Total Degrees and Nonsplitting Properties of Σ_2^0 Enumeration Degrees

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This paper continues the project, initiated in [ACK], of describing general conditions under which relative splittings are derivable in the local structure of the enumeration degrees.

The main results below include a proof that any high total e-degree below $\mathbf{0}'_e$ is splittable over any low e-degree below it, and a construction of a Π_1^0 e-degree unsplittable over a Δ_2 e-degree below it.

In [ACK] it was shown that using semirecursive sets one can construct minimal pairs of e-degrees by both effective and uniform ways, following which new results concerning the local distribution of total e-degrees and of the degrees of semirecursive sets enabled one to proceed, via the natural embedding of the Turing degrees in the enumeration degrees, to results concerning embeddings of the diamond lattice in the e-degrees. A particularly striking application of these techniques was a relatively simple derivation of a strong generalisation of the Ahmad Diamond Theorem.

This paper extends the known constraints on further progress in this direction, such as the result of Ahmad and Lachlan [AL98] showing the existence of a nonsplitting Δ_2^0 e-degree $> \mathbf{0}_e$, and the recent result of Soskova [Sos07] showing that $\mathbf{0}'_e$ is unsplittable in the Σ_2^0 e-degrees above some Σ_2^0 e-degree $< \mathbf{0}'_e$. This work also relates to results (e.g. Cooper and Copestake [CC88]) limiting the local distribution of total e-degrees.

For further background concerning enumeration reducibility and its degree structure, the reader is referred to Cooper [Co90], Sorbi [Sor97] or Cooper [Co04], chapter 11.

Theorem 1 *If* $\mathbf{a} < \mathbf{h} \leq \mathbf{0}'$, $\mathbf{a}$ *is low and* $\mathbf{h}$ *is total and high then there is a low total e-degree* $\mathbf{b}$ *such that* $\mathbf{a} \leq \mathbf{b} < \mathbf{h}$.

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Corollary 2 *Let* $\mathbf{a} < \mathbf{h} \leq \mathbf{0}'$, $\mathbf{h}$ *be a high total e-degree,* $\mathbf{a}$ *be a low e-degree. Then there are* Δ^0_2 *e-degrees* $\mathbf{b}_0 < \mathbf{h}$ *and* $\mathbf{b}_1 < \mathbf{h}$ *such that* $\mathbf{a} = \mathbf{b}_0 \cap \mathbf{b}_1$ *and* $\mathbf{h} = \mathbf{b}_0 \cup \mathbf{b}_1$.

Proof. Immediately follows from Theorem 1, and Theorem 6 of [ACK]. □
Proof of Theorem 1. Assume A has low e-degree, $H \oplus \overline{H}$ has high e-degree (i.e., H has high Turing degree) and $A \leq_e H \oplus \overline{H}$.

We want to construct an H-computable increasing sequence of initial segments $\{\sigma_s\}_{s\in\omega}$ such that the set $B = \cup_s \sigma_s$ satisfies the requirements

$$P_n : n \in A \iff (\exists y)[\langle n, y\rangle \in B]$$

and

$$R_n : (\exists \sigma \subset B)[n \in W^\sigma_n \vee (\forall \tau \supset \sigma)[\tau \in S^A \implies n \notin W^\tau_n]]$$

for each $n \in \omega$, where

$$S^A = \{\tau : (\forall x)(\forall y)[\tau(\langle x, y\rangle) \downarrow = 1 \implies x \in A]\}.$$

Note that P_n-requirements guarantee that $A \leq_e B$, and hence $A \leq_e B \oplus \overline{B}$. To prove that the R_n-requirements provide $B' \equiv_T \emptyset'$, first note that $S^A \equiv_e A$, which has low e-degree, and

$$X = \{\langle \sigma, n\rangle : (\exists \tau \supset \sigma)[\tau \in S^A \;\&\; n \in W^\tau_n]\} \leq_e S^A.$$

Then $X \in \Delta^0_2$ and

$$n \notin B' \iff (\exists \sigma \subset B)[\langle \sigma, n\rangle \notin X],$$

so that B' is co-c.e. in $B \oplus \emptyset' \equiv_T \emptyset'$. Thus $B' \leq_T \emptyset'$ by Post's Theorem.

Since the set B will be computable in H, the set

$$Q = \{n : (\forall \sigma \subset B)(\exists \tau \supset \sigma)[\tau \in S^A \;\&\; n \in W^\tau_n]\}$$

will be computable in $(H \oplus \emptyset')' \equiv_T H'$ – indeed, we have $n \in Q \iff (\forall \sigma \subset B)[\langle \sigma, n\rangle \in X]$, so that Q is co-c.e. in $H \oplus \emptyset'$. Now to construct the desired set B we can apply the Recursion Theorem and fix an H-computable function g such that $Q(x) = \lim_s g(x, s)$.

Let $\{A_s\}_{s\in\omega}$ and $\{S^A_s\}_{s\in\omega}$ be respective H-computable enumerations of A and S^A.

Construction.

Stage $s = 0$. $\sigma_0 = \lambda$.

Stage $s + 1 = 2\langle n, z\rangle$ (to satisfy P_n). Given σ_s define $l = |\sigma_s|$.

If $n \notin A_s$, then let $\sigma_{s+1} = \sigma_s{}^\frown 0$.

If $n \in A_s$, then choose the least $k \geq l$ such that $k = \langle n, y \rangle$ for some $y \in \omega$ and define $\sigma_{s+1} = \sigma_s{}^\frown 0^{k-l}{}^\frown 1$ (so that $\sigma_{s+1}(k) = 1$).

Stage $s+1 = 2\langle n, z \rangle + 1$ (to satisfy R_n). H-computably find the least stage $t \geq s$ such that either $g(n,t) = 0$, or $n \in W^\tau_{n,t}$ for some τ satisfying $\tau \in S^A_t$ and $\tau \supset \sigma_s$. (Such stage t exists since if $\lim_s g(n,s) = 1$ then $n \in Q$, and hence there exists some $\tau \supset \sigma_s$ such that $n \in W^\tau_n$ and $\tau \in S^A$.)

If $g(n,t) = 0$ then define $\sigma_{s+1} = \sigma_s{}^\frown 0$.

Otherwise, choose the first $\tau \supset \sigma_s$ such that $\tau \in S^A_t$ and $n \in W^\tau_{n,t}$. Define $\sigma_{s+1} = \tau$.

This completes the description of the construction.

Let $B = \cup_s \sigma_s$. Clearly $B \leq_T H$ since each σ_s is obtained effectively in H. Each P_n-requirement is satisfied by the even stages of the construction since $\sigma_s \in S^A$ for any $s \in \omega$.

To prove that each R_n-requirement is met suppose that

$$(\forall \sigma \subset B)(\exists \tau \supseteq \sigma)[\tau \in S^A \;\&\; n \in W^\tau_n]$$

for some n. This means that $n \in Q$. Choose any odd stage $s = 2\langle n, z \rangle + 1$ such that $g(n,t) = 1$ for all $t \geq s$. Then by the construction $n \in W^{\sigma_s}_n$.

Hence $A \leq_e B \oplus \overline{B} \leq_e H \oplus \overline{H}$, and $\deg_e(B \oplus \overline{B})$ is low. □

Theorem 3 *There is a Π^0_1 e-degree* $\mathbf{a}$ *and a* 3*-c.e. e-degree* $\mathbf{b} < \mathbf{a}$ *such that* $\mathbf{a}$ *is not splittable over* $\mathbf{b}$.

Proof. We construct a Π^0_1 set A and 3-c.e. set B satisfying both the global requirement:

$G : B = \Omega(A)$,

and the requirements

$$R_{\Xi,\Psi,\Theta} : A = \Xi(\Psi(A) \oplus \Theta(A)) \Longrightarrow (\exists \text{ e-operator } \Gamma) A = \Gamma(\Psi(A) \oplus B) \vee$$
$$(\exists \text{ e-operator } \Lambda) A = \Lambda(\Theta(A) \oplus B)$$

for each triple of e-operators Ξ, Ψ, Θ, and

$$N_\Phi : A \neq \Phi(B)$$

for each e-operator Φ.

In fact A will be constructed as a 2-c.e. set. Note that the e-degrees of Π_1 sets coincide with the e-degrees of 2-c.e. sets. Hence this will still produce the desired enumeration degrees.

Basic Strategies

Suppose we have an effective listing of all requirements $R_1, R_2, \ldots$ and $N_1, N_2, \ldots$ The requirements will then be arranged by priority in the following way: $G < R_1 < N_1 < R_2 < N_2 < \ldots$

To satisfy the requirement G we will make sure that every time we enumerate an element into the set B, we enumerate a corresponding axiom into the set Ω; and every time we extract an element from B, we make the corresponding axiom invalid by extracting elements from A. More precisely every element y that enters B will have a corresponding marker m in A and an axiom $\langle y, \{m\}\rangle$ in Ω. If y is extracted from B then we extract m from A. If y is later re-enumerated into B – this can happen since B is 3-c.e. – then we will just enumerate the axiom $\langle y, \emptyset\rangle$ into Ω.

To satisfy the requirements R_i we will initially try to construct an operator Γ using information from both of sets B and $\Psi(A)$. Again, enumeration of elements into A is always accompanied by enumeration of axioms into Γ, and extraction of elements from A can be rectified via B-extractions.

The N-strategies follow a variant of the Friedberg -Muchnik strategy while at the same time respecting the Γ-rectification, so we will call them (N_Φ, Γ)-strategies. They choose a follower x, enumerate it in A, then wait until $x \in \Phi(B)$. If this happens - they extract the element x from A while restraining $B \upharpoonright \varphi(x)$ in B. The need to rectify Γ after the extraction of the follower x from A can be in conflict with the restraint on B. To resolve this conflict we try to obtain a change in the set $\Psi(A)$ which would enable us to rectify Γ without any extraction from the set B. To do this we monitor the length of agreement

$$l_{\Xi,\Psi,\Theta}(s) = \max\{y : (\forall y < x)[y \in A[s] \iff y \in \Xi(\Psi(A) \oplus \Theta(A))[s]]\}.$$

We only proceed with actions directed at a particular follower once it is below the length of agreement. This ensures that the extraction of x from A will have one of the following consequences

1. The length of agreement will never return so long as at least one of the axioms that ensure $x \in \Xi(\Psi(A) \oplus \Theta(A))$ remains valid.
2. There is a useful change in the set $\Psi(A)$.
3. There is a useful change in the set $\Theta(A)$.

We will initially assume that it is the case that the third consequence is true and commence a backup strategy (N_Φ, Λ) which is devoted to building an enumeration operator Λ with information from A and $\Theta(A)$. This is a new copy of the N-strategy working with the same follower. It will try to make use of this change in $\Theta(A)$ to satisfy the requirement. Only when we are provided with evidence that our assumption is wrong will we return to the initial strategy (N_Φ, Γ).

Basic module for an N_Φ-strategy below one $R_{\Xi,\Psi,\Theta}$-strategy

We will first consider the simple case involving just two requirements. Assume we have N_Φ, which we refer to as the N-requirement, below $R_{\Xi,\Psi,\Theta}$, which we refer to as the R-requirement.

At the root we have the R-strategy denoted by (R, Γ). It will have two outcomes $e <_L gw$. The R-strategy will monitor all elements $x \notin A$. In the case in which there is an element $x \notin A$ such that $x \in \Gamma(\Psi(A) \oplus B)$ the operator Γ cannot be rectified. The (R, Γ)-strategy will then have outcome gw, and we will be able to argue that $x \in \Xi(\Psi(A) \oplus \Theta(A))$, which indicates a global win for the R-requirement. Strategies working below this outcome will follow a simple Friedberg-Muchnik strategy and preserve the difference at x by using followers of big enough value. In case there is no such x the operator, Γ can be rectified and the (R, Γ)-strategy will have outcome e.

Below e we will try to meet N satisfying $A = \Gamma(\Psi(A) \oplus B)$. The (N, Γ)-strategy will have four outcomes: three finitary outcomes, f, w and l, and one infinitary outcome λ. The outcomes are arranged in the following way: $\lambda <_L f <_L w <_L l$. Outcome l indicates that at that node the R-requirement is globally satisfied since the follower x enumerated in A is not in $\Xi(\Psi(A) \oplus \Theta(A))$. Outcome w indicates that Γ is correct on x and the N-requirement is satisfied as $x \in A - \Phi(B)$. Outcome f is only accessible once a follower x has been returned. It will indicate that Γ is again correct on x and the N-requirement is satisfied via $x \in \Phi(B) - A$.

Below outcome λ strategies will be devoted to constructing an operator Λ with $A = \Lambda(\Theta(A) \oplus B)$ where they will receive their followers from (N, Γ). Again we have a controlling strategy (R, Λ) with only one outcome e which makes sure that the operator Λ can be rectified at all times. In case it sees an element $x \notin A$ for which the axiom enumerated in Λ is valid, it will send x back to (N, Γ). We will be able to argue that x has provided evidence of a useful change in $\Psi(A)$.

Below (R, Λ)'s only outcome e we try to meet N by (N, Λ) with $A = \Lambda_\Phi(\Theta(A) \oplus B)$. The strategy below the outcome λ acts only when the (N, Γ)-strategy *sends* its follower x. It performs similar actions with regard to (N, Γ) and has two outcome $f <_L w$ both indicating that the N-requirement is satisfied and the operator Λ remains intact.

The R strategy:

1. Scan all followers $x \notin A$ defined up to the current stage.
2. If $x \in \Gamma(\Psi(A) \oplus B)$, then let the outcome be $o = gw$.
3. If all followers are scanned and none has produced an outcome $o = gw$, then let the outcome be $o = e$.

The (N, Γ) strategy:

At stage s the strategy will start its work at the step of the module indicated at the previous stage.

Setup 1) Choose a new follower x as a fresh number (bigger than any previously set up restraint). Enumerate it into A_s.

2) If there are finite sets $G(x), H(x), L(x)$ with $x \in \Xi(G(x) \oplus H(x))$, $G(x) \subset \Psi(L(x))$, $H(x) \subset \Theta(L(x))$ and $L(x) \subset A$ then restrain A on $\max(L(x))$ and go to *Setup 3*. Otherwise let the outcome be $o = l$ and return to *Setup*2) at the next stage.

3) Define $x's$ B-marker $y(x)$, along with its corresponding A-marker $m(x)$, as fresh numbers bigger than any previously set restraint on A or B. Enumerate $y(x)$ in B_s and $m(x)$ in A_s. Define a new axiom $\langle y(x), \{m(x)\}\rangle$ for Ω_s. Enumerate each $\langle z, G_x \oplus B \restriction y(x)\rangle$ into Γ where z is either x, or $m(x)$, or a follower $z \in A$ from a previous cycle of the strategy. Note that we enumerate axioms for previous followers as well. So at this point the operator Γ is rectified. Let the outcome be $o = w$. Go to $Wait$ at the next stage.

Wait If $x \in \Phi(B_s)$ then go to *Attack*. Otherwise let the outcome be $o = w$ and return to $Wait$ at the next stage.

Attack 1) Check if any previously sent follower has been returned. If so go to *Result*. Otherwise go to *Attack*2.

2)Let $v(x) = \max(\varphi(x), y(x))$ and restrain B on $v(x)$. Extract $y(x)$ from B_s and $m(x)$ from A_s, noting that x is still in $\Xi(\Psi(A) \oplus \Theta(A))$ as the marker $m(x)$ is chosen as a fresh number after $G(x)$ and $H(x)$ are already defined. *Send* x. Let the outcome be $o = \lambda$. At the next stage start from *Setup*1, choosing a new current follower. The strategy working below outcome λ will believe B only below a right boundary $R_s = y(x)$. Note that the next follower will choose its B-marker of greater value. So if the outcome λ is visited infinitely often then the right boundary R will grow unboundedly.

Result Let the returned follower be x. Put $y(x)$ into B_s and $\langle y(x), \emptyset\rangle$ into Ω_s. For each follower z of this strategy such that $z \in A$ put the axiom $\langle z, \emptyset\rangle$ into Γ_s.

1) For the returned follower we know that $x \notin A_s$ and $H(x) \subset \Theta(A_s)$. The outcome λ will not be accessible anymore so we can preserve $H(x) \subseteq \Theta(A_t)$ at further stages t. Also if $G(x) \subseteq \Psi(A_s)$ then the (R, Γ)-strategy would have outcome gw preserving the difference and satisfying R globally. The (N, Γ)-strategy would not be accessible any longer. Otherwise $G(x) \nsubseteq A$ and the outcome is $o = f$. Return at $Result1$ at the next stage.

The (R, Λ)-strategy below outcome λ :

1. Scan all followers $x \notin A$.
2. If $x \in \Lambda(\Theta(A) \oplus B)$ then return x. End this stage.
3. If all followers are scanned and none have been returned then let the outcome be e.

The (N, Λ)-strategy below outcome λ :

Setup 1) Let $x \in A$ be a new integer which was sent by the (N, Γ)-strategy. Now x becomes the *follower* of the (N, Λ)-strategy. Go to $Setup2$.
2) Put $\langle x, H_x \oplus B \restriction v(x) \rangle$ into Λ. Go to $Wait$.

Wait If $x \in \Phi(B)$ with use $\varphi(x) < R_s$ then go to *Attack*. Otherwise the outcome is $o = w$, return to $Wait$ at the next stage.

Attack Extract x from A. Go to result.

Result Let the outcome be $o = f$. Return to *Result* at the next stage.

The (N, FM)-strategy below outcome l or gw :

Setup Choose a new follower x bigger than any previously set restraint on A and enumerate it into A. Go to $Wait$.

Wait If $x \in \Phi(B)$ go to *Attack*. Otherwise the outcome is $o = w$, return to $Wait$ at the next stage.

Attack Extract x from A and go to *Result*.

Result Let the outcome be $o = f$. Return to *Result* at the next stage.

Now the (N, FM) strategy below outcome l will also be changing A. To keep Γ and Λ rectified, every time we initialise the (N, FM)-strategy and cancel its follower x, if $x \in A$ we will add the axiom $\langle x, \emptyset \rangle$ in Γ and Λ.

If the (R, Γ)-strategy has outcome gw on stage s for the first time, then the (N, FM)-strategy working below will will be initialised on the previous stage and will choose its follower x anew, respecting the restraint on A that (N, Γ) has set up. So (R, Γ) will have outcome gw on all further stages and B will not be modified any longer. The (N, FM)-strategy will be able to satisfy its requirement.

Suppose that (R, Γ)-strategy never has outcome gw. We will analyse all possible outcomes of the N-strategies and see that in each case the requirements are satisfied.

Consider first the possible outcomes of the strategy (N, Γ). If one of the cycles stops at $Setup2$, i.e. on all stages $t > s$ the strategy has outcome l, then the true outcome will be $(o = l)$. The length of agreement $l_{\Xi,\Psi,\Theta}(s) = \max\{y : (\forall y < x)[y \in A[s] \iff y \in \Xi(\Psi(A) \oplus \Theta(A))[s]]\}$ is bounded and hence the requirement R is trivially satisfied.

The set B is not modified after stage s and the simple strategy (N, FM), active on all stages $t \geq s$ succeeds to satisfy the requirement N.

Suppose now that no cycle of the (N, Γ)-strategy stops at $Setup2$. In this case the (N, FM)-strategy may be activated infinitely many times and will be initialised every time the $(N\Gamma)$-strategy moves on to $Wait$. The current follower x of the (N, FM)-strategy will be cancelled and if it is not yet extracted from A the corresponding axiom $\langle x, \emptyset\rangle$ will be enumerated in Γ and Λ. This ensures that both operators will be correct at x for all cancelled followers x of the strategy (N, FM).

We first consider the case when the (N, Γ)-strategy during its work sends only finitely many integers. Then some cycle with a follower x stops either at $Wait$ or reaches $Result$. If the cycle stops at $Wait$ then the outcome is $o = w$ and $x \in A - \Phi(B)$, hence the N-requirement is satisfied. On the other hand for all followers z we have $z \in A \iff z \in \Gamma(\Psi(A) \oplus B)$ and $m(z) \in A \iff m(z) \in \Gamma(\Psi(A) \oplus B)$ since $y(z) \in B \iff z = x$. Hence Γ is correct at all followers z.

If the cycle reaches $Result$ then we have $y(x) \in B$ and hence $x \in \Phi(B) - A$, so N is satisfied. Also $H_x \subseteq \Theta(A)$ via some finite set $P_x \subset A$. If $G_x \subseteq \Psi(A)$ then this will be apparent at some finite stage s, i.e. on stage s we will see a finite set $Q_x \subset A$ such that $G_x \subseteq \Psi(Q_x)$. Then from stage s on the (R, Γ)-strategy will have outcome $o = gw$, contradicting our assumption. So $G_x \not\subseteq \Psi(A)$ giving $x \notin \Gamma(\Psi(A) \oplus B)$. Since again $y(z) \in B \iff z = x$ we have $z \in A \iff z \in \Gamma(\Psi(A) \oplus B)$ and $m(z) \in A \iff m(z) \in \Gamma(\Psi(A) \oplus B)$ for any follower z. Hence the operator Γ remains correct at all further stages.

Suppose now that the (N, Γ)-strategy during its work sends infinitely many integers. In particular, no x is returned to (N, Γ). Then the true outcome is $o = \lambda$ and we will see that the (N, Λ)-strategy is successful.

If the (N, Λ)- strategy stops at $Wait$ then $x \in A - \Phi(B)$. Indeed if we assume that $x \in \Phi(B)$ then there is some finite $M_x \subset B$ such that $x \in \Phi(M_x)$. The right boundary R grows unboundedly, so eventually there will be a stage s with $R_s > \max(M_x)$ and the strategy will move on to *Attack*.

The second case is if the strategy reaches *Result*. Then $x \in \Phi(B) - A$ because at some stage s we found a set $M_x \subset B_s$ with $\max M_x < R$ such that $x \in \Phi(M_x)$. The strategy (N, Γ) will not extract any more markers from B after stage s that are below the right boundary R_s, hence $x \in \Phi(B)$.

At this stage of the construction we can only prove that Λ will be correct at the follower x and all cancelled followers of the strategy (N_Φ, FM). To prove that the operator is correct at the rest of the followers enumerated in A by the (N, Γ)-strategy we will need to consider how all N-strategies will work together.

Basic module for many N_Φ-strategies under one $R_{\Xi,\Psi,\Theta}$-strategy

We will try to meet all requirements $N_{\Phi_1}, N_{\Phi_2}, \ldots$. Each requirement N_{Φ_j} will be denoted by N_j and met by one of the following strategies:

1. (N_j, Γ) with outcomes λ, f, w and l;
2. (N_j, FM) with outcomes f and w and situated in the subtree of the strategy (N_i, Γ) with outcome l, where $i \leq j$.
3. (N_j, Λ) with outcomes f and w and situated in the subtree of the strategy (N_i, Γ) with outcome λ where where $i \leq j$.

We now need to be more careful as more strategies will enumerate and extract markers from A and B. We will have to ensure that the operator constructed on the true path is correct and manages to satisfy the R-requirement.

The first rule that we will implement in order to achieve this follows the idea of cancelling followers of the (N, FM)-strategy from the previous section. Namely, whenever we initialise a strategy (N_j, S) on an node α in the tree of strategies whose follower x is in A we will enumerate an axiom $\langle x, \emptyset \rangle$ into all operators Γ and Λ that are constructed on nodes $\beta < \alpha$. If $m(x)$ is in A we will also enumerate an axiom $\langle m(x), \emptyset \rangle$ into these operators.

Secondly we will be more careful when enumerating axioms in the corresponding operators. Instead of just using the sets $G(x)$ and $H(x)$, we will use the information from all axioms defined up until now. More precisely we will modify the modules of the strategies from the previous section in the following way:

The (N_j, Γ)**-strategy** is the same as the as the (N_Φ, Γ)-strategy with the exception of step *Setup*3, which is now as follows:

*Setup*3) Enumerate all $\langle z, G_x \oplus B \upharpoonright y_x \cup U\rangle$ into Γ where z is either x, or m_x, or a follower $z \in A$ from a previous cycle of the strategy and U is the union of all sets D such that $\langle v, D\rangle$ is a valid axiom in Γ, where $v \in A$ is a follower of the strategy (N_i, Γ) with $i < j$.

The (N_{Φ_j}, Λ)**-strategy** is the same as the (N_Φ, Λ)-strategy with the exception of *Setup*2), which is now as follows:

*Setup*2) Enumerate $\langle x, (H_x \oplus B \upharpoonright v(x)) \cup U\rangle$ into Λ where U is the union of all finite sets D such that $\langle v, D\rangle \in \Lambda$ for some follower $v \in A$ of an (N_k, Λ)-strategy with $k < j$.

The main idea behind the added sets U in the axioms is that a strategy α working below another strategy β where α and β construct the same operator O believes that $\beta' s$ work is final and the axioms enumerated in O by β will remain true. In the case that β changes its mind and invalidates one of these axioms α will be initialised as β will have an outcome to the left of α. If $\alpha' s$ followers are still in A then an axioms for them will be enumerated in the operator as stated in above. But if $\alpha' s$ follower is not in A, then we need to ensure that there isn't a valid axiom in O for it. α will not be able to monitor this follower any longer, so the job is going to be transferred to β automatically via the set U which includes an axiom for $\beta' s$ follower, which β observes and makes sure is invalid.

Two *R*-requirements

Now we need to consider the case when there are two R-requirements. Corresponding to them there are nodes on the tree: an (R_1, Γ_1)-strategy and an (R_2, Γ_2)-strategy along each path, scanning for an appropriate global win for the R-requirements. Below outcome gw for an R_i-strategy the N-requirements simply ignore the requirement R_i and act as in the previous section.

There now more possibilities for an N-strategy working below outcomes e of both (R_i, Γ_i)-strategies depending on how it believes the R_i-requirements will be satisfied.

The main strategy will be again the one that deals with operators Γ_1 and Γ_2. It will try to obtain the necessary changes in the sets $\Psi_1(A)$ and $\Psi_2(A)$ using backup strategies that try to satisfy the R requirements in a different manner. The requirement R_1 is of higher priority. The method for satisfying the lower priority requirement R_2 will be decided after we have established the method for satisfying R_1 unless we have already evidence that the R_2-requirement is trivially satisfied. The N-strategy starts off assuming that the requirements will be satisfied via operators Γ_1 and Γ_2. It will be denoted by (N, Γ_1, Γ_2). Its outcomes are $\lambda_2 <_L f <_L w <_L l_2 <_L l_1$. Outcomes w and f will represent the fact that the strategy has succeeded in satisfying its requirement while keeping both operators rectified.

Outcome l_1 will represent a global win for R_1. The price we pay for it is that the operator Γ_2 will not be rectified. Below this outcome there will be a backup (N, FM_1, Γ_2')-strategy. It will construct a new operator Γ_2' and meet the requirement N. Its outcomes are $\lambda_2 <_L f <_L w <_L l_2$ and it acts just as the (N, Γ)-strategy from the previous section.

Outcome l_2 will represent a global win for R_2. Below it we have a strategy (N, Γ_1, FM_2) which continues to construct the same operator Γ_1 as the (N, Γ_1, Γ_2)-strategy. Strategies below will simply treat R_2 as satisfied - that is, this requirement will be invisible to them.

Below outcome λ_2 is the (R_2, Λ_2)-strategy followed by a backup strategy (N, Γ_1, Λ_2). It continues to construct the same operator for the first strategy Γ_1 but switches the method for the second strategy to Λ_2. Its outcomes are $\lambda_1 <_L f <_L w$.

Below outcome λ_1 is the (R_1, Λ_1)-strategy a backup strategy that changes the method for satisfying the requirement R_1. As a consequence the method for R_2 must be decided again. The strategy is $(N, \Lambda_2, \Gamma_2'')$ with outcomes $\lambda_2 <_L f <_L <_w <_L l_2$. The method for satisfying R_1 cannot be switched anymore. The method for R_2 can be further switched via (N, Λ_1, FM_2) below l_2 and to $(N, \Lambda_1, \Lambda_2'')$ below outcome λ_2.

In this way all possible combinations of methods for satisfying the two R-requirements are distributed through the tree.

The modules for each of the described strategies above follow the basic steps as outlined in the previous section. The (N, Γ_1, Γ_2) strategy chooses a follower x. It tries to define the parameters for R_1 - $H_1(x)$, $G_1(x)$, $y_1(x)$ and $m_1(x)$ and rectifies Γ_1. Then it focuses on the second requirement R_2. Once R_2's parameters are defined a new element $m_2(x)$ will be enumerated in A. The new point is that this new change in A must be reflected in the definition of Γ_1. So an axiom $\langle m_2(x), G_1(x) \oplus \{y_2(x)\}\rangle$ is enumerated in Γ_1. If $m_2(x)$ is extracted from A then we will extract $y_2(x)$ from B and this axiom will not be valid. We will enumerate $y_2(x)$ back in B only if x has been returned in which case $G_1(x) \not\subseteq \Psi_1(A)$.

The axioms enumerated in Γ_2 will have to include additionally $m_1(x)$ and all $m_1(z)$ for previously defined followers of this strategy from previous cycles, that are still in A.

Once we have established that $x \in \Phi(B)$, we start the attack by sending the follower x with defined $v(x) = \max(\varphi(x), y_1(x), y_2(x))$ to (N, Γ_1, Λ_2). This strategy will need to get further permission from Γ_1. An axiom $\langle z, H_2(x) \oplus B \upharpoonright v(x)\rangle$

will be enumerated for each z which is a follower from a previous cycle, x or $m_1(x)$. This strategy also starts an attack by sending x to $(N, \Lambda_1, \Gamma_2'')$ and extracting $y_1(x)$ and $m_1(x)$ from A once it has observed that $x \in \Phi(B)$. Note that this will make the axiom for x in Λ_2 invalid.

The $(N, \Lambda_1, \Gamma_2'')$-strategy now must define parameters $G_2''(x)$ and $H_2''(x)$, markers $y_2''(x)$ and $m_2''(x)$. And then it will initiate the last attack sending x to $(N, \Lambda_1, \Lambda_2'')$.

Once the follower is extracted from A it can climb back up these strategies. (R_2, Λ_2'') will send it back to $(N, \Lambda_1, \Gamma_2'')$ in case $H_2''(x) \subset \Theta_2(A)$.

(R_1, Λ_1) will send the follower x back to (N, Γ_1, Λ_2) in case $H_1(x) \subset \Theta_1(A)$.

Then (R, Λ_2) will send it back to $(N, \Gamma_1\Gamma_2)$ in case $H_2(x) \subset \Theta_2(A)$.

When the $(N, \Gamma_1\Gamma_2)$-strategy re-receives x it will have proof that $H_1(x) \subseteq \Theta_1(A)$, so that $G_1(x) \not\subseteq \Psi_1(A)$ and Γ_1 is rectified and $H_2(x) \subset \Theta_2(A)$, so $G_2(x) \not\subseteq \Psi_2(A)$ and Γ_2 is rectified.

Considering two requirements we can justify the need for the (R_i, Λ_i)-strategies. Suppose $\alpha\hat{\ }l_2 \subset \beta$ and β is sharing the same method Λ_1 as α. If a follower x of β is extracted from A we must ensure that the axioms for x defined in the operator Λ_1 are invalid. It could be the case that α moves on to outcome w and initialises β. The follower x will not be observed any longer. But as $\Theta_1(A)$ is not in our control it is possible that $H_1(x) \subset \Theta_1(A)$ and this is revealed at a later stage after x has been cancelled. If x is not sent back, then Λ_1 will not be correct. This is why we need the (R_1, Λ_1) strategy which observes all followers. It will return x even after x is cancelled.

The (R, Γ_1) strategy plays a similar role. Suppose that $\alpha\hat{\ }l_2 \subset \beta$. Now β is sharing the same method Γ_1 as α. If a follower x of β is extracted from A we must ensure that the axioms for x defined in the operator Γ_1 are invalid. If α moves on to outcome w thereby initialising β we lose control on x and it could happen that $G_1(x) \subset \Psi_1(A)$ at a later stage. We will be able to argue that if the axiom for x in Γ_1 is valid, then $H_1(x) \subset \Theta_1(A)$ and (R, Γ_1) will have outcome gw at all further stages.

In [ACKS] we combine the ideas from the above description to obtain the construction that meets all requirements. □

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