf{b}$. □

3.3. Proposition. *Let $\mathcal{A}, \mathcal{B} \in \mathcal{S}$. Then the following assertions are true:*

(J0) $\mathcal{A} \lneq_u \mathcal{A}'$
(J1) $\mathcal{A} \leq_u \mathcal{B} \Rightarrow \mathcal{A}' \leq_u \mathcal{B}'$

Proof. Clearly $\mathcal{A} \leq_e \mathcal{P}(\mathcal{A}) \leq_e \mathcal{A}'$. Hence $\mathcal{A} \leq_u \mathcal{A}'$. Assume that $\mathcal{A}' \leq_u \mathcal{A}$. Then $\mathcal{A}' \leq_e \mathcal{P}(\mathcal{A})$ and hence $\mathcal{P}_1(\mathcal{A}) = \mathcal{P}_0(\mathcal{A})' \oplus A_1 \leq_e \mathcal{P}_0(\mathcal{A})$. By the properties of the enumeration jump the last is not possible.

The condition (J1) follows by Proposition 3.2. □

From (J1) it follows that $\mathcal{A} \equiv_u \mathcal{B} \Rightarrow \mathcal{A}' \equiv_u \mathcal{B}'$. So we may define a jump operation on the ω-enumeration degrees by $d_\omega(\mathcal{A})' = d_\omega(\mathcal{A}')$.

From Proposition 3.2 we get immediately the following characterization of the jump:

3.4. Proposition. *Let $\mathbf{a}, \mathbf{b} \in \mathcal{D}_\omega$. Then*

$$\mathbf{a} \leq_\omega \mathbf{b}' \iff (\forall \mathbf{x} \in \mathcal{D}_T)(\mathbf{b} \leq_\omega \lambda(\mathbf{x}) \Rightarrow \mathbf{a} \leq_\omega \lambda(\mathbf{x}')).$$

Proof. Let $\mathcal{A} \in \mathbf{a}$ and $\mathcal{B} \in \mathbf{b}$. Then

$$\mathbf{a} \leq_\omega \mathbf{b}' \iff \mathcal{A} \leq_u \mathcal{B}' \iff J_{\mathcal{B}'} \subseteq J_\mathcal{A} \iff \{\mathbf{x}' : \mathbf{x} \in J_\mathcal{B}\} \subseteq J_\mathcal{A}.$$

□

Next we show that the jump on the ω-enumeration degrees agrees with the enumeration jump and hence with the Turing jump.

3.5. Proposition. *Let $\mathbf{x} \in \mathcal{D}_e$. Then $\kappa(\mathbf{x}') = \kappa(\mathbf{x})'$.*

Proof. Let $\mathbf{x} \in \mathcal{D}_e$ and $X \in \mathbf{x}$. Clearly

$$\mathcal{P}(X' \uparrow \omega) \equiv_e \{X^{(k+1)}\}_{k<\omega} \equiv_e \{\mathcal{P}_{1+k}(X \uparrow \omega)\}_{k<\omega} = (X \uparrow \omega)'$$

Hence $\kappa(\mathbf{x}') = \kappa(\mathbf{x})'$. □

Using the agreement of the enumeration jump with the Turing jump under the Rogers' embedding we obtain and the following:

3.6. Corollary. *For every* $\mathbf{x} \in \mathcal{D}_T$, $\lambda(\mathbf{x}') = \lambda(\mathbf{x})'$

Combining Proposition 3.5 and Proposition 3.4 we get also

3.7. Proposition. *For any two ω-enumeration degrees* $\mathbf{a}$ *and* $\mathbf{b}$,

$$\mathbf{a} \leq_\omega \mathbf{b}' \iff (\forall \mathbf{x} \in \mathcal{D}_e)(\mathbf{b} \leq_\omega \kappa(\mathbf{x}) \Rightarrow \mathbf{a} \leq_\omega \kappa(\mathbf{x}')).$$

Proof. Let $\mathbf{a} \leq_\omega \mathbf{b}'$. Consider a $\mathbf{x} \in \mathcal{D}_e$ and suppose that $\mathbf{b} \leq_\omega \kappa(\mathbf{x})$. Then $\mathbf{b}' \leq_\omega \kappa(\mathbf{x})' = \kappa(\mathbf{x}')$. Hence $\mathbf{a} \leq_\omega \mathbf{b}' \leq_\omega \kappa(\mathbf{x}')$.

Suppose now that for all $\mathbf{x} \in \mathcal{D}_e$, $\mathbf{b} \leq_\omega \kappa(\mathbf{x})$ implies $\mathbf{a} \leq_\omega \kappa(\mathbf{x}')$. Then for all $\mathbf{x} \in \mathcal{D}_T$, $\mathbf{b} \leq_\omega \lambda(\mathbf{x})$ implies $\mathbf{a} \leq_\omega \lambda(\mathbf{x}')$. Hence $\mathbf{a} \leq_\omega \mathbf{b}'$. □

Given $n \geq 0$, set $\mathcal{A}^{(n)} = \{\mathcal{P}_{n+k}(\mathcal{A})\}_{k<\omega}$. One can easily check that $\mathcal{A}^{(0)} \equiv_e \mathcal{P}(\mathcal{A})$ and for all $n \geq 0$, $\mathcal{A}^{(n+1)} \equiv_e (\mathcal{A}^{(n)})'$.

For every ω-enumeration degree $\mathbf{a} = d_\omega(\mathcal{A})$, let $\mathbf{a}^{(n)} = d_\omega(\mathcal{A}^{(n)})$. Then $\mathbf{a}^{(0)} = \mathbf{a}$ and for all n, $\mathbf{a}^{(n+1)} = (\mathbf{a}^{(n)})'$.

Next we turn to the jump inversion problem.

Let us fix a sequence $\mathcal{A} = \{A_k\}_{k<\omega}$ of sets of natural numbers.

3.8. Definition. Let $\mathcal{B} \in \mathcal{S}$ and $n \geq 1$. Then set $I^n_{\mathcal{A}}(\mathcal{B}) = \{C_k\}_{k<\omega}$, where $(\forall k < n)(C_k = A_k)$ and $(\forall k \geq n)(C_k = \mathcal{P}_{n-k}(\mathcal{B}))$.

3.9. Proposition. *Let $\mathcal{A}^{(n)} \leq_u \mathcal{B}$. Then the following assertions hold:*

(1) $\mathcal{A} \leq_u I^n_{\mathcal{A}}(\mathcal{B})$.
(2) $I^n_{\mathcal{A}}(\mathcal{B})^{(n)} \equiv_u \mathcal{B}$.
(3) *If $\mathcal{A} \leq_u \mathcal{C}$ and $\mathcal{B} \leq_u \mathcal{C}^{(n)}$ then $I^n_{\mathcal{A}}(\mathcal{B}) \leq_u \mathcal{C}$.*

Proof. The assertions (1) and (2) follow directly from the definitions. To prove (3) suppose that $\mathcal{A} \leq_u \mathcal{C}$ and $\mathcal{B} \leq_u \mathcal{C}^{(n)}$. Then for all k, $\mathcal{P}_k(\mathcal{B}) \leq_e \mathcal{P}_{n+k}(\mathcal{C})$ uniformly in k. Since $\mathcal{A} \leq_u \mathcal{C}$, for all $k < n$, $A_k \leq_e \mathcal{P}_k(\mathcal{C})$. Thus $I^n_{\mathcal{A}}(\mathcal{B}) \leq_e \mathcal{P}(\mathcal{C})$ and hence $I^n_{\mathcal{A}}(\mathcal{B}) \leq_u \mathcal{C}$. □

Let us mention some other obvious but useful properties of the invert operation $I^n_{\mathcal{A}}$:

(I0) $I^n_{\mathcal{A}}(\mathcal{A}^{(n)}) \equiv_u \mathcal{A}$.
(I1) Let $\mathcal{A}, \mathcal{A}^* \in \mathcal{S}$. If for some $\mathcal{B}, \mathcal{C} \in \mathcal{S}$, $I^n_{\mathcal{A}}(\mathcal{B}) \equiv_u I^n_{\mathcal{A}^*}(\mathcal{C})$, then
$$(\forall k < n)(\mathcal{P}_k(\mathcal{A}) \equiv_e \mathcal{P}_k(\mathcal{A}^*)).$$
(I2) If $\mathcal{B} \equiv_u \mathcal{C}$ then $I^n_{\mathcal{A}}(\mathcal{B}) \equiv_u I^n_{\mathcal{A}}(\mathcal{C})$.
(I3) If $(\forall k < n)(\mathcal{P}_k(\mathcal{A}) \equiv_e \mathcal{P}_k(\mathcal{A}^*))$ then for all $\mathcal{B} \in \mathcal{S}$, $I^n_{\mathcal{A}}(\mathcal{B}) \equiv_u I^n_{\mathcal{A}^*}(\mathcal{B})$.

Let $\mathbf{a}, \mathbf{b} \in \mathcal{D}_\omega$ and $n \geq 1$. Let $\mathcal{A} \in \mathbf{a}$ and $\mathcal{B} \in \mathbf{b}$. Set $I^n_{\mathbf{a}}(\mathbf{b}) = d_\omega(I^n_{\mathcal{A}}(\mathcal{B}))$. By (I2) and (I3) $I^n_{\mathbf{a}}(\mathbf{b})$ is a correctly defined binary operation on $\mathcal{D}_\omega$.

Proposition 3.9 has several corollaries which look surprising and show that the jump operator on the ω-enumeration degrees possesses some nice properties which are not true neither for the Turing nor for the enumeration jump.

3.10. Proposition. *Let $\mathbf{a}, \mathbf{b} \in \mathcal{D}_\omega$ and $\mathbf{a}^{(n)} \leq_\omega \mathbf{b}$. Then $I^n_{\mathbf{a}}(\mathbf{b})$ is the least element of the set $\{\mathbf{x} : \mathbf{a} \leq_\omega \mathbf{x}\ \&\ \mathbf{x}^{(n)} = \mathbf{b}\}$.*

3.11. Proposition. *For every* $\mathbf{a} \in \mathcal{D}_\omega$ *and* $n \geq 1$,

$$\{\mathbf{x}^{(n)} : \mathbf{a} \leq_\omega \mathbf{x} \leq_\omega \mathbf{a}'\} = \{\mathbf{y} : \mathbf{a}^{(n)} \leq_\omega \mathbf{y} \leq_\omega \mathbf{a}^{(n+1)}\}.$$

Proof. Clearly for every $\mathbf{x} \in [\mathbf{a}, \mathbf{a}']$, $\mathbf{x}^{(n)} \in [\mathbf{a}^{(n)}, \mathbf{a}^{(n+1)}]$.

Suppose now that $\mathbf{a}^{(n)} \leq_\omega \mathbf{y} \leq_\omega \mathbf{a}^{(n+1)}$ and set $\mathbf{x} = I^n_{\mathbf{a}}(\mathbf{y})$. Then $\mathbf{a} \leq_\omega \mathbf{x}$ and $\mathbf{x}^{(n)} = \mathbf{y}$. It remains to show that $\mathbf{x} \leq_\omega \mathbf{a}'$. Indeed, we have that $\mathbf{a}^{(n)} \leq_\omega \mathbf{y}$ and $\mathbf{y} \leq_\omega \mathbf{a}^{(n+1)} = (\mathbf{a}')^{(n)}$. Hence, by Proposition 3.9, $\mathbf{x} \leq_\omega \mathbf{a}'$. $\square$

Given ω-enumeration degrees $\mathbf{a} \leq_\omega \mathbf{b}$, denote by $\mathcal{D}_\omega[\mathbf{a}, \mathbf{b}]$ the structure $(\{\mathbf{x} : \mathbf{a} \leq_\omega \mathbf{x} \leq_\omega \mathbf{b}\}, \leq_\omega \restriction [\mathbf{a}, \mathbf{b}])$.

3.12. Proposition. *Let* $\mathbf{a} \in \mathcal{D}_\omega$ *and* $n \geq 1$. *Then*

$$\mathcal{D}_\omega[\mathbf{a}^{(n)}, \mathbf{a}^{(n+1)}] \simeq \mathcal{D}_\omega[\mathbf{a}, I^n_{\mathbf{a}}(\mathbf{a}^{(n+1)})].$$

Proof. It follows easily from Proposition 3.9 that if $\mathbf{a}^{(n)} \leq_\omega \mathbf{x}, \mathbf{y}$ then

$$\mathbf{x} \leq_\omega \mathbf{y} \iff I^n_{\mathbf{a}}(\mathbf{x}) \leq_\omega I^n_{\mathbf{a}}(\mathbf{y}).$$

So to conclude the proof it is enough to show that if $\mathbf{a} \leq_\omega \mathbf{x} \leq_\omega I^n_{\mathbf{a}}(\mathbf{a}^{(n+1)})$ then $\mathbf{x} = I^n_{\mathbf{a}}(\mathbf{x}^{(n)})$. Indeed, let $\mathcal{A} \in \mathbf{a}$ and $\mathcal{X} \in \mathbf{x}$. Then $\mathcal{A} \leq_u \mathcal{X}$ and $\mathcal{X} \leq_u I^n_{\mathcal{A}}(\mathcal{A}^{(n+1)})$. From here it follows that for all $k < n$, $\mathcal{P}_k(\mathcal{A}) \leq_e \mathcal{P}_k(\mathcal{X}) \leq_e \mathcal{P}_k(\mathcal{A})$. Therefore $(\forall k < n)(\mathcal{P}_k(\mathcal{X}) \equiv_e \mathcal{P}_k(\mathcal{A}))$. Then $\mathcal{X} \equiv_u I^n_{\mathcal{A}}(\mathcal{X}^{(n)})$ and hence $\mathbf{x} = I^n_{\mathbf{a}}(\mathbf{x}^{(n)})$. $\square$

The last Proposition shows that $\mathcal{D}_\omega[\mathbf{a}, \mathbf{a}']$ contains a substructure isomorphic to $\mathcal{D}_\omega[\mathbf{a}^{(n)}, \mathbf{a}^{(n+1)}]$.

Denote by $\mathcal{D}_\omega{}'$ the structure $(\mathcal{D}_\omega; \mathbf{0}_\omega; \leq_\omega; ')$ of the ω-enumeration degrees augmented by the jump operation.

In the remaining part of this section we shall show that $\mathcal{D}_1$ is first order definable in $\mathcal{D}_\omega{}'$.

3.13. Definition. Given a $\mathbf{a}, \mathbf{x} \in \mathcal{D}_\omega$, set $I_{\mathbf{a}}(\mathbf{x}) = I^1_{\mathbf{a}}(\mathbf{x})$ and let

$$\mathcal{I}_{\mathbf{a}} = \{I_{\mathbf{a}}(\mathbf{x}) : \mathbf{a}' \leq_\omega \mathbf{x}\}.$$

Notice that

$$\mathbf{z} \in \mathcal{I}_{\mathbf{a}} \iff \mathbf{a} \leq_\omega \mathbf{z} \;\&\; (\forall \mathbf{y})(\mathbf{a} \leq_\omega \mathbf{y} \;\&\; \mathbf{y}' = \mathbf{z}' \Rightarrow \mathbf{z} \leq_\omega \mathbf{y}).$$

Hence there exists a fist order formula Φ with two free variables such that

$$\mathcal{D}_\omega{}' \models \Phi(\mathbf{z}, \mathbf{a}) \iff \mathbf{z} \in \mathcal{I}_{\mathbf{a}}.$$

3.14. Proposition. *Let* $\mathbf{a} = d_\omega(\mathcal{A})$ *and* $\mathbf{b} = d_\omega(\mathcal{B})$. *Then*

$$\mathcal{I}_{\mathbf{a}} \subseteq \mathcal{I}_{\mathbf{b}} \iff \mathbf{b} \leq_\omega \mathbf{a} \;\&\; A_0 \equiv_e B_0.$$

Proof. Let $\mathcal{I}_{\mathbf{a}} \subseteq \mathcal{I}_{\mathbf{b}}$. By (I0) $\mathbf{a} \in \mathcal{I}_{\mathbf{a}}$ and hence $\mathbf{a} \in \mathcal{I}_{\mathbf{b}}$. Then $\mathbf{a} = I_{\mathbf{b}}(\mathbf{x})$ for some $\mathbf{x}$ such that $\mathbf{b}' \leq_\omega \mathbf{x}$. Therefore $\mathbf{b} \leq_\omega \mathbf{a}$. On the other hand $\mathbf{a} = I_{\mathbf{a}}(\mathbf{a}') = I_{\mathbf{b}}(\mathbf{x})$. Hence by (I1) $A_0 = P_0(\mathcal{A}) \equiv_e \mathcal{P}_0(\mathcal{B}) = B_0$.

Suppose now that $\mathbf{b} \leq_\omega \mathbf{a}$ and $A_0 \equiv_e B_0$. We have to show that for every $\mathbf{x}$ such that $\mathbf{a}' \leq_\omega \mathbf{x}$, $I_{\mathbf{a}}(\mathbf{x}) \in \mathcal{I}_{\mathbf{b}}$. Indeed, note that $\mathbf{b}' \leq_\omega \mathbf{a}' \leq_\omega \mathbf{x}$ and hence $I_{\mathbf{b}}(\mathbf{x}) \in \mathcal{I}_{\mathbf{b}}$. From $A_0 \equiv_e B_0$ by (I3) we get that $I_{\mathbf{b}}(\mathbf{x}) = I_{\mathbf{a}}(\mathbf{x})$. $\square$

3.15. Corollary. *If* $\mathcal{I}_{\mathbf{a}} = \mathcal{I}_{\mathbf{b}}$ *then* $\mathbf{a} = \mathbf{b}$.

3.16. Proposition. *For all* $\mathbf{a} \in \mathcal{D}_\omega$,

$$\mathbf{a} \in \mathcal{D}_1 \iff (\forall \mathbf{b})(\mathcal{I}_{\mathbf{a}} \subseteq \mathcal{I}_{\mathbf{b}} \Rightarrow \mathcal{I}_{\mathbf{a}} = \mathcal{I}_{\mathbf{b}}).$$

Proof. Let $\mathbf{a} = d_\omega(A \uparrow \omega) \in \mathcal{D}_1$. Suppose that $\mathbf{b} = d_\omega(\mathcal{B})$ and $\mathcal{I}_\mathbf{a} \subseteq \mathcal{I}_\mathbf{b}$. Then $A \equiv_e B_0$ and hence $A \uparrow \omega \leq_u \mathcal{B}$. So $\mathbf{a} \leq_\omega \mathbf{b}$. By the proposition above $\mathcal{I}_\mathbf{b} \subseteq \mathcal{I}_\mathbf{a}$.

Suppose now that $(\forall \mathbf{b})(\mathcal{I}_\mathbf{a} \subseteq \mathcal{I}_\mathbf{b} \Rightarrow \mathcal{I}_\mathbf{a} = \mathcal{I}_\mathbf{b})$. Consider a sequence $\mathcal{A} \in \mathbf{a}$. Set $\mathcal{B} = A_0 \uparrow \omega$ and let $\mathbf{b} = d_\omega(\mathcal{B})$. Notice that $\mathbf{b} \in \mathcal{D}_1$. Clearly $\mathbf{b} \leq_\omega \mathbf{a}$. Therefore by the proposition above $\mathcal{I}_\mathbf{a} \subseteq \mathcal{I}_\mathbf{b}$. Then $\mathcal{I}_\mathbf{a} = \mathcal{I}_\mathbf{b}$. From here we get that $\mathbf{a} = \mathbf{b}$ and hence $\mathbf{a} \in \mathcal{D}_1$ □

3.17. Corollary. *$\mathcal{D}_1$ is first order definable in ${\mathcal{D}_\omega}'$.*

4. The automorphisms of ${\mathcal{D}_\omega}'$

The definability of $\mathcal{D}_1$ shows that every automorphism of ${\mathcal{D}_\omega}'$ induces an automorphism of the structure $\mathcal{D}_1$ and hence of the structure $\mathcal{D}_e$. On the other hand, since $\mathcal{D}_1$ is a base of the automorphisms of $\mathcal{D}_\omega$ we have that if two automorphisms of ${\mathcal{D}_\omega}'$ induce the same automorphism of $\mathcal{D}_e$ then they coincide. In particular every nontrivial automorphism of ${\mathcal{D}_\omega}'$ induces a nontrivial automorphism of $\mathcal{D}_e$.

Now we shall show that every automorphism of $\mathcal{D}_e$ can be extended to an automorphism of ${\mathcal{D}_\omega}'$. We start by recalling some facts about the automorphisms of $\mathcal{D}_T$.

Denote by ${\mathcal{D}_T}'$ the structure of the Turing degrees augmented by the Turing jump operator and by ${\mathcal{D}_e}'$ the structure of the enumeration degrees augmented by the enumeration jump.

The following Theorem is proved by Richter[**?**], see also [**?**]:

4.1. Theorem. *Let $\mathbf{a}, \mathbf{b} \in \mathcal{D}_T$. Suppose that ${\mathcal{D}_T}'[\mathbf{a}, \infty] \simeq {\mathcal{D}_T}'[\mathbf{b}, \infty]$. Then $\mathbf{a}^{(2)} \leq_T \mathbf{b}^{(3)}$.*

As a corollary Richter obtained the following fact about the automorphisms of ${\mathcal{D}_T}'$:

4.2. Theorem. *Let φ be an automorphism of ${\mathcal{D}_T}'$. Then $\varphi(\mathbf{a}) = \mathbf{a}$ for all $\mathbf{a}$ above $\mathbf{0}^{(3)}$.*

Using Theorem 4.1 one cane obtain similar results about ${\mathcal{D}_e}'$.

4.3. Theorem. *Let $\mathbf{a}, \mathbf{b} \in \mathcal{D}_e$ are such that ${\mathcal{D}_e}'[\mathbf{a}, \infty] \simeq {\mathcal{D}_e}'[\mathbf{b}, \infty]$. Then $\mathbf{a}^{(3)} \leq_e \mathbf{b}^{(4)}$.*

Proof. Let φ be an isomorphism from ${\mathcal{D}_e}'[\mathbf{a}, \infty]$ to ${\mathcal{D}_e}'[\mathbf{b}, \infty]$.

We shall show that φ maps the total enumeration degrees above $\mathbf{a}'$ onto the total enumeration degrees above $\mathbf{b}'$. Indeed, consider a total degree $\mathbf{x}$ above $\mathbf{a}'$. By Theorem 2.3 there exists a $\mathbf{y}$ such that $\mathbf{a} \leq_e \mathbf{y}$ and $\mathbf{y}' = \mathbf{x}$. Then

$$\varphi(\mathbf{x}) = \varphi(\mathbf{y}') = \varphi(\mathbf{y})'.$$

Since every jump is a total degree $\varphi(\mathbf{x})$ is total. Clearly $\mathbf{b}' = \varphi(\mathbf{a}') \leq_e \varphi(\mathbf{y}') = \varphi(\mathbf{x})$.

Suppose now that $\mathbf{b}' \leq_e \mathbf{y}$ and $\mathbf{y}$ is total. Since φ^{-1} is an isomorphism from ${\mathcal{D}_e}'[\mathbf{b}, \infty]$ to ${\mathcal{D}_e}'[\mathbf{a}, \infty]$, $\varphi^{-1}(\mathbf{y})$ is total and $\mathbf{a}' \leq_e \varphi^{-1}(\mathbf{y})$.

Define the mapping γ on $\mathcal{D}_T[\iota^{-1}(\mathbf{a}'), \infty]$ by $\gamma(\mathbf{x}) = \iota^{-1}(\varphi(\iota(\mathbf{x})))$. Clearly γ is an isomorphism from ${\mathcal{D}_T}'[\iota^{-1}(\mathbf{a}'), \infty]$ to ${\mathcal{D}_T}'[\iota^{-1}(\mathbf{b}'), \infty]$. By Theorem 4.1 $\iota^{-1}(\mathbf{a}')^{(2)} \leq_T \iota^{-1}(\mathbf{b}')^{(3)}$. Hence $\mathbf{a}^{(3)} \leq_e \mathbf{b}^{(4)}$. □

As a corollary we obtain the following property of the automorphisms of ${\mathcal{D}_e}'$ whose proof follows along the lines the proof of Theorem 4.2 presented in [**?**].

4.4. Theorem. *Let φ be an automorphism of ${\mathcal{D}_e}'$. Then $\varphi(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x}$ above ${\mathbf{0}_e}^{(4)}$.*

Proof. Consider first a total degree $\mathbf{c}$ greater than ${\mathbf{0}_e}^{(4)}$. By Theorem 2.3 there exists an enumeration degree $\mathbf{a}$ such that $\mathbf{c} = \mathbf{a} \cup {\mathbf{0}_e}^{(4)} = \mathbf{a}^{(4)}$.

Let $\mathbf{d} = \varphi(\mathbf{c})$ and $\mathbf{b} = \varphi(\mathbf{a})$. By the previous Theorem $\mathbf{b} \leq_e \mathbf{b}^{(3)} \leq_e \mathbf{a}^{(4)}$.

Clearly $\mathbf{b}^{(4)} = \varphi(\mathbf{a}^{(4)}) = \varphi(\mathbf{c}) = \mathbf{d}$.

On the other hand,

$$\mathbf{b}^{(4)} = \varphi(\mathbf{a}^{(4)}) = \varphi(\mathbf{a} \cup {\mathbf{0}_e}^{(4)}) = \varphi(\mathbf{a}) \cup {\mathbf{0}_e}^{(4)} = \mathbf{b} \cup {\mathbf{0}_e}^{(4)}.$$

Hence $\mathbf{d} = \mathbf{b} \cup {\mathbf{0}_e}^{(4)} \leq_e \mathbf{a}^{(4)} = \mathbf{c}$.

Using the fact that φ^{-1} is also an automorphism of ${\mathcal{D}_e}'$ we obtain by the same reasoning that $\mathbf{c} \leq_e \mathbf{d}$. Thus $\mathbf{c} = \mathbf{d}$.

Let $\mathbf{x}$ be an arbitrary enumeration degree greater than ${\mathbf{0}_e}^{(4)}$. By ROZINAS[**?**] there exist total enumeration degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{x} = \mathbf{a} \cap \mathbf{b}$. Then

$$\varphi(\mathbf{x}) = \varphi(\mathbf{a} \cap \mathbf{b}) = \varphi(\mathbf{a}) \cap \varphi(\mathbf{b}) = \mathbf{a} \cap \mathbf{b} = \mathbf{x}.$$

□

Now we are ready to show that every automorphism of ${\mathcal{D}_e}'$ can be extended to an automorphism of ${\mathcal{D}_\omega}'$. Let us fix an automorphism φ of ${\mathcal{D}_e}'$.

Consider a sequence $\mathcal{A} = \{A_k\}_{k<\omega}$ of sets of natural numbers. We shall show, that one can construct a sequence $\mathcal{B}$ such that $J^e_{\mathcal{B}} = \{\varphi(\mathbf{x}) : \mathbf{x} \in J^e_{\mathcal{A}}\}$. Indeed, let $\mathbf{p}_k = d_e(\mathcal{P}_k(\mathcal{A}))$. Notice that if $k \geq 4$ then $\mathbf{p_k} \geq \mathbf{0}^{(4)}$ and hence $\varphi(\mathbf{p_k}) = \mathbf{p_k}$.

Fix some elements B_0, B_1, B_2, B_3 of $\varphi(\mathbf{p_0}), \varphi(\mathbf{p_1}), \varphi(\mathbf{p_2})$ and $\varphi(\mathbf{p_3})$ respectively and let for $k \geq 4$, $B_k = \mathcal{P}_k(\mathcal{A})$.

4.5. Lemma. $J^e_{\mathcal{B}} = \{\varphi(\mathbf{x}) : \mathbf{x} \in J^e_{\mathcal{A}}\}$.

Proof. Let $\mathbf{x} \in J^e_{\mathcal{A}}$ and let $X \in \mathbf{x}$. Then $\mathcal{A} \leq_u X \uparrow \omega$ and hence $\mathcal{P}(\mathcal{A}) \leq_e \{X^{(k)}\}_{k<\omega}$. Consider a set $Y \in \varphi(\mathbf{x})$. By Theorem 4.4 $X^{(4)} \equiv_e Y^{(4)}$. Therefore for all $k \geq 4$, $X^{(k)} \equiv_e Y^{(k)}$ uniformly in k. Clearly $B_k \leq_e Y^{(k)}$ for $k \leq 3$. So, $\mathcal{B} \leq_u \{Y^{(k)}\}_{k<\omega}$. Thus $\varphi(\mathbf{x}) \in J^e_{\mathcal{B}}$.

Suppose now that $\mathbf{y} \in J^e_{\mathcal{B}}$ and let $\mathbf{y} = \varphi(\mathbf{x})$. Let $X \in \mathbf{x}$ and $Y \in \mathbf{y}$. Then again $X^{(4)} \equiv_e Y^{(4)}$. From here it follows as in he previous case that $\mathcal{P}(\mathcal{A}) \leq_e \{X^{(k)}\}_{k<\omega}$ and hence $\mathbf{x} \in J^e_{\mathcal{A}}$. □

Let us define the mapping Φ on $\mathcal{D}_\omega$ as follows. Given an element $\mathbf{a} \in \mathcal{D}_\omega$, consider a sequence $\mathcal{A} \in \mathbf{a}$ and construct the sequence $\mathcal{B}$ as above. Let $\Phi(\mathbf{b}) = d_\omega(\mathcal{B})$. By the Lemma the mapping Φ is correctly defined, it is injective and preserves the partial ordering " $\leq_\omega$ ". So to prove that Φ is an automorphism of $\mathcal{D}_\omega$ it is enough to show that Φ is onto. Indeed, let $\mathbf{b} = d_\omega(\mathcal{B})$. Since φ^{-1} is an automorphism of ${\mathcal{D}_e}'$ there exist a sequence $\mathcal{A}$ such that $J^e_{\mathcal{A}} = \{\varphi^{-1}(\mathbf{x}) : \mathbf{x} \in J^e_{\mathcal{B}}\}$. Let $\mathbf{a} = d_\omega(\mathcal{A})$ and $\Phi(\mathbf{a}) = d_\omega(\mathcal{B}^*)$, where $J^e_{\mathcal{B}^*} = \{\varphi(\mathbf{x}) : \mathbf{x} \in J^e_{\mathcal{A}}\}$. Then $J^e_{\mathcal{B}} = J^e_{\mathcal{A}}$ and hence $\Phi(\mathbf{a}) = \mathbf{b}$.

The following Lemma follows directly from the definition of Φ:

4.6. Lemma. *For every $\mathbf{a} \in \mathcal{D}_\omega$,*

$$\{\mathbf{y} : \mathbf{y} \in \mathcal{D}_e \ \& \ \Phi(\mathbf{a}) \leq_\omega \kappa(\mathbf{y})\} = \{\varphi(\mathbf{x}) : \mathbf{x} \in \mathcal{D}_e \ \& \ \mathbf{a} \leq_\omega \kappa(\mathbf{x})\}.$$

4.7. Corollary. *For every $\mathbf{a} \in \mathcal{D}_e$, $\Phi(\kappa(\mathbf{a})) = \kappa(\varphi(\mathbf{a}))$.*

Proof. Let $\mathbf{a} \in \mathcal{D}_e$. Clearly for every $\mathbf{y} \in \mathcal{D}_e$,

$$\Phi(\kappa(\mathbf{a})) \leq_\omega \kappa(\mathbf{y}) \iff \kappa(\mathbf{a}) \leq_\omega \kappa(\varphi^{-1}(\mathbf{y})) \iff$$
$$\mathbf{a} \leq_e \varphi^{-1}(\mathbf{y}) \iff \varphi(\mathbf{a}) \leq_e \mathbf{y} \iff \kappa(\varphi(\mathbf{a})) \leq_\omega \kappa(\mathbf{y}).$$

Thus $\Phi(\kappa(\mathbf{a})) = \kappa(\varphi(\mathbf{a}))$. □

4.8. Corollary. *For every* $\mathbf{a} \in \mathcal{D}_e$*,* $\Phi^{-1}(\kappa(\mathbf{a})) = \kappa(\varphi^{-1}(\mathbf{a}))$*.*

Proof. Let $\mathbf{a} \in \mathcal{D}_e$. Then $\kappa(\mathbf{a}) \in \mathcal{D}_1$ and hence by the definability of $\mathcal{D}_1$, $\Phi^{-1}(\kappa(\mathbf{a})) \in \mathcal{D}_1$. Then $\varphi(\kappa^{-1}(\Phi^{-1}(\kappa(\mathbf{a})))) = \kappa^{-1}(\Phi(\Phi^{-1}(\kappa(\mathbf{a})))) = \mathbf{a}$.

Hence $\kappa^{-1}(\Phi^{-1}(\kappa(\mathbf{a}))) = \varphi^{-1}(\mathbf{a})$. From the last equality it follows immediately that $\Phi^{-1}(\kappa(\mathbf{a})) = \kappa(\varphi^{-1}(\mathbf{a}))$. □

It remains to show that Φ preserves the jump operator.

4.9. Lemma. *For every* $\mathbf{a} \in \mathcal{D}_\omega$*,* $\Phi(\mathbf{a}') = \Phi(\mathbf{a})'$*.*

Proof. Let us fix an element $\mathbf{a}$ of $\mathcal{D}_\omega$. First we shall show that $\Phi(\mathbf{a}') \leq_\omega \Phi(\mathbf{a})'$. For we are going to use Proposition 3.7. We need to show that for all $\mathbf{x} \in \mathcal{D}_e$,

$$\Phi(\mathbf{a}) \leq_\omega \kappa(\mathbf{x}) \Rightarrow \Phi(\mathbf{a}') \leq_\omega \kappa(\mathbf{x}').$$

Notice that φ^{-1} is an automorphism of $\mathcal{D}_e{}'$. Let $\mathbf{x} \in \mathcal{D}_e$. Then

$$\Phi(\mathbf{a}) \leq_\omega \kappa(\mathbf{x}) \Rightarrow \mathbf{a} \leq_\omega \Phi^{-1}(\kappa(\mathbf{x})) \Rightarrow \mathbf{a} \leq_\omega \kappa(\varphi^{-1}(\mathbf{x})) \Rightarrow$$
$$\mathbf{a}' \leq_\omega \kappa(\varphi^{-1}(\mathbf{x}))' \Rightarrow \mathbf{a}' \leq_\omega \kappa(\varphi^{-1}(\mathbf{x}')) \Rightarrow \Phi(\mathbf{a}') \leq_\omega \kappa(\mathbf{x}').$$

To prove the reverse inequality we shall show that for all $\mathbf{x} \in \mathcal{D}_T$,

$$\Phi(\mathbf{a}') \leq_\omega \lambda(\mathbf{x}) \Rightarrow \Phi(\mathbf{a})' \leq_\omega \lambda(\mathbf{x}).$$

Let $\mathbf{x} \in \mathcal{D}_T$ and $\Phi(\mathbf{a}') \leq_\omega \lambda(\mathbf{x})$. We have that $\mathbf{0}_\omega{}' \leq_\omega \mathbf{a}'$ and hence

$$\Phi(\mathbf{0}_\omega{}') = \kappa(\varphi(\mathbf{0}_e{}')) = \kappa(\mathbf{0}_e{}') \leq_\omega \Phi(\mathbf{a}') \leq_\omega \lambda(\mathbf{x}).$$

So $\kappa(\mathbf{0}_e{}') \leq_\omega \lambda(\mathbf{x})$. Since $\lambda(\mathbf{x}) = \kappa(\iota(\mathbf{x}))$ and $\mathbf{0}_e{}' = \iota(\mathbf{0}_T{}')$, we get from here that $\mathbf{0}_T{}' \leq_T \mathbf{x}$. By the Friedberg's Jump inversion Theorem there exists a $\mathbf{y} \in \mathcal{D}_T$ such that $\mathbf{y}' = \mathbf{x}$. Then

$$\Phi^{-1}(\lambda(\mathbf{x})) = \kappa(\varphi^{-1}(\iota(\mathbf{y}'))) = \kappa(\varphi^{-1}(\iota(\mathbf{y}))').$$

Clearly $\mathbf{b} = \varphi^{-1}(\iota(\mathbf{y}))'$ is a total enumeration degree and $\mathbf{a}' \leq_\omega \kappa(\mathbf{b})$. By Theorem 2.3 there exists a total enumeration degree $\mathbf{z}$ such that $\mathbf{z}' = \mathbf{b}$ and $\mathbf{a} \leq_\omega \kappa(\mathbf{z})$. So

$$\Phi(\mathbf{a})' \leq_\omega \Phi(\kappa(\mathbf{z}))' = \kappa(\varphi(\mathbf{z}))' = \kappa(\varphi(\mathbf{z}')) = \kappa(\varphi(\mathbf{b})) = \lambda(\mathbf{x}).$$

□

Combining all sofar proved properties of Φ we obtain the following:

4.10. Theorem. *For every isomorphism* φ *of* $\mathcal{D}_e{}'$ *there exists a unique automorphism* Φ *of* $\mathcal{D}_\omega{}'$ *such that:*

$$(\forall \mathbf{x} \in \mathcal{D}_e)(\Phi(\kappa(\mathbf{x})) = \kappa(\varphi(\mathbf{x}))). \tag{1}$$

Proof. We need to show only that Φ is unique. Indeed let us suppose that Φ_1 and Φ_2 are automorphisms of $\mathcal{D}_\omega{}'$ satisfying (1). Then for all $\mathbf{y} \in \mathcal{D}_1$, $\Phi_1(\mathbf{y}) = \Phi_2(\mathbf{y})$. Since $\mathcal{D}_1$ is a base of the automorphisms of $\mathcal{D}_\omega$, $\Phi_1 = \Phi_2$. □

4.11. Corollary. *The groups of the automorphisms of $\mathcal{D}_e{}'$ and of the automorphisms of $\mathcal{D}_\omega{}'$ are isomorphic.*

Proof. Given an automorphism φ of $\mathcal{D}_e{}'$, let $\Lambda(\varphi)$ be the automorphism Φ of $\mathcal{D}_\omega{}'$ satisfying (1). Clearly Λ is well defined and injective.

Suppose that Φ is an automorphism of $\mathcal{D}_\omega{}'$. By the definability of $\mathcal{D}_1$, $\Phi(\mathbf{y}) \in \mathcal{D}_1$ for every $\mathbf{y} \in \mathcal{D}_1$. Define φ on $\mathcal{D}_e$ by

$$\varphi(\mathbf{x}) = \kappa^{-1}(\Phi(\kappa(\mathbf{x}))).$$

On can easily see that φ is an automorphism of $\mathcal{D}_e{}'$ and that φ and Φ satisfy (1). So Λ is one to one.

It remains to show that for any two automorphisms φ_1 and φ_2 of $\mathcal{D}_e{}'$,

$$\Lambda(\varphi_1 \circ \varphi_2) = \Lambda(\varphi_1) \circ \Lambda(\varphi_2).$$

Set $\Phi = \Lambda(\varphi_1 \circ \varphi_2)$, $\Phi_1 = \Lambda(\varphi_1)$ and $\Phi_2 = \Lambda(\varphi_2)$. It is enough to show that for all $\mathbf{x} \in \mathcal{D}_e$, $\Phi(\kappa(\mathbf{x})) = \Phi_2(\Phi_1(\kappa(\mathbf{x}))$. Indeed, let $\mathbf{x} \in \mathcal{D}_e$. Then

$$\Phi(\kappa(\mathbf{x})) = \kappa(\varphi_2(\varphi_1(\mathbf{x}))) = \Phi_2(\kappa(\varphi_1(\mathbf{x}))) = \Phi_2(\Phi_1(\kappa(\mathbf{x}))).$$

□

In [**?**] Kalimullin proved that the enumeration jump operator is first order definable in $\mathcal{D}_e$. Hence the groups of the automorphisms of $\mathcal{D}_e$ and $\mathcal{D}_e{}'$ coincide. So we may reformulate the last Corollary as follows:

4.12. Theorem. *The groups of the automorphisms of $\mathcal{D}_e$ and of $\mathcal{D}_\omega{}'$ are isomorphic.*

The established connection between the automorphisms of $\mathcal{D}_\omega{}'$ and $\mathcal{D}_e{}'$ has the following corollary which shows that every automorphism of $\mathcal{D}_\omega{}'$ is the identity on the cone above $\mathbf{0}_\omega{}^{(4)}$.

4.13. Theorem. *Let Φ be an automorphism of $\mathcal{D}_\omega{}'$. Then $\Phi(\mathbf{a}) = \mathbf{a}$ for all $\mathbf{a}$ greater than $\mathbf{0}_\omega{}^{(4)}$.*

Proof. Let φ be an automorphism of $\mathcal{D}_e{}'$ such that for all $\mathbf{x} \in \mathcal{D}_e$, $\Phi(\kappa(\mathbf{x})) = \kappa(\varphi(\mathbf{x}))$. Let $\mathbf{0}_\omega{}^{(4)} \leq_\omega \mathbf{a}$. Clearly $\mathbf{0}_\omega{}^{(4)} \leq_\omega \Phi(\mathbf{a})$. Then for all $\mathbf{x} \in \mathcal{D}_e$,

$$\mathbf{a} \leq_\omega \kappa(\mathbf{x}) \iff \Phi(\mathbf{a}) \leq_\omega \Phi(\kappa(\mathbf{x})) \iff \Phi(\mathbf{a}) \leq_\omega \kappa(\varphi(\mathbf{x})) \iff \Phi(\mathbf{a}) \leq_\omega \kappa(\mathbf{x}).$$

□

5. Jumps of the ω-enumeration degrees below $\mathbf{0}_\omega{}'$

The sofar obtained results show that the structures $\mathcal{D}_e{}'$ and $\mathcal{D}_\omega{}'$ are closely related but not elementary equivalent. As we shall see in this section the structure $\mathcal{D}_\omega{}'$ contains new explicitly defined elements which can be used to characterize the low and the high degrees not only in $\mathcal{D}_\omega$ but also in $\mathcal{D}_e$ and $\mathcal{D}_T$.

5.1. Definition. Let $n \geq 1$. An ω-enumeration degree $\mathbf{a} \leq \mathbf{0}_\omega{}'$ is high n if $\mathbf{a}^{(n)} = \mathbf{0}_\omega{}^{(n+1)}$. The degree $\mathbf{a}$ is low n if $\mathbf{a}^{(n)} = \mathbf{0}_\omega{}^{(n)}$.

Denote by H_n the set of all high n degrees and by L_n set of all low n degrees. Clearly a Turing degree $\mathbf{x}$ is high (low) n if and only if $\lambda(\mathbf{x}) \in H_n(L_n)$ and an enumeration degree $\mathbf{y}$ is high (low) n if and only if $\kappa(\mathbf{x}) \in H_n(L_n)$.

Set

$$H = \bigcup_{n\geq 1} H_n;\ L = \bigcup_{n\geq 1} L_n \text{ and } I = \{\mathbf{a} \leq_\omega \mathbf{0}_\omega{}' : \mathbf{a} \notin (H \cup L)\}.$$

Clearly the classes H, L and I are invariant under the automorphisms of $\mathcal{D}_\omega{}'$ and hence one can expect that they admit a natural characterization.

Given a $n \geq 1$ set $o_n = I^n_{\mathbf{0}_\omega}(\mathbf{0}_\omega{}^{(n+1)})$. In other words o_n is the least among the degrees $\mathbf{a}$ such that $\mathbf{a}^{(n)} = \mathbf{0}_\omega{}^{(n+1)}$. Clearly if $\mathbf{a} \leq_\omega \mathbf{0}_\omega{}'$, then $\mathbf{a} \in H_n \iff o_n \leq_\omega \mathbf{a}$.

It follows from the definition of the invert operation that for every $n \geq 1$, $o_n = d_\omega(\{O^n_k\}_{k<\omega})$, where $O^n_k = \emptyset$ if $k < n$ and $O^n_k = \emptyset^{(k+1)}$ if $n \leq k$.

Set $o_0 = \mathbf{0}_\omega{}'$. The following facts are immediate from the explicit definition of the degrees o_n:

(O1) $(\forall n)(o_{n+1} \leq_\omega o_n\ \&\ o_{n+1} \neq o_n)$.
(O2) $(\forall n)(\mathcal{D}_\omega[o_{n+1}, o_n] \simeq \mathcal{D}_e[\mathbf{0}_e{}^{(n)}, \mathbf{0}_e{}^{(n+1)}])$.
(O3) If $\mathbf{x} \in \mathcal{D}_e$ and $\kappa(\mathbf{x}) \leq_\omega o_1$ then $\mathbf{x} = \mathbf{0}_e$.

5.2. Definition. An ω-enumeration degree $\mathbf{a}$ is *almost zero* (a.z.) if $(\forall n)(\mathbf{a} \leq_\omega o_n)$.

Clearly $\mathbf{0}_\omega$ is a.z. Actually there exist infinitely many a.z. degrees. To prove this we need the following explicit characterization of the a.z. degrees:

5.3. Proposition. *A degree $\mathbf{x}$ is a.z. if $\mathbf{x} \leq_\omega \mathbf{0}_\omega{}'$ and there exists a sequence $\{X_k\}_{k<\omega} \in \mathbf{x}$ such that $(\forall k)(X_k \leq_e \emptyset^{(k)})$.*

Proof. Suppose that $\mathbf{x}$ is a.z. Clearly $\mathbf{x} \leq_\omega \mathbf{0}_\omega{}'$. Let $\{X_k\}_{k<\omega} \in \mathbf{x}$. Fix a k. Since $\mathbf{x} \leq_\omega o_{k+1}$, $X_k \leq_e \mathcal{P}_k(\{O^{k+1}_n\}_{n<\omega})$ and hence $X_k \leq_e \emptyset^{(k)}$.

Now let $\{X_k\}_{k<\omega}$ be a sequence of sets of natural numbers which is uniformly reducible to $\emptyset_\omega{}'$ and such that $(\forall n)(X_k \leq_e \emptyset^{(k)})$. We shall show that for all $n \geq 1$, $\{X_k\} \leq_u \{O^n_k\}_{k<\omega}$. Indeed, fix a $n \geq 1$ and set $\mathcal{O}^n = \{O^n_k\}_{k<\omega}$. Clearly for all $k \geq n$, $X_k \leq_e \mathcal{P}_k(\emptyset_\omega{}') \equiv_e \mathcal{P}_k(\mathcal{O}^n)$ uniformly in k. If $k < n$ then $\mathcal{P}_k(\mathcal{O}^n) \equiv_e \emptyset^{(k)}$ and hence $X_k \leq_e \mathcal{P}_k(\mathcal{O}^n)$. Thus $\{X_k\} \leq_u \mathcal{O}^n$. □

Using Proposition 5.3 and the definition of the invert operation we obtain immediately the following property of the a.z. degrees:

5.4. Proposition. *Let $\mathbf{d}$ be a.z. then $(\forall n)(I^n_{\mathbf{0}_\omega}(\mathbf{d}^{(n)}) = \mathbf{d})$.*

5.5. Corollary. *Let $\mathbf{d} \neq \mathbf{0}_\omega$ be a.z. Then $\mathbf{d} \in I$.*

Proof. Since $(\forall n)(\mathbf{d} \leq_\omega o_n)$, $\mathbf{d} \notin H$. Assume that $\mathbf{d} \in L$ and let $\mathbf{d}^{(n)} = \mathbf{0}_\omega{}^{(n)}$. Then $\mathbf{d} = I^n_{\mathbf{0}_\omega}(\mathbf{0}_\omega{}^{(n)}) = \mathbf{0}_\omega$. A contradiction. □

5.6. Proposition. *There exist nonzero a.z. degrees.*

Proof. We shall construct a sequence $\mathcal{D} = \{D_k\}_{k<\omega}$ of finite sets so that $\mathcal{D} \not\leq_u \emptyset_\omega$ and $\mathcal{D} \leq_u \emptyset_\omega{}'$.

Let $g_0, \ldots, g_k, \ldots$ be an effective enumeration of all primitive recursive functions and $W_0, \ldots, W_k \ldots$ be a Gödel enumeration of the r.e. sets.

Set

$$D_k = \begin{cases} \emptyset, & \text{if } 0 \in W_{g_k(k)}(\emptyset^{(k)}); \\ \{0\}, & \text{if } 0 \notin W_{g_k(k)}(\emptyset^{(k)}); \end{cases}$$

Let $\mathcal{D} = \{D_k\}_{k<\omega}$. From the definition of the sets D_k it follows that there does not exist a primitive recursive function g such that $(\forall k)(D_k = W_{g(k)}(\emptyset^{(k)}))$. Thus

$\mathcal{D} \not\leq_e \mathcal{P}(\emptyset_\omega)$ and hence $\mathcal{D} \not\leq_u \emptyset_\omega$. On the other hand, using the oracle $\emptyset^{(k+1)}$ on can decide uniformly in k whether $0 \in W_{g_k(k)}(\emptyset^{(k)})$. Therefore $\mathcal{D} \leq_e \mathcal{P}(\emptyset_\omega{}')$ and hence $\mathcal{D} \leq_u \emptyset_\omega{}'$. □

5.7. Corollary. *There exist infinitely many a.z. degrees.*

Proof. Let $\mathbf{d} \neq \mathbf{0}_\omega$ be a.z. By the density of the ω-enumeration degrees below $\mathbf{0}_\omega{}'$, see [?], there exists a $\mathbf{x}$ such that $\mathbf{0}_\omega <_\omega \mathbf{x} <_\omega \mathbf{d}$. Clearly $\mathbf{x}$ is also a.z. □

In the rest of the paper we are going to prove the following two theorems which characterize the classes H and L by means of the almost zero degrees:

5.8. Theorem. *Let* $\mathbf{a} \leq_\omega \mathbf{0}_\omega{}'$. *Then* $\mathbf{a} \in H \iff (\forall$ *a.z.* $\mathbf{d})(\mathbf{d} \leq_\omega \mathbf{a})$.

5.9. Theorem. *Let* $\mathbf{a} \leq_\omega \mathbf{0}_\omega{}'$. *Then* $\mathbf{a} \in L \iff (\forall$ *a.z.* $\mathbf{d})(\mathbf{d} \leq_\omega \mathbf{a} \Rightarrow \mathbf{d} = \mathbf{0}_\omega)$.

Before starting with the proofs let us mention the following corollary of Theorem 5.8:

5.10. Corollary. *The ideal of all a.z. degrees does not have a minimal upper bound below* $\mathbf{0}_\omega{}'$.

Proof. Let $\mathbf{a} \leq_\omega \mathbf{0}_\omega{}'$ be an upper bound of all a.z. degrees. By Theorem 5.8 $\mathbf{a} \in H$ and hence $\mathbf{a} \in H_n$ for some $n \geq 1$. Then $o_n \leq \mathbf{a}$ and hence $o_{n+1} <_\omega \mathbf{a}$. Clearly o_{n+1} is an upper bound of all a.z. degrees. □

The proofs of Theorem 5.8 and Theorem 5.9 use the notion of *good approximation* of a sequence of sets of natural numbers. This notion is introduced in [?] and is based on the notion of good approximation of a set of natural numbers from [?].

5.11. Definition. Let $\mathcal{B} = \{B_k\}_{k<\omega}$ be a sequence of sets of natural numbers. A sequence $\{B_k^s\}$ of finite sets recursive in k and s is a *good approximation of* $\mathcal{B}$ if the following three conditions are satisfied:

(i) $(\forall s)(\forall k)[B_k^s \subseteq B_k \Rightarrow (\forall r \leq k)(B_r^s \subseteq B_r)]$.
(ii) $(\forall n)(\forall k)(\exists s)(\forall r \leq k)(B_r \restriction n \subseteq B_r^s \subseteq B_r)$.
(iii) $(\forall n)(\forall k)(\exists s)(\forall t \geq s)[B_k^t \subseteq B_k \Rightarrow (\forall r \leq k)(B_r \restriction n \subseteq B_r^t)]$.

If $\{B_k^s\}$ is a good approximation of the sequence $\mathcal{B} = \{B_k\}_{k<\omega}$, then by G_k we shall denote the set of all *k-good stages*, i.e the set of all s such that $B_k^s \subseteq B_k$. Clearly $G_r \supseteq G_k$ for all $r \leq k$.

5.12. Definition. Let $\mathcal{A} = \{A_k\}_{k<\omega}$ and $\mathcal{B} = \{B_k\}_{k<\omega}$ be sequences of sets of natural numbers and let $\{B_k^s\}$ be a good approximation of $\mathcal{B}$. A sequence $\{A_k^s\}$ of finite sets recursive in s and k is a *correct (with respect to $\{B_k^s\}$) approximation of* $\mathcal{A}$ if the following two conditions hold:

(C1) $(\forall k, s)(B_k^s \subseteq B_k \Rightarrow (\forall r \leq k)(A_r^s \subseteq A_r))$.
(C2) For all natural numbers k, n there exists a v such that if $s \geq v$ and $B_k^s \subseteq B_k$, then $(\forall r \leq k)(A_r \restriction n \subseteq A_r^s)$.

Given an r.e. set W_a and $s \in \mathbb{N}$, set

$$W_{a,s} = \{x : x \leq s \ \&\ \{a\}(x) \text{ halts in less than } s \text{ steps}\}.$$

The following lemma is an analogue of Lemma 2.2 from [?] and can be proved by similar arguments.

5.13. Lemma. *Let $\{B_k^s\}$ be a good approximation of the sequence $\mathcal{B}$. Let W_a be an r.e. set. Then $\{W_{a,s}[k](B_k^s)\}$ is a correct approximation of $W_a(\mathcal{A})$.*

The proof of the following proposition can be found in [**?**].

5.14. Proposition. *Let $\mathcal{A} \leq_u {\emptyset_\omega}'$. Then there exists a sequence $\mathcal{P}$ of sets of natural numbers such that $\mathcal{P}(\mathcal{A}) \equiv_e \mathcal{P}$ and $\mathcal{P}$ has a good approximation.*

5.15. Theorem. *Let $\mathbf{a} \in I$. Then there exists an a.z. degree $\mathbf{d}$ such that $\mathbf{d} \not\leq_\omega \mathbf{a}$.*

Proof. Let $\mathbf{a} \in I$ and $\mathcal{A} \in \mathbf{a}$. Clearly $\mathcal{P}(\mathcal{A}) \leq_e \mathcal{P}({\emptyset_\omega}')$. Fix a sequence $\mathcal{P} = \{P_k\}_{k<\omega}$ such that $\mathcal{P} \equiv_e \mathcal{P}({\emptyset_\omega}')$ and there exists a good approximation $\{P_k^s\}$ of $\mathcal{P}$. Clearly $\mathcal{P}(\mathcal{A}) \leq_e \mathcal{P}$ and hence there exist a correct (with respect to $\{P_k^s\}$) approximation $\{P_k^s(\mathcal{A})\}$ of $\mathcal{P}(\mathcal{A})$.

We have that for all k, $P_k \not\leq_e \mathcal{P}_k(\mathcal{A})$. Indeed, assume that for some k, $P_k \equiv_e \mathcal{P}_k(\mathcal{A})$. Since $\mathcal{P} \equiv_e \mathcal{P}({\emptyset_\omega}')$ and $\mathcal{P}({\emptyset_\omega}') \equiv_e \{\emptyset^{(k+1)}\}_{k<\omega}$, we get that $\emptyset^{(k+1)} \equiv_e \mathcal{P}_k(\mathcal{A})$. Then for all $r \geq k$, $\emptyset^{(k+1+r)} \leq_e \mathcal{P}_{k+r}(\mathcal{A})$ uniformly in r which shows that $\mathcal{A}^{(k)} \equiv_e {\emptyset_\omega}^{(k+1)}$. Hence $\mathbf{a} \in H$. A contradiction.

5.16. Lemma. *Let V be an r.e. set satisfying the following requirements for all $k < \omega$:*

(F_k) *$V[k](P_k)$ is a finite set.*

(N_k) *$W_k(\mathcal{P}_k(\mathcal{A})) \neq V[k](P_k)$.*

Then $\mathbf{d} = d_\omega(V(\mathcal{P}))$ is a.z. and $\mathbf{d} \not\leq_\omega \mathbf{a}$.

Proof. Clearly the sequence $V(\mathcal{P}) = \{V[k](P_k)\}$ is uniformly reducible to ${\emptyset_\omega}'$ and $(\forall k)(V[k](P_k) \leq_e \emptyset^{(k)})$. Thus by Proposition 5.3 $\mathbf{d}$ is a.z. Assume that $\mathbf{d} \leq_\omega \mathbf{a}$. Then $V(\mathcal{P}) \leq_e \mathcal{P}(\mathcal{A})$ and hence there exist a primitive recursive function g such that for all k, $V[k](P_k) = W_{g(k)}(\mathcal{P}_k(\mathcal{A}))$. By the Recursion Theorem there exists a k such that $W_k = W_{g(k)}$ and hence $V[k](P_k) = W_k(\mathcal{P}_k(\mathcal{A}))$. A contradiction. $\square$

So to conclude the proof of the Theorem it is enough to construct an r.e. set V satisfying the requirements (F_k) and (N_k) for all k.

The construction of V will be performed on stages. At every stage s we shall construct effectively a finite set V_s so that $V_s \subseteq V_{s+1}$ and set $V = \bigcup V_s$.

Let $V_0 = \emptyset$ and suppose that V_s is constructed.

5.17. Definition. Given two sets X and Y of natural number let

$$l^s(X, Y) = \max\{n \leq s : (\forall x \leq n)(x \in X \iff x \in Y)\}.$$

For every $k \leq s$ we act for the requirement (N_k) as follows. Let

$$l_k^s = l^s(W_{k,s}(P_k^s(\mathcal{A})), V_s(P_k^s)).$$

For every $x \leq l_k^s$ if $x \in P_k^s$ then we enumerate $\langle\langle k, x\rangle, P_k^s\rangle$ in $V[k]$, i.e. we put $\langle k, \langle\langle k, x\rangle, P_k^s\rangle\rangle$ in V_{s+1}.

End of the construction

5.18. Lemma. *All requirements (N_k) are satisfied.*

Proof. Fix a k. Assume that $W_k(\mathcal{P}_k(\mathcal{A})) = V[k](P_k)$. Recall that a stage s is k-good if $P_k^s \subseteq P_k$.

We shall show that $(\forall x)(\langle k, x\rangle \in V[k](P_k) \iff x \in P_k)$. Indeed, let $\langle k, x\rangle \in V[k](P_k)$. Then there exists an axiom $\langle\langle k, x\rangle, D\rangle \in V[k]$ such that $D \subseteq P_k$. From the construction of V it follows that this axiom is enumerated by the requirement (N_k) an hence for some s, $D = P_k^s$ and $x \in P_k^s$. Since $P_k^s = D \subseteq P_k$, $x \in P_k$.

Suppose now that $x \in P_k$. Since $W_k(\mathcal{P}_k(\mathcal{A})) = V[k](P_k)$ there exists a k-good stage s such that $x \leq l_k^s$ and $x \in P_k^s$. Then, by the construction of V, $\langle k, x\rangle \in V_{s+1}[k](P_k^s)$ and hence $\langle k, x\rangle \in V[k](P_k)$.

Thus $(\forall x)(x \in P_k \iff \langle k, x\rangle \in V[k](P_k) \iff \langle k, x\rangle \in W_k(\mathcal{P}_k(\mathcal{A})))$. Hence $P_k \leq_e \mathcal{P}_k(\mathcal{A})$. A contradiction. $\square$

5.19. Lemma. *All requirements (F_k) are satisfied.*

Proof. Fix a k. By the construction of V for all y,

$$y \in V[k](P_k) \iff (\exists x, s)(y = \langle k, x\rangle \ \&\ \langle y, P_k^s\rangle \in V[k] \ \&\ P_k^s \subseteq P_k).$$

Notice that an axiom of the form $\langle\langle k, x\rangle, P_k^s\rangle$ can be enumerated in $V[k]$ only by the requirement (N_k).

By the previous Lemma $W_k(\mathcal{P}_k(\mathcal{A})) \neq V[k](P_k)$. Fix a n such that

$$W_k(\mathcal{P}_k(\mathcal{A}))(n) \neq V[k](P_k)(n).$$

By the definition of the good approximations there exist a stage v such that for all k-good stages $s \geq v$, $l_k^s < n$. Hence if at a k-good stage s, $\langle\langle k, x\rangle, P_k^s\rangle$ is enumerated in $V[k]$, then $x \leq s < v$ or $x < n$. Thus $V[k](P_k)$ is finite. $\square$

The proof of the Theorem is completed. $\square$

Proof of Theorem 5.8. Let $\mathbf{a} \leq_\omega \mathbf{0}_\omega{}'$. Assume that $\mathbf{a} \in H$. Then $\mathbf{a} \in H_n$ for some $n \geq 1$ and hence $o_n \leq_\omega \mathbf{a}$. Therefore for all a.z. $\mathbf{d}$, $\mathbf{d} \leq_\omega o_n \leq_\omega \mathbf{a}$.

Assume now that $\mathbf{a}$ is above all a.z. degrees. Let $\mathbf{d}$ be a nonzero a.z. degree. Then for all n, $\mathbf{0}_\omega{}^{(n)} <_\omega \mathbf{d}^{(n)} \leq_\omega \mathbf{a}^{(n)}$ and hence $\mathbf{a} \notin L$. By the previous Theorem, $\mathbf{a} \notin I$. Thus $\mathbf{a} \in H$. $\square$

5.20. Theorem. *Let $\mathbf{a} \in I$. There exists a nonzero a.z. degree $\mathbf{d} \leq_\omega \mathbf{a}$.*

Proof. Fix a sequence $\mathcal{A} \in \mathbf{a}$ and let $\{P_k^s\}$ be a good approximation of a sequence $\mathcal{P}$ such that $\mathcal{P} \equiv_e \mathcal{P}(\mathcal{A})$. Clearly $\mathcal{P}(\emptyset_\omega) \leq_e \mathcal{P}$ and hence there exists a correct (with respect to $\{P_k^s\}$) approximation $\{Z_k^s\}$ of $\mathcal{P}(\emptyset_\omega)$.

Given a sequence $\mathcal{B} = \{B_k\}_{k<\omega}$, let $\mathcal{B}^* = \{B_{k+1}\}_{k<\omega}$. Notice that $\mathcal{B}' = \mathcal{P}(\mathcal{B})^*$. Set $\mathcal{B}^{(0*)} = \mathcal{B}$ and $\mathcal{B}^{((n+1)*)} = \mathcal{B}^{(n*)^*}$.

Clearly if $\mathcal{B} \leq_e \mathcal{C}$ then for all n, $\mathcal{B}^{(n*)} \leq_e \mathcal{C}^{(n*)}$ and there exists a recursive function g such that $(\forall n)(\mathcal{B}^{(n*)} = W_{g(n)}(\mathcal{C}^{(n*)}))$. In particular, for all n, $\mathcal{B}^{(n*)} \leq_e \mathcal{B}^{(n)} = \mathcal{P}(\mathcal{B})^{(n*)}$.

Clearly $(\forall n)(\mathcal{A}^{(n)} = \mathcal{P}(\mathcal{A})^{(n*)} \equiv_e \mathcal{P}^{(n*)})$.

Notice that if $\{B_k^s\}$ is a good approximation of $\mathcal{B}$ then $\{B_{n+k}^s\}$ is a good approximation of $\mathcal{B}^{(n*)}$. Hence $\{P_{n+k}^s\}$ is a good approximation of $\mathcal{P}^{(n*)}$ and $\{Z_{n+k}^s\}$ is a correct (with respect to $\{P_{n+k}^s\}$) approximation of $\emptyset_\omega{}^{(n)}$.

We shall construct an r.e. set V satisfying the following requirements for all $i \in \mathbb{N}$:

(F_i) $V[i](\mathcal{P}_i) \leq_e \emptyset^{(i)}$.

(N_i) $W_i(\emptyset_\omega{}^{(i)}) \neq V(\mathcal{P})^{(i*)}$.

5.21. Lemma. *Suppose that V is an r.e. set satisfying for all i the requirements (F_i) and (N_i). Then $\mathbf{d} = d_\omega(V(\mathcal{P}))$ is a nonzero a.z. degree below $\mathbf{a}$.*

Proof. Clearly $\mathbf{d} \leq_\omega \mathbf{a}$. Since V satisfies the requirements (F_i) the degree $\mathbf{d}$ is a.z. It remains to show that $\mathbf{d} \neq \mathbf{0}_\omega$. Assume that $\mathbf{d} = \mathbf{0}_\omega$. Then $V(\mathcal{P}) \leq_e \mathcal{P}(\emptyset_\omega)$ and hence there exists a recursive function g such that for all i,

$$V(\mathcal{P})^{(i*)} = W_{g(i)}({\emptyset_\omega}^{(i)}).$$

By the Recursion Theorem there exists an i such that $W_i = W_{g(i)}$. Then

$$V(\mathcal{P})^{(i*)} = W_i({\emptyset_\omega}^{(i)}).$$

A contradiction. □

We shall construct V on stages. At every stage s we shall define effectively a finite set V_s so that $V_s \subseteq V_{s+1}$ and let $V = \bigcup V_s$.

Set $V_0 = \emptyset$ and suppose that V_s is defined.

5.22. Definition. Given sequences $\mathcal{X} = \{X_k\}$ and $\mathcal{Y} = \{Y_k\}$, let

$$l^s(\mathcal{X}, \mathcal{Y}) = \max\{u : u \leq s \ \& \ (\forall \langle k, x\rangle \leq u)(X_k(x) = Y_k(x))\}.$$

For every $i \leq s$ we act for the requirement (N_i) as follows. Let

$$l_i^s = l_s(W_{i,s}(\{Z_{i+k}^s\}_{k<\omega}), \{V_s[i+k](P_{i+k}^s)\}_{k<\omega}).$$

For every pair $\langle k, x\rangle \leq l_i^s$ such that $x \in P_{i+k}^s$ we enumerate the axiom $\langle\langle i, x\rangle, P_{i+k}^s\rangle$ in $V[i+k]$.

End of the construction.

Notice that for every j the set $V[j]$ consists of pairs $\langle\langle i, x\rangle, P_j^s\rangle$, where $i \leq j$.

5.23. Lemma. *All requirements (N_i) are satisfied.*

Proof. Fix an i and suppose that $W_i({\emptyset_\omega}^{(i)}) = V(\mathcal{P})^{(i*)}$. We shall show that for all k,

$$\langle i, x\rangle \in V[i+k](\mathcal{P}_{i+k}) \iff x \in \mathcal{P}_{i+k}. \tag{2}$$

Let $\langle i, x\rangle \in V[i+k](\mathcal{P}_{i+k})$. Then there exists an axiom $\langle\langle i, x\rangle, D\rangle \in V[i+k]$ such that $D \subseteq \mathcal{P}_{i+k}$. By the construction of V, $D = P_{i+k}^s$ for some s such that $x \in P_{i+k}^s$. Hence $x \in \mathcal{P}_{i+k}$.

Assume now that $x \in \mathcal{P}_{i+k}$. There exists an $(i+k)$-good stage s such that $i \leq s$, $\langle k, x\rangle \leq l_i^s$ and $x \in P_{i+k}^s$. Then $\langle\langle i, x\rangle, P_{i+k}^s\rangle \in V_{s+1}[i+k]$ and hence $\langle i, x\rangle \in V[i+k](\mathcal{P}_{i+k})$.

It follows from (2) that

$$(\forall k, x)(\langle i, x\rangle \in W_i[k](\mathcal{P}_{i+k}(\emptyset_\omega)) \iff x \in \mathcal{P}_{i+k}).$$

and hence $\mathcal{A}^{(i)} \equiv_e \mathcal{P}^{(i*)} \leq_e {\emptyset_\omega}^{(i)}$. The last shows that $\mathbf{a} \in L$. A contradiction. □

5.24. Lemma. *All requirements (F_j) are satisfied.*

Proof. Let us fix a $j \in \mathbb{N}$. We need to show that $V[j](\mathcal{P}_j) \leq_e \emptyset^{(j)}$. Clearly

$$V[j](\mathcal{P}_j) = \bigcup_{i \leq j} \{\langle i, x\rangle : \langle i, x\rangle \in V[j](\mathcal{P}_j)\}.$$

So it is enough to show that for every $i \leq j$,

$$X_i = \{x : \langle i, x\rangle \in V[j](\mathcal{P}_j)\} \leq_e \mathcal{P}_j(\emptyset_\omega).$$

Fix an $i \leq j$ and set $k = j - i$. We shall consider two cases:

a) There exists a $u \in \mathbb{N}$ such that for all j-good stages $s \geq i$, $l_i^s \leq u$. Suppose that $\langle i, x\rangle \in V[j](\mathcal{P}_j)$. Then there exist a j-good stage $s \geq i$ such hat $\langle k, x\rangle \leq l_i^s \leq u$. Hence X_i is finite.

b) For every u there exists a j-good stage $s \geq i$ such that $u < l_i^s$.

We shall show that $V[j](\mathcal{P}_j) = W_i[k](\mathcal{P}_j(\emptyset_\omega))$.

Let $x \in W_i[k](\mathcal{P}_j(\emptyset_\omega))$. By the properties of the correct approximations there exists a v such that for all j-good stages $s \geq v$, $x \in W_{i,s}[k](Z_j^s)$. Let s be a j-good stage such that $\max(v, \langle k, x\rangle) \leq l_i^s$. Then $v \leq l_i^s \leq s$. Clearly $x \in W_{i,s}[k](Z_j^s)$. Hence $x \in V_s[j](P_j^s)$ and therefore $x \in V[j](\mathcal{P}_j)$.

Let $x \in V[j](\mathcal{P}_j)$. Fix a v such that for all j-good stages $s \geq v$, $x \in V_s[j](P_j^s)$. Consider a j-good stage $s \geq v$ such that $\langle k, x\rangle \leq l_i^s$. Ten $x \in W_{i,s}[k](Z_j^s)$ and hence $x \in W_i[k](\mathcal{P}_j(\emptyset_\omega))$.

So we obtain that

$$x \in X_i \iff \langle i, x\rangle \in W_i[k](\mathcal{P}_j(\emptyset_\omega))$$

Hence $X_i \leq_e \mathcal{P}_j(\emptyset_\omega)$. □

The proof of the Theorem is concluded. □

Proof of Theorem 5.9. Let $\mathbf{a} \leq_\omega \mathbf{0}_\omega{}'$.

Assume that the only a.z. degree below $\mathbf{a}$ is $\mathbf{0}_\omega$. Since there exist nonzero a.z. degrees $\mathbf{a} \notin H$. By the previous Theorem $\mathbf{a} \notin I$. Thus $\mathbf{a} \in L$.

Suppose now that $\mathbf{a} \in L$. Let $\mathbf{d}$ be an a.z. degree below $\mathbf{a}$. Then for some n, $\mathbf{d}^{(n)} \leq_\omega \mathbf{a}^{(n)} = \mathbf{0}_\omega{}^{(n)}$. Hence $\mathbf{d}^{(n)} = \mathbf{0}_\omega{}^{(n)}$. Therefore $\mathbf{d} = \mathbf{0}_\omega$. □

References

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