Splitting and Nonsplitting in the Σ_2^0 Enumeration Degrees*

M. M. Arslanov[1], S. B. Cooper[2], I. Sh. Kalimullin[1]
and M. I. Soskova[3]
[1]Department of Mathematics, Kazan State University,
420008 Kazan, Russia
[2]School of Mathematics, University of Leeds,
Leeds LS2 9JT, U.K.
[3]Faculty of Mathematics and Informatics,
Sofia University, 1164 Sofia, Bulgaria.

Abstract

This paper continues the project, initiated in [ACK], of describing general conditions under which relative splittings are derivable in the local structure of the enumeration degrees, for which the Ershov hierarchy provides an informative setting.

The main results below include a proof that any high total e-degree below $\mathbf{0}'_e$ is splittable over any low e-degree below it, a non-cupping result in the high enumeration degrees which occurs at a low level of the Ershov hierarchy, and a $\emptyset'''$-priority construction of a Π_1^0 e-degree unsplittable over a 3-c.e. e-degree below it.

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1 Introduction

Following Friedberg and Rogers [FR], A is said to be *enumeration reducible* to B ($A \leq_e B$) if there exists an effective procedure for obtaining an enumeration of A from *any* enumeration of B. It turned out that this relation is the most general well-behaved means of computably comparing the positive information content of sets. Indeed, Selman proved in [Se71] that this reducibility is a maximal transitive relation of the relation "*is* Σ^0_1 *in*".

Enumeration reducibility can also be thought of as a fundamental form of non deterministic reducibility: $A \leq_e B$ iff there exists a non deterministic oracle Turing machine M that, when equipped with the semi-characteristic function of B computes the semi-characteristic function of A (see [Mc84]). On the other hand Scott [Sc75, Sc76] showed that the operators that arise naturally from the above definition coincide precisely with the denotation of closed terms of the type free lambda calculus under the *graph model* interpretation first suggested by Plotkin in [Pl72]. Moreover, as Scott pointed out, enumeration reducibility is tantamount, under this interpretation, to application by a closed lambda term (see [Sc75, p. 538]). However much of the present interest in enumeration reducibility stems from its relationship with the most widely studied relation in computability theory, Turing reducibility ($\leq_T$) and the latter's degree structure, the *Turing degrees*. In effect, being transitive and reflexive $\leq_e$ itself induces an equivalence relation ($\equiv_e$) on the powerset of $\mathbb{N}$. As a result, two sets belong to the same equivalence class if they contain the same positive information content as stipulated by $\leq_e$. We call the structure of these equivalence classes, under the relation induced by $\leq_e$, the *enumeration degrees*. This structure is an upper semi-lattice with zero degree corresponding to the class of c.e. sets. Moreover, there is a natural isomorphic embedding (ι) of the Turing degrees into the enumeration degrees. We call the degrees belonging to this substructure *total* (since any such degree is characterised by the fact that it contains the graph of a total function). Accordingly, the enumeration degrees and its total substructure can be considered as a more general setting for the study of the Turing degrees. [SC08] represents work in this direction, and illustrates the potentialities of such a viewpoint.

A jump operation for the enumeration degrees (with the same notation as that for the Turing degrees) was defined by McEvoy and Cooper in [MC85, Mc84]. This is defined in such a way that the jump is preserved under the natural embedding. The jump operation gives rise to the local structure of the enumeration degrees consisting of all enumeration degrees reducible to $0'_e$, the enumeration jump of the zero degree. Cooper [Co90] proves that the enumeration degrees in the local structure are exactly those containing Σ_2^0 sets. Furthermore the images of the computably enumerable Turing degrees under the natural embedding are the Π_1^0 enumeration degrees and the Δ_2^0 Turing degrees embed onto a proper subset of the Δ_2^0 enumeration degrees. Thus the local structure of the enumeration degrees itself can be considered as a proper extension of the local structure of the Turing degrees.

This paper continues the project, initiated in [ACK], of describing general conditions under which relative splittings are derivable in the local structure of the enumeration degrees.

The main results below include a proof that any high total e-degree below $\mathbf{0}_e'$ is splittable over any low e-degree below it, a proof that there exists within the high e-degrees a 3-c.e. e-degree which cannot be cupped to some 2-c.e. (and so total) e-degree above it, and a $\emptyset'''$-priority construction of a Π_1^0 e-degree unsplittable over a Δ_2 e-degree below it.

In [ACK] it was shown that using semirecursive sets one can construct minimal pairs of e-degrees by both effective and uniform ways, following which new results concerning the local distribution of total e-degrees and of the degrees of semirecursive sets enabled one to proceed, via the natural embedding of the Turing degrees in the enumeration degrees, to results concerning embeddings of the diamond lattice in the e-degrees. A particularly striking application of these techniques was a relatively simple derivation of a strong generalisation of the Ahmad Diamond Theorem.

This paper extends the known constraints on further progress in this direction, such as the result of Ahmad and Lachlan [AL98] showing the existence of a nonsplitting Δ_2^0 e-degree $> \mathbf{0}_e$, and the recent result of Soskova [Sos07] showing that $\mathbf{0}_e'$ is unsplittable in the Σ_2^0 e-degrees above some Σ_2^0 e-degree $< \mathbf{0}_e'$. This work also relates to results (e.g. Cooper and Copestake [CC88]) limiting the local distribution of total e-degrees.

For further background concerning enumeration reducibility and its degree structure, the reader is referred to Cooper [Co90], Sorbi [Sor97] or Cooper [Co04, chapter 11].

2 Splitting high degrees

We first show, building on [ACK], that suitably extensive intervals of enumeration degrees below $\mathbf{0}_e'$ can accommodate diamond lattice embeddings. The

Ahmad Diamond Theorem [Ah91] then appears as a special case.

Theorem 1 *If* $\mathbf{a} < \mathbf{h} \leq \mathbf{0}'_e$, $\mathbf{a}$ *is low and* $\mathbf{h}$ *is total and high then there is a low total enumeration degree* $\mathbf{b}$ *such that* $\mathbf{a} \leq \mathbf{b} < \mathbf{h}$.

Corollary 2 *Let* $\mathbf{a} < \mathbf{h} \leq \mathbf{0}'_e$, $\mathbf{h}$ *be a high total e-degree, and* $\mathbf{a}$ *be a low e-degree. Then there are* Δ^0_2 *e-degrees* $\mathbf{b}_0 < \mathbf{h}$ *and* $\mathbf{b}_1 < \mathbf{h}$ *such that* $\mathbf{a} = \mathbf{b}_0 \cap \mathbf{b}_1$ *and* $\mathbf{h} = \mathbf{b}_0 \cup \mathbf{b}_1$.

Proof of Corollary. Immediately follows from Theorem 1, and Theorem 6 of [ACK]. $\square$

Proof of Theorem 1. Assume A has low e-degree, $H \oplus \overline{H}$ has high e-degree (i.e., H has high Turing degree) and $A \leq_e H \oplus \overline{H}$.

We want to construct an H-computable increasing sequence of strings $\{\sigma_s\}_{s\in\omega}$ such that the set $B = \cup_s \sigma_s$ satisfies the requirements

$$\mathcal{P}_n : n \in A \iff (\exists y)[\langle n, y\rangle \in B]$$

and

$$\mathcal{R}_n : (\exists \sigma \subset B)[n \in W^\sigma_n \vee (\forall \tau \supset \sigma)[\tau \in S^A \implies n \notin W^\tau_n]]$$

for each $n \in \omega$, where

$$S^A = \{\tau : (\forall x)(\forall y)[\tau(\langle x, y\rangle) \downarrow = 1 \implies x \in A]\}.$$

Note that $\mathcal{P}_n$-requirements guarantee that $A \leq_e B$, and hence $A \leq_e B \oplus \overline{B}$. To prove that the $\mathcal{R}_n$-requirements provide $B' \equiv_T \emptyset'$, first note that $S^A \equiv_e A$, which has low e-degree, and

$$X = \{\langle \sigma, n\rangle : (\exists \tau \supset \sigma)[\tau \in S^A \ \&\ n \in W^\tau_n]\} \leq_e S^A.$$

Then $X \in \Delta^0_2$ and

$$n \notin B' \iff (\exists \sigma \subset B)[\langle \sigma, n\rangle \notin X],$$

so that B' is co-c.e. in $B \oplus \emptyset' \equiv_T \emptyset'$. Thus $B' \leq_T \emptyset'$ by Post's Theorem.

Since the set B will be computable in H, the set

$$Q = \{n : (\forall \sigma \subset B)(\exists \tau \supset \sigma)[\tau \in S^A \ \&\ n \in W^\tau_n]\}$$

will be computable in $(H \oplus \emptyset')' \equiv_T H'$ – indeed, we have $n \in Q \iff (\forall \sigma \subset B)[\langle \sigma, n\rangle \in X]$, so that Q is co-c.e. in $H \oplus \emptyset'$. Now to construct the desired set B we can apply the Recursion Theorem and fix an H-computable function g such that $Q(x) = \lim_s g(x, s)$.

Let $\{A_s\}_{s\in\omega}$ and $\{S^A_s\}_{s\in\omega}$ be respective H-computable enumerations of A and S^A.

The Construction:

Stage $s = 0$. $\sigma_0 = \emptyset$.

Stage $s+1 = 2\langle n, z\rangle$ (to satisfy $\mathcal{P}_n$). Given σ_s define $l = |\sigma_s|$.

If $n \notin A_s$, then let $\sigma_{s+1} = \sigma_s{}^\frown 0$.

If $n \in A_s$, then choose the least $k \geq l$ such that $k = \langle n, y\rangle$ for some $y \in \omega$ and define $\sigma_{s+1} = \sigma_s{}^\frown 0^{k-l\frown}1$ (so that $\sigma_{s+1}(k) = 1$).

Stage $s+1 = 2\langle n, z\rangle + 1$ (to satisfy $\mathcal{R}_n$). H-computably find the least stage $t \geq s$ such that either $g(n,t) = 0$, or $n \in W^\tau_{n,t}$ for some τ satisfying $\tau \in S^A_t$ and $\tau \supset \sigma_s$. (Such stage t exists since if $\lim_s g(n,s) = 1$ then $n \in Q$, and hence there exists some $\tau \supset \sigma_s$ such that $n \in W^\tau_n$ and $\tau \in S^A$.)

If $g(n,t) = 0$ then define $\sigma_{s+1} = \sigma_s{}^\frown 0$.

Otherwise, choose the first $\tau \supset \sigma_s$ such that $\tau \in S^A_t$ and $n \in W^\tau_{n,t}$. Define $\sigma_{s+1} = \tau$.

This completes the description of the construction.

Let $B = \cup_s \sigma_s$. Clearly $B \leq_T H$ since each σ_s is obtained effectively in H. Each $\mathcal{P}_n$-requirement is satisfied via the even stages of the construction since $\sigma_s \in S^A$ for any $s \in \omega$.

To prove that each $\mathcal{R}_n$-requirement is met suppose that

$$(\forall \sigma \subset B)(\exists \tau \supseteq \sigma)[\tau \in S^A \ \&\ n \in W^\tau_n]$$

for some n. This means that $n \in Q$. Choose any odd stage $s = 2\langle n, z\rangle + 1$ such that $g(n,t) = 1$ for all $t \geq s$. Then by the construction $n \in W^{\sigma_s}_n$.

Hence $A \leq_e B \oplus \overline{B} \leq_e H \oplus \overline{H}$, and $\deg_e(B \oplus \overline{B})$ is low. □

3 Non-cupping and the Ershov hierarchy

Cooper, Sorbi and Yi [CSY] constructed below $\mathbf{0}'_e$ an enumeration degree not cuppable to $\mathbf{0}'_e$, but showed that every non-zero Δ^0_2 e-degree is cuppable to $\mathbf{0}'_e$. In particular, every non-zero low e-degree is so cuppable. They also showed that there is a low e-degree $\mathbf{c}$ bounding a non-zero e-degree $\mathbf{b}$ which is not cuppable to $\mathbf{c}$. The following result establishes a non-cupping result at the other end of the high-low hierarchy, and at a surprisingly low level of the Ershov hierarchy.

Theorem 3 *There are high enumeration degrees* $\mathbf{h} < \mathbf{a}$ *such that* $\mathbf{h}$ *is* 3*-c.e.,* $\mathbf{a}$ *is* 2*-c.e. (and hence total) and* $\mathbf{h}$ *is not cupped to* $\mathbf{a}$.

Proof. We will enumerate c.e. sets A and B such that $\mathbf{a} = \deg(\overline{A})$ and $\mathbf{h} = \deg(H)$ are the required degrees, where $H = \overline{A} \cup B$. Note that we automatically have $H \leq_e \overline{A}$. The symbols A_s and B_s will denote finite sets of elements enumerated in A and B respectively at stages $\leq s$. Let H_s be $\overline{A}_s \cup B_s$. We meet the requirements

$$\begin{aligned} \mathcal{N}_i: &\qquad \overline{A} = \Phi_i(\Theta_i^{\overline{A}} \oplus H) \Longrightarrow \overline{A} \leq_e \Theta_i^{\overline{A}}, \\ \mathcal{Q}_i: &\qquad \varphi_i \text{ total} \Longrightarrow (\exists z)(\forall x > z)[\varphi_i(x) \leq c_H(x)], \end{aligned}$$

where $\{\Phi_i, \Theta_i\}_{e\in\omega}$ is some effective listing of all pairs of e-operators, $\{\varphi_i\}_{i\in\omega}$ is an effective listing of all p.r. functions and

$$c_H(x) = (\mu s \geq x)[H_s \restriction x \subseteq H \restriction x].$$

By [MC85] the $\mathcal{Q}$-requirements imply highness of the e-degree of the set H.

The strategy for an $\mathcal{N}_i, i \in \omega$, requirement acts as follows:

- Wait for a stage s such that for some integer y and finite sets $F \subseteq \overline{A_s}$ and G we have $y \in A \cap \Phi_i^{\Theta_i^F \oplus G}[s]$.
- Enumerate G in B and restrain F from being enumerated in A.

If there is a stage s with such y, F and G then we were successful in satisfying the $\mathcal{N}_i$-requirement diagonalizing $\overline{A}$ against $\Phi_i^{\Theta_i^{\overline{A}} \oplus H}$ via y. Otherwise (if there are no such y, F, G) the assumption $\overline{A} = \Phi_i^{\Theta_i^{\overline{A}} \oplus H}$ would imply $\overline{A} \leq_e \Theta^{\overline{A}}$.

The strategy for a $\mathcal{Q}_i, i \in \omega$, requirement acts as follows:

With this requirement we associate the column $\{\langle i, n\rangle \mid n \in \omega\}$. Then,

- Wait for a stage s_1 such that $\varphi_i(x) \downarrow < s_1$ for each $x \leq \langle i, 1\rangle$.
- Enumerate $\langle i, 0\rangle$ in A. Restrain $\langle i, 0\rangle$ from being enumerated in B.
- Wait for a stage $s_2 > s_1$ such that $\varphi_i(x) \downarrow < s_2$ for each $x \leq \langle i, 2\rangle$.
- Enumerate $\langle i, 1\rangle$ in A. Restrain $\langle i, 1\rangle$ from being enumerated in B.

$\vdots$

- Wait for a stage $s_{k+1} > s_k$ such that $\varphi_i(x) \downarrow < s_{k+1}$ for each $x \leq \langle e, k+1\rangle$.
- Enumerate $\langle i, k\rangle$ in A. Restrain $\langle i, k\rangle$ from being enumerated in B.

Now, if ϕ_i is total then c_H would dominate ϕ_i beginning at $\langle i, 0\rangle$.

There is an obvious conflict between $\mathcal{N}$ and $\mathcal{Q}$-requirements (a $\mathcal{Q}$-requirement restrains an element $\langle i, k\rangle$ from being enumerated in B, but an $\mathcal{N}$-requirement enumerates it in B). This conflict is solved by an ordering of the strategies on the priority tree ($\mathcal{Q}$-strategies of can guess the result of an $\mathcal{N}$-strategy of higher priority, which produces either empty, or co-finite column in H).

Let $T = \omega^{<\omega}$ be the *tree* of nodes (strings) of our construction with the root node $\emptyset$, the concatenation $\widehat{\ }$, and the usual orderings $\subset$, $<_L$ and $\prec$:

$$\begin{aligned} \sigma \subset \tau &\iff (\exists \rho \neq \emptyset)[\tau = \sigma\widehat{\ }\rho], \\ \sigma <_L \tau &\iff (\exists \rho \in T)(\exists m)(\exists n < m)[\rho\widehat{\ }n \subseteq \sigma \;\&\; \rho\widehat{\ }m \subseteq \tau], \\ \sigma \prec \tau &\iff \rho \subset \tau \vee \rho <_L \tau. \end{aligned}$$

We also can consider the reflexive versions of these orderings: $\subseteq$, $\leq_L$ and $\preceq$. Fix some $1-1$ computable map $n : T \longrightarrow \omega$.

We attach each node σ with the even length $|\sigma| = 2i$ with the requirement $\mathcal{N}_i$, and we attach each node σ with the odd length $|\sigma| = 2i+1$ with the requirement $\mathcal{Q}_i$.

Notation. For every set $X \subseteq \omega$ and $\sigma \in T$ let

$$X^{[\prec\sigma]} = \bigcup\{X^{[n(\tau)]} : \tau \prec \sigma\}$$

and $S_0^\sigma(X) = \bigcup\{X^{[n(\tau)]} : \tau\widehat{\ }0 \subseteq \sigma \;\&\; |\tau| \text{ is odd}\}$.

Given A_s and H_s at some stage s we define the following parameters:

$$l_N(\sigma, s) = \max\{x{\leq}s : (\forall y{<}x)(\forall t{<}x)[y \notin A_s \cap \Phi_{i,s}(\Theta_{i,s}(\bigcap_{u=t}^{s} \overline{A}_s) \oplus \bigcap_{u=t}^{s} H_s)]\}$$

if $|\sigma| = 2i$, and

$$l_Q(\sigma, s) = \max(\{0\} \cup \{x \leq s : (\forall y \leq \langle n(\sigma), x\rangle)[\varphi_{i,s}(y)\downarrow < s]\}) \text{ if } |\sigma| = 2i+1.$$

The Construction.

The *initialization* of a node σ at stage $s \in \omega$ just means that we mark the node as *initialized* commencing with this stage.

Stage $s = 0$. Set $A_0 = B_0 = \emptyset$ and $\delta_0 = \emptyset$. No node is initialized at stage $s = 0$.

Stage $s+1$.

Step 1. (The definition of δ_{s+1}.) Define the string $\delta_{s+1} \in T$ with the length $s+1$ by the induction below. Assume $\delta \restriction n = \sigma$ is defined and $n \leq s$.

Suppose $n = 2i$ (i.e. σ is a N-node.) Let $\delta_{s+1}(n) = m > 0$ if
1) $l_N(\sigma, s) \leq \max\{l_N(\sigma, t) : t < s \ \&\ \sigma \subseteq \delta_t\}$,
2) $m = (\mu k > 0)[\sigma\hat{\ }k$ is not initialized at stages $\leq s]$.
Otherwise $\delta_{s+1}(n) = 0$.

Suppose now that $n = 2i + 1$ (i.e. σ is a Q-node.) Then define $\delta_{s+1}(n)$ exactly as above but with l_Q instead l_N.

Step 2. (The action.) A node σ *requires attention* at stage $s + 1$ if
1) $|\sigma| = 2i$,
2) $\sigma\hat{\ }0 \subseteq \delta_{s+1}$,
3) there is $y \leq s$, such that $y \in A_s \cap \Phi_{i,s}(\Theta^F_{i,s} \oplus G)$ for some finite F, G such that $F \subseteq \overline{A}_s$, $G^{[\prec\sigma]} \subseteq H_s$ and $S^\sigma_0(F) = S^\sigma_0(G) = \emptyset$.

Case 1. There is a node σ which requires attention. Then fix one such σ_0 with the least length; choose the corresponding finite sets F and G (with the least sum of their canonical indices); enumerate the set G into B.

Also, for all odd nodes σ (i.e. $|\sigma| = 2i+1$ for some i), such that $\sigma\hat{\ }0 \subset \sigma_0$, enumerate into A all pairs $\langle n(\sigma), x\rangle$ for each $x < l_Q(\sigma, s)$. Choose a sufficiently large z (in particular, greater than all elements of F and G) and initialize all nodes $\alpha \succ \sigma_0$ such that $n(\alpha) < z$.

We say σ_0 *receives attention* at stage $s + 1$.

Case 2. There is no node which requires attention. Then for all odd nodes σ, such that $\sigma\hat{\ }0 \subset \delta_{s+1}$, enumerate in A all pairs $\langle n(\sigma), x\rangle$ for each $x < l_Q(\sigma, s)$. Choose a sufficient large z and initialize all nodes α, such that $\delta_{s+1} <_L \sigma$ and $n(\alpha) < z$.

Then for all odd nodes σ (i.e. $|\sigma| = 2i + 1$ for some i), such that $\sigma\hat{\ }0 \subset \sigma_0$, enumerate in A all pairs $\langle n(\sigma), x\rangle$ for each $x < l_Q(\sigma, s)$. Choose a sufficiently large z and initialize all nodes $\alpha \succ \sigma_0$ such that $n(\alpha) < z$. Go to the next stage.

Let $\sigma \subset \delta$ indicate that $\sigma \subseteq \delta_s$ for infinitely many s and $\delta_s <_L \sigma$ for only finitely many s.

Lemma 4 *a) No node $\sigma \subset \delta$ can be initialized during the construction.*
b) $S^0_\sigma(\overline{A}) = S^0_\sigma(H) = \emptyset$ for every $\sigma \subset \delta$.
c) $H^{[\prec\sigma]}$ is computable for every $\sigma \subset \delta$.
d) There is the true path δ, namely the infinite path containing all σ such that $\sigma \subset \delta$.

Proof. a) Suppose not. Let σ be the $\subset$-least node, such that $\sigma \subset \delta$, which is initialized at some stage. Let this stage be stage $s + 1$, say.

If Case 2 holds at this stage then $\delta_{s+1} <_L \sigma$. Hence, for some $\rho \subset \delta_{s+1}$ and $m > 0$ we have $\rho^\frown m \subseteq \sigma$. Since by the construction $\sigma \not\subseteq \delta_t$ for any $t > s$, this contradicts $\sigma \subset \delta$.

Suppose now that Case 1 holds at stage $s+1$, and the node σ_0 receives attention at this stage. Let $|\sigma_0| = 2i$. Again, if $\sigma_0 <_L \sigma$ or ${\sigma_0}^\frown m \subseteq \sigma$, with $m > 0$, then $\sigma \not\subseteq \delta_t$ for every $t > s$, which is impossible. Hence, ${\sigma_0}^\frown 0 \subseteq \sigma$.

By the choice of σ, node σ_0 cannot be initialized. Hence, for some $y \leq s$ we have $y \in A_{s+1} \cap \Phi_{i,s+1}(\Theta^F_{i,s+1} \oplus G)$, where $F \subseteq \overline{A}_t$ and $G \subseteq H_t$ for every $t > s$. It follows that $l_N(\sigma_0, t) \leq s$ for all $t > s$. But this contradicts the fact that ${\sigma_0}^\frown 0 \subseteq \sigma \subset \delta$.

b) If $\tau^\frown 0 \subseteq \sigma \subset \delta$ and $|\tau|$ is odd then $\lim_s l_Q(\tau, s) = \infty$ so that each element of $\omega^{[n(\tau)]}$ will be enumerated into A during the construction. No element from $\omega^{[n(\tau)]}$ will be enumerated into B since τ cannot be initialized.

c) Since $\sigma \subset \delta = \lim_s \delta_s$ we have $H^{[n(\tau)]} = \omega^{[n(\tau)]}$ for almost every $\tau \prec \sigma$ (that is, apart from finitely many). Furthermore, for each $\tau \prec \sigma$ either the set $H^{[n(\tau)]}$ is finite or the set $\omega^{[n(\tau)]} - H^{[n(\tau)]}$ is finite.

d) Suppose that there is a $\subset$-maximal $\sigma \subset \delta$. By a) σ cannot be initialized, and can receive attention at only finitely many stages (if $|\sigma|$ is even). By the choice of σ we have $\sigma^\frown 0 \subseteq \delta_s$ at only finitely many stages. Let s_0 be a stage greater than all these above mentioned stages such that $\sigma^\frown m \subseteq \delta_{s_0}$ for some $m > 0$. Then $\sigma^\frown m \subset \delta$. Which gives a contradiction. □

Lemma 5 *$\mathcal{N}_i$ is satisfied for each $i \in \omega$.*

Proof. Suppose $\overline{A} = \Phi_i(\Theta_i^{\overline{A}} \oplus H)$ and choose $\sigma \subset \delta$ such that $|\sigma| = 2i$. Then $\lim_s l_N(\sigma, s) = \infty$, $\sigma^\frown 0 \subset \delta$, and σ never receives attention.

Then for all $y \in \omega$

$$y \in \overline{A} \iff (\exists \text{ finite } G, R)[y \in \Phi_i^{R \oplus G} \ \&\ R \subseteq \Theta_i^{\overline{A}} \ \&\ G^{[\prec\sigma]} \subseteq H^{[\prec\sigma]}].$$

Indeed, the left-to-right implication is evident. For the reverse direction suppose that $y \in A \cap \Phi_i^{R \oplus G}$, where $R \subseteq \Theta_i^{\overline{A}}$ and $G^{[\prec\sigma]} \subseteq H^{[\prec\sigma]}]$. Let $F \subseteq \overline{A}$ be such finite set that $R \subseteq \Theta^F$. By Lemma 1 b) we have $S^0_\sigma(F) = S^0_\sigma(G) = \emptyset$. Then σ requires and receives attention at some stage, which is impossible.

Since $H^{[\prec\sigma]}$ is computable by Lemma 1 c), we have $\overline{A} \leq_e \Theta_i(\overline{A})$. □

Lemma 6 *$\mathcal{Q}_i$ is satisfied for each $i \in \omega$.*

Proof. Let $\sigma \subset \delta$ be such node that $|\sigma| = 2i+1$. Suppose that φ_i is total. Then $\lim_s l_Q(\sigma, s) = \infty$, and therefore $\sigma^\frown 0 \subset \delta$ and $H^{[n(\sigma)]} = B^{[n(\sigma)]} = \emptyset$. It will suffice to prove that $\varphi_i(y) < c_H(y) = (\mu s \geq y)[H_s \upharpoonright y \subseteq H \upharpoonright y]$ for every $y > \langle n(\sigma), 0\rangle$. Suppose not, so that $c_H(y) \leq \varphi_i(y)$ for some $y > \langle n(\sigma), 0\rangle$.

Let $\langle n(\sigma), x-1\rangle < y \leq \langle n(\sigma), x\rangle$ for some $x > 0$. Then there is a stage $s_y + 1 \leq \varphi_i(y)$ at which $\langle n(\sigma), x-1\rangle$ was enumerated into A, that is at which we have $\langle n(\sigma), x-1\rangle \in H_{s_y} - H_{s_y+1}$. Then by the construction $\sigma\hat{\ }0 \subseteq \delta_{s_y+1}$ and $x - 1 < l_Q(\sigma, s_y)$. But then $x \leq l_Q(\sigma, s_y)$, so that $\varphi_i(y) < s_y$ by the definition of l_Q, a contradiction. □

This completes the proof of the theorem. □

4 Non-splitting and the Ershov hierarchy

It is easy to see, using the natural embedding of splitting results from the Turing degrees, that the nonsplitting degree $> \mathbf{0}_e$ given by the Ahmad-Lachlan nonsplitting theorem is necessarily properly Δ^0_2. While previous splitting results from [ACK] show that the nonsplitting base given by the Soskova [Sos07] nonsplitting theorem for $\mathbf{0}'_e$ is at best properly Σ^0_2. We show below that, surprisingly, there is a Π^0_1 e-degree which is not splittable over some Δ^0_2 e-degree — in fact, unsplittable over one which is 3-c.e.

Theorem 7 *There is a Π^0_1 e-degree $\mathbf{a}$ and a 3-c.e. e-degree $\mathbf{b} < \mathbf{a}$ such that $\mathbf{a}$ is not splittable over $\mathbf{b}$.*

Proof. Cooper [Co90] has shown that the class of the Π^0_1 enumeration degrees coincides with the class of the 2-c.e. enumeration degrees. We shall therefore construct a 2-c.e. set A and 3-c.e. set B satisfying the following list of requirements:

1. We have a global requirement which ensures that $B \leq_e A$ via an enumeration operator Ω constructed by us:
$$\mathcal{S} : B = \Omega^A.$$

2. To ensure the non-splitting property of the degree of A consider a computable enumeration of all triples of enumeration operators $\{(\Xi, \Psi, \Theta)_i\}_{i<\omega}$. We denote the members of the i-th triple by Ξ_i, Ψ_i and Θ_i. For every i we shall have a requirement:
$$\mathcal{P}_i : A = \Xi_i^{\Psi_i^A, \Theta_i^A} \Rightarrow (\exists \Gamma_i, \Lambda_i)[A = \Gamma_i^{\Psi_i^A, B} \vee A = \Lambda_i^{\Theta_i^A, B}].$$

3. Finally we need to ensure that the degree of A is strictly greater than the degree of B. Let $\{\Phi_e\}_{e<\omega}$ be a computable enumeration of all enumeration operators. For every e we shall have a requirement:
$$\mathcal{N}_e : A \neq \Phi_e^B.$$

An overview of the strategies

The requirements shall be given the priority ordering:

$$\mathcal{S} < \mathcal{P}_0 < \mathcal{N}_0 < \mathcal{P}_1 < \mathcal{N}_2 < \dots$$

In the course of the construction whenever we enumerate an element in the set B, we will enumerate a corresponding axiom in the set Ω. Whenever we extract an element from B, we invalidate the corresponding axiom by extracting an element from A. Thus the global requirement $\mathcal{S}$ shall be satisfied without an explicit strategy on the tree ensuring this. More precisely every element n that enters B will be assigned a marker $\omega(n)$ in A and an axiom $\langle n, \{\omega(n)\}\rangle$ in Ω. If n is extracted from B then we extract $\omega(n)$ from A. This can happen only once as we will be constructing a 3-c.e. approximation to the set B. If n is later re-enumerated in B, it will remain in B forever and we can just enumerate the axiom $\langle n, \emptyset\rangle$ in Ω.

To satisfy a $\mathcal{P}$-requirement working with the triple (Ξ, Ψ, Θ) we will initially attempt to reduce A to the set $\Psi^A \oplus B$ by constructing an e-operator Γ to witness this. In this case as well the enumeration of elements in A is always accompanied by an enumeration of axioms in Γ, and extraction of elements from A can be rectified via B-extractions.

The $\mathcal{N}$-strategies follow a variant of the Friedberg-Muĉnik strategy (*FM*-strategy) while at the same time respecting the rectification of the operators constructed by higher priority strategies. We shall use labels for $\mathcal{N}$-strategies which clarify with respect to which constructed operators they work. An $\mathcal{N}$-strategy working with respect to the initial $\mathcal{P}$-strategy, for example, shall be denoted by $(\mathcal{N}, \Gamma)$. The $(\mathcal{N}, \Gamma)$-strategy working with the operator Φ shall choose a witness x, enumerate it in A and then wait until $x \in \Phi^B$. If this happens it shall extract the element x from A while restraining $B \upharpoonright use(\Phi, B, x)$ in B.

The need to rectify Γ after the extraction of the witness x from A can be in conflict with the restraint on B. To resolve this conflict we try to obtain a change in the set Ψ^A which would enable us to rectify Γ without any extraction from the set B. We introduce an explicit $\mathcal{P}$-strategy on the tree whose only job will be to monitor the length of agreement $l(\Xi^{\Psi^A, \Theta^A}, A)[s]$ at every stage s. The $(\mathcal{N}, \Gamma)$-strategy will proceed with actions directed at a particular witness once it is below the length of agreement. This ensures that the extraction of x from A will have one of the following consequences.

1. The length of agreement will never return to its previous value as long as at least one of the axioms that ensure $x \in \Xi^{\Psi^A, \Theta^A}$ remains valid. In this

case the $\mathcal{P}$-requirement is satisfied and we can use the simple FM-strategy for $\mathcal{N}$.

2. The length of agreement returns and there is a useful extraction from the set Ψ^A rectifying Γ. The $\mathcal{P}$-strategy remains intact while the $(\mathcal{N}, \Gamma)$-strategy is successful.

3. The length of agreement returns and there is an extraction from the set Θ^A.

We will initially assume that the third consequence is true and commence a backup strategy $(\mathcal{N}, \Lambda)$ which is devoted to building an enumeration operator Λ attempting to reduce A to $\Theta^A \oplus B$. This strategy will work with the same witness which it receives from $(\mathcal{N}, \Gamma)$. It will use the change in Θ^A in order to satisfy its own requirement. Only when we are provided with evidence that our assumption is wrong will we return to the initial strategy $(\mathcal{N}, \Gamma)$-strategy.

Basic cases

To provide the reader with more intuition about the construction we shall discuss a few simpler cases before we proceed with the general construction. We start off with the simplest case of just one $\mathcal{N}$-requirement below one $\mathcal{P}$-requirement. Then we shall explain how we can deal with all $\mathcal{N}$-requirements below a single $\mathcal{P}$-requirement. Finally we will discuss how to handle an $\mathcal{N}$-requirement working with respect to two $\mathcal{P}$-requirements.

One $\mathcal{N}$-requirement below one $\mathcal{P}$-requirement

Consider a $\mathcal{P}$-requirement associated with the triple (Ξ, Ψ, Θ) and an $\mathcal{N}$-requirement associated with the enumeration operator Φ. We describe the strategies associated with each requirement and at the same time define the first few levels of the tree of strategies.

The $(\mathcal{P}, \Gamma)$-strategy

The root of the tree is associated with the $(\mathcal{P}, \Gamma)$-strategy. We will denote it by α. It will have two outcomes $e <_L l$. At stage s the strategy α will monitor all elements $x \notin A[s]$. If there is an element $x \notin A[s]$ such that $x \in \Gamma^{\Psi^A, B}[s]$ then the operator Γ cannot be rectified. We shall later see that this yields $x \in \Xi^{\Psi^A, \Theta^A}[s]$ and the $\mathcal{P}$-requirement is satisfied. The strategy α shall have outcome l in this case. Strategies working below this outcome will follow the simple FM-strategy. If for every element $x \notin A \Rightarrow x \in \Gamma^{\Psi^A,B}$ the strategy shall have outcome e and the $(\mathcal{N},\Gamma)$-strategy shall be activated.

At stage s the strategy α acts as follows:

1. Scan all witnesses $x \notin A[s]$ defined at stages $t \leq s$.
2. If $x \in \Gamma^{\Psi^A,B}[s]$, then let the outcome be $o = l$.
3. If all witnesses are scanned and none has produced an outcome $o = l$, then let the outcome be $o = e$.

The $(\mathcal{N},\Gamma)$-strategy

The $\mathcal{N}$-requirement below outcome e will be assigned to an $(\mathcal{N},\Gamma)$-strategy denoted by β. It will have four outcomes: three finitary outcomes, f, w and l, and one infinitary outcome g, arranged in the following way: $g <_L f <_L w <_L l$.

The strategy first defines a witness x, enumerates it in the set A and then waits for this witness to enter the set Ξ^{Ψ^A,Θ^A}. While it waits the outcome is l indicating a global win for the $\mathcal{P}$-requirement as $A(x) \neq \Xi^{\Psi^A,\Theta^A}(x)$.

If the witness x enters the set Ξ^{Ψ^A,Θ^A} then there is a valid axiom of the form $\langle x, G(x) \oplus H(x)\rangle \in \Xi$ with $G(x) \subseteq \Psi^A$ and $H(x) \subseteq \Theta^A$. The strategy β shall then define a B-marker for x, $\gamma(x)$ and enumerate it in the set B. This is accompanied by enumerating a corresponding axiom for $\gamma(x)$ in Ω. Then it shall define a new axiom for x in Γ of the form $\langle x, G(x) \oplus (B \upharpoonright \gamma(x) + 1)\rangle$. While $x \notin \Phi^B$ it has outcome w. Finally if $x \in \Phi^B$ the strategy shall perform *capricious destruction* on the operator Γ by extracting the marker $\gamma(x)$ from B. Then instead of extracting the witness x from the set A, it shall *send* the witness x to a backup $(\mathcal{N},\Lambda)$-strategy which will be described in detail later and have outcome g. After this β starts a new cycle with a new witness x_1. As the old witness x is still in the set A but has no valid axiom in the operator Γ, the strategy shall rectify the operator Γ at x, using the axiom that will be defined for the new witness x_1. If the old witness x is later returned by the backup strategy then it was extracted from the set A with no useful extraction from the set $H(x)$. Thus if $x \notin \Xi^{\Psi^A,\Theta^A}$ then there is a useful extraction in $G(x)$. The strategy β shall then restore the set B by reenumerating the marker $\gamma(x)$. If at the next stage the $(\mathcal{P},\Gamma)$-strategy α does not see a global win for its requirement then $G(x) \nsubseteq \Psi^A$, the operator Γ is rectified and β can successfully preserve $x \in \Phi^B \setminus A$ at further stages. It will have outcome f in this case.

Every witness or marker that we define shall be selected as a fresh number, one that has not yet appeared in the construction so far under any form.

At stage s the strategy β will initially start its work at *Setup* and then later from the step of the module indicated at the previous stage.

- **Setup:**

  1. Choose a new current witness x as a fresh number. Enumerate x in $A[s]$.
  2. If $x \notin \Xi^{\Psi^A,\Theta^A}[s]$ then let the outcome be l and return to this step at the next stage. Otherwise define $G(x)$ and $H(x)$ to be finite sets such that $x \in \Xi^{G(x),H(x)}[s]$, $G(x) \subseteq \Psi^A[s]$, $H(x) \subseteq \Theta^A[s]$. Go to the next step.
  3. Define the B-marker $\gamma(x)$, along with its A-marker $\omega(\gamma(x))$, as fresh numbers. Enumerate $\gamma(x)$ in $B[s]$ and $\omega(\gamma(x))$ in $A[s]$. Enumerate a new axiom $\langle \gamma(x), \{\omega(\gamma(x))\}\rangle$ in $\Omega[s]$.

     Enumerate each $\langle z, G_x \oplus (B \upharpoonright \gamma(x)+1)\rangle$ in Γ, where $z \in A[s]$ is either x, or $\omega(\gamma(x))$, or a witness from a previous cycle of the strategy for which there is no valid axiom in Γ. This axiom for x shall be called *the main axiom* for x in Γ. Let the outcome be $o = w$. Go to $Waiting$ at the next stage.

- **Waiting:** If $x \in \Phi^B[s]$ then go to *Attack*. Otherwise let the outcome be $o = w$ and return to $Waiting$ at the next stage.

- **Attack:**

  1. Check if any previously sent witness has been returned. If so go to *Result*. Otherwise go to the next step.
  2. Define $\lambda(x) = \max(use(\Phi, B, x)[s], \gamma(x) + 1)$ and $R[s] = \gamma(x)$. Extract $\gamma(x)$ from $B[s]$ and $\omega(\gamma(x))$ from $A[s]$. Note that the extraction of $\omega(\gamma(x))$ does not injure $x \in \Xi^{\Psi^A,\Theta^A}[s]$ as the marker is defined as a fresh number larger than $\max(use(\Psi, A, G(x)), use(\Theta, A, H(x)))$.

     Send x. Let the outcome be $o = g$. At the next stage start from *Setup*, choosing a new current witness. The strategy working below outcome g will work under the assumption that B does not change below the right boundary $R[s]$.

- **Result:** Let the returned witness be x. Enumerate $\gamma(x)$ back in $B[s]$ and $\langle \gamma(x), \emptyset\rangle$ in $\Omega[s]$. *Cancel* each witness $z \in A[s]$ of this strategy by enumerating the axiom $\langle z, \emptyset\rangle$ in $\Gamma[s]$. Let the outcome be $o = f$. Return to *Result* at the next stage.

The backup strategies

We have two backup strategies: a $(\mathcal{P},\Lambda)$-strategy $\widehat{\alpha}$ and an $(\mathcal{N},\Lambda)$-strategy $\widehat{\beta}$.

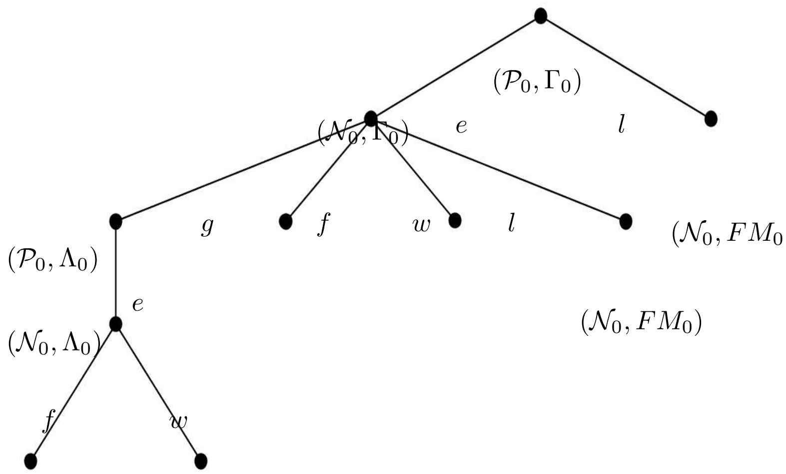
The $(\mathcal{P},\Lambda)$-strategy $\widehat{\alpha}$ will only monitor the status of the sent witnesses. If it spots a witness that is ready to be sent back it will do so ending the stage prematurely. It has only one outcome e. At stage s it operates as follows:

1. Scan all sent witnesses $x \notin A[s]$.
2. If $x \in \Lambda^{\Theta^A,B}[s]$ then return x. End this stage.
3. If all witnesses are scanned and none are returned then let the outcome be e.

The $(\mathcal{N},\Lambda)$-strategy $\widehat{\beta}$ shall wait for an available witness x to be sent by β. It shall enumerate the axiom $\langle x, H(x) \oplus (B \restriction \lambda(x))\rangle$ in the operator Λ and carry on with the usual FM-strategy: wait for $x \in \Phi^B$ with outcome w, then extract x from A. If this does not entail a useful extraction from the set $H(x)$ then $\widehat{\alpha}$ shall send the witness x back and $\widehat{\beta}$ shall not be accessible at further stages. If $\widehat{\beta}$ is visited again then it shall have outcome f. At stage s the $(\mathcal{N},\Lambda)$-strategy $\widehat{\beta}$ operates as follows:

- **Setup:** Let $x \in A[s]$ be a new witness which was sent by the $(\mathcal{N},\Gamma)$-strategy. Now x becomes the *witness* of the $(\mathcal{N},\Lambda)$-strategy. Enumerate $\langle x, H(x) \oplus (B[s] \restriction \lambda(x)+1)\rangle$ in $\Lambda[s]$. This is *the main axiom* for x in Λ. Go to $Waiting$.
- **Waiting:** If $x \in \Phi^B[s]$ and $use(\Phi, B, x)[s] < R[s]$ then go to *Attack*. Otherwise the outcome is $o = w$, return to $Waiting$ at the next stage.
- **Attack:** Extract x from $A[s]$. Go to *Result*.
- **Result:** Let the outcome be $o = f$. Return to *Result* at the next stage.

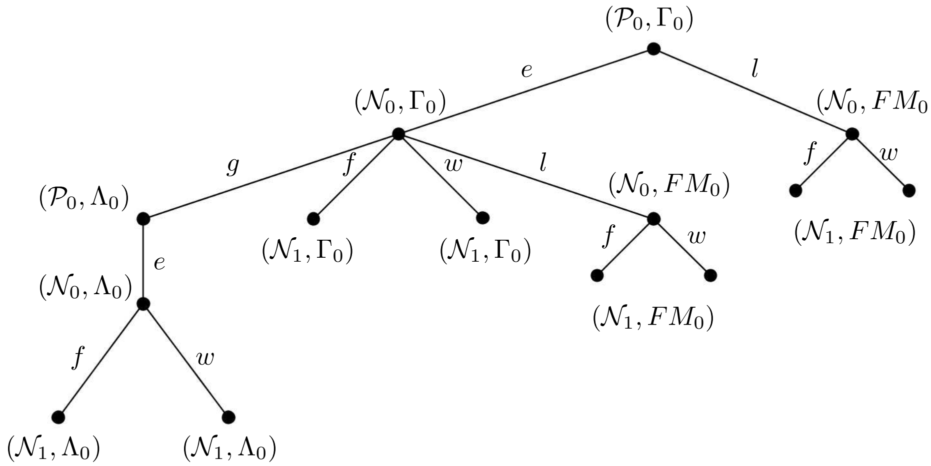
The next picture shows the first few levels of the tree of strategies:



When we inspect the tree in detail we notice that we might visit an $(\mathcal{N}, FM)$-strategy on several occasions, allow it to enumerate its own witness in the set A and then initialize it. In the design of the operators Γ and Λ we have neglected to enumerate axioms for such elements. If the $(\mathcal{N}, FM)$-strategy manages to extract from A its witness before it is initialized then this will not cause any errors in the constructed operators. If the element is still in A then we could have a problem. To avoid this every time we initialize an $(\mathcal{N}, FM)$-strategy we will enumerate axioms $\langle x, \emptyset \rangle$ in both Γ and Λ for every witness x of this strategy which is not extracted from the set A. This extra action will keep Γ and Λ always rectified.

Many $\mathcal{N}$-strategies below one $\mathcal{P}$-strategy

To incorporate a further $\mathcal{N}$-strategy in the construction described in the previous section we use the same basic ideas. The second $\mathcal{N}$-requirement $\mathcal{N}_1$ shall be assigned to an $(\mathcal{N}_1, FM)$-strategy below the l-outcomes of both α and β. Below β's outcomes w and f we have $(\mathcal{N}_1, \Gamma)$-strategies $\beta\hat{\ }w$ and $\beta\hat{\ }f$ which operate just like the strategy β described above. Similarly below the outcome f and w of the backup strategy $\widehat{\beta}$ we have $(\mathcal{N}_1, \Lambda)$-strategies $\widehat{\beta}\hat{\ }w$ and $\widehat{\beta}\hat{\ }f$ which operate just like the strategy $\widehat{\beta}$.



We only need to take extra care to keep the constructed operators Γ and Λ rectified at elements enumerated in A by strategies that are later initialized. Firstly we will use the initialization rule inspired by the $(\mathcal{N}, FM)$-strategy described in the previous section. Whenever we initialize an $\mathcal{N}$-strategy α we will enumerate axioms $\langle x, \emptyset \rangle$ in all operators constructed by higher priority strategies $\beta < \alpha$ for every witness x of α which is not extracted from the set A.

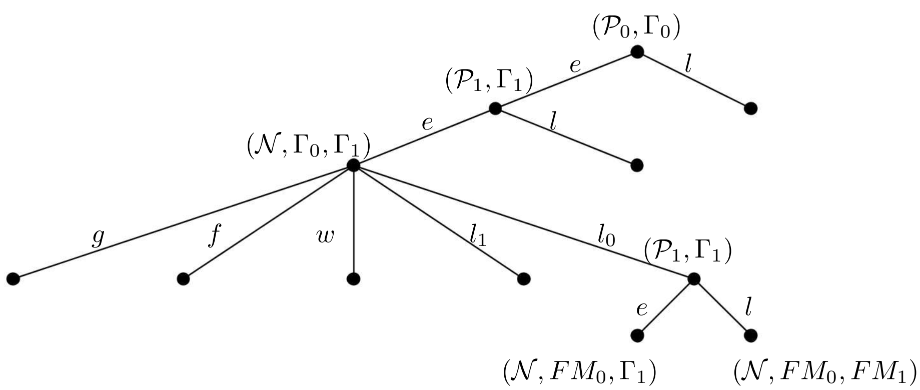
This action is sufficient if the initialized strategy does not enumerate axioms in any of the constructed operators. An $(\mathcal{N}, \Gamma)$-strategy such as $\beta\hat{\ }w$ or $\widehat{\beta}\hat{\ }w$ however enumerates axioms in the operator Γ. When it is initialized it will stop monitoring the correctness of Γ at its witnesses. We will therefore enumerate an axiom $\langle z, \emptyset \rangle$ in Γ if $z \in A$ is a witness of the initialized strategy or an Ω-marker defined by this strategy.

If a witness of the initialized strategy is already extracted from the set A we need to ensure that there are no valid axioms for it in Γ. We will modify the axioms a bit to ensure this. We will transfer the responsibility for the rectification of an operator at witnesses of initialized strategies to the strategy which initializes them. We notice that an $\mathcal{N}$-strategy such as β initializes the $(\mathcal{N}, \Gamma)$-strategies below its outcome w only when it invalidates an axiom for its witness. The axiom for this witness will continue to be invalid at all further stages at which β is visited. So whenever we define an axiom for a witness x of a strategy extending $\beta\hat{\ }w$ it shall have the form $\langle x, G(x) \oplus (B \upharpoonright \gamma(x)+1) \cup U \rangle$, where U is the union of all sets D such that $\langle v, D \rangle$ is a valid axiom in Γ and $v \in A$ is a witness of a higher priority $(\mathcal{N}, \Gamma)$ strategy constructing the same operator Γ. Thus if β with current witness v initializes the strategies extending $\beta\hat{\ }w$ which had enumerated an axiom for a witness x, then this axiom contains an axiom for v which will be invalid at further stages, making the axiom for x invalid as well.

Similarly the axioms enumerated in Λ shall have the form $\langle x, (H(x) \oplus B \upharpoonright \lambda(x)) \cup U \rangle$, where U is the union of all finite sets D such that $\langle v, D \rangle \in \Lambda$ and $v \in A$ is a witness of a higher priority $(\mathcal{N}, \Lambda)$-strategy, constructing the same operator Λ.

One $\mathcal{N}$-requirement below two $\mathcal{P}$-requirements

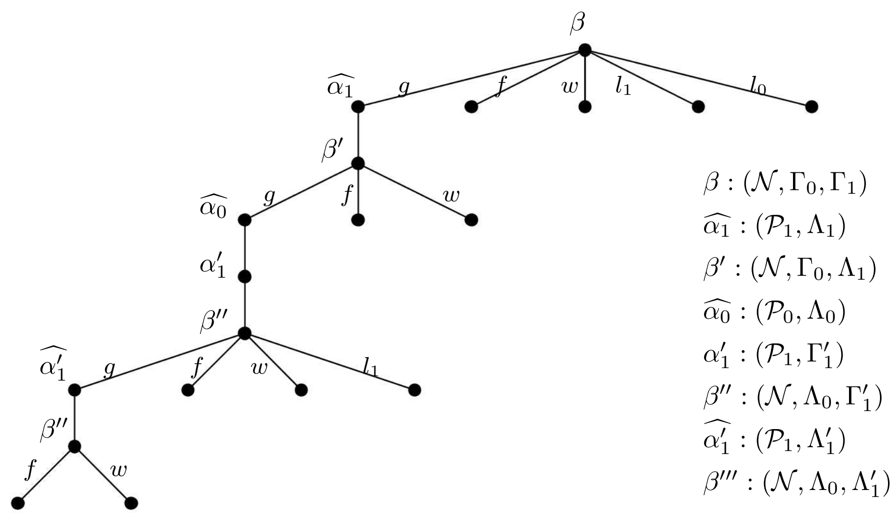
Before we present the full construction we shall discuss the design of an $\mathcal{N}$-strategy working with respect to two $\mathcal{P}$-requirements. Each new $\mathcal{P}_i$-requirement is initially assigned a $(\mathcal{P}_i, \Gamma_i)$-strategy. Suppose we have two such successive strategies α_0 and α_1 working on the requirements $\mathcal{P}_0$ and $\mathcal{P}_1$ and with the operators Γ_0 and Γ_1, respectively. The most general of the strategies for an $\mathcal{N}$-requirement below $\mathcal{P}_0$ and $\mathcal{P}_1$ is the one placed below both e-outcomes, denote it by β. This is an $(\mathcal{N}, \Gamma_0, \Gamma_1)$-strategy which now needs to respect the rectification of both constructed operators Γ_0 and Γ_1.



The strategy β selects a witness x which is enumerated in A. Before x can start its journey along the tree β needs to setup its axioms in both operators Γ_0 and Γ_1. The setup module comes in two copies, one for each operator. The rectification of the operator Γ_0 has higher priority, so β first tries to find a valid axiom for x in $\Xi_0^{\Psi_0^A, \Theta_0^A}$. If the strategy is unsuccessful it has true outcome l_0 and $\mathcal{P}_0$ is globally satisfied. The operator Γ_1 will remain unrectified at this point and therefore we need to restart the $\mathcal{P}_1$-strategy below outcome l_0. Once the sets $G_0(x)$ and $H_0(x)$ are successfully defined the strategy defines the markers $\gamma_0(x)$

and $\omega(\gamma_0(x))$ and enumerates the necessary axioms in the operators Γ_0 and Ω. The strategy β then proceeds to search for a valid axiom for x in $\Xi_1^{\Psi_1^A,\Theta_1^A}$. If it cannot find such an axiom the outcome is l_1, $\mathcal{P}_1$ is satisfied and the operator Γ_0 is correct. After β has successfully defined the sets $G_1(x)$ and $H_1(x)$ as well it defines markers $\gamma_1(x)$ and $\omega(\gamma_1(x))$ and enumerates the necessary axioms in the operators Γ_1 and Ω for x and for both markers $\omega(\gamma_1(x))$ and $\omega(\gamma_0(x))$. Finally we need to enumerate an axiom in Γ_0 for the newly defined $\omega(\gamma_1(x))$. The marker $\omega(\gamma_1(x))$ belongs to A if and only if the marker $\gamma_1(x)$ belongs to B and x belongs to A. Thus we enumerate an axiom which reflects this - constructed from the axiom enumerated in Γ_0 for x by adding the marker $\gamma_1(x)$.

The strategy β then waits for x to enter Φ^B with outcome w while $x \notin \Phi^B$. Once x enters the set Φ^B the strategy β needs to ensure useful extractions from both sets $G_0(x)$ and $G_1(x)$. Of course the extraction of x from A might cause changes in any of the combinations $[G_0(x), G_1(x)]$, $[G_0(x), H_1(x)]$, $[H_0(x), G_1(x)]$, $[H_0(x), H_1(x)]$. Therefore we will need a backup strategy for each of these combinations.



The strategy β performs capricious destruction only on the operator Γ_1 by extracting the marker $\gamma_1(x)$ from B and correspondingly $\omega(\gamma_1(x))$ from A. Note that this action does not injure $x \in \Xi_0^{\Psi_0^A,\Theta_0^A}$ as the marker $\omega(\gamma_1(x))$ is defined as fresh number after the definition of $G_0(x)$ and $H_0(x)$. The strategy then sends the witness x to the first backup strategy β', an $(\mathcal{N}, \Gamma_0, \Lambda_1)$-strategy which constructs the same operator Γ_0 and uses the set $H_1(x)$ to enumerate an axiom for x in the new operator Λ_1. This strategy requires for success the second combination of useful changes $[G_0(x), H_1(x))]$. If the witness x reappears in Φ^B the strategy β' performs capricious destruction on the operator Γ_0 and sends the witness further to a second backup strategy β''. Before the second backup strategy is activated we need to restart the $\mathcal{P}$-strategy on a node α_1', as the original operator Λ_1 might be destroyed: β' extracts the marker $\omega(\gamma_0(x))$, possibly injuring $H_1(x) \subseteq \Theta_1(A)$. The second backup strategy has the form $(\mathcal{N}, \Lambda_0, \Gamma_1')$ and constructs two new operators: Λ_0 using the set $H_0(x)$ to define an axiom for x and Γ_1' for which the setup process is repeated and new finite sets $G_1'(x)$ and $H_1'(x)$ are defined if possible. Finally if x enters the set Φ^B again it is sent to the last backup strategy β''', which is of the form $(\mathcal{N}, \Lambda_0, \Lambda_1')$. It is the strategy that will extract x from A if it reenters Φ^B for the third time.

Depending on the changes that this extraction causes we have the following cases:

- $H_0(x) \nsubseteq A \setminus \{x\}$: If there is no change in either $G_1'(x)$ or $H_1'(x)$, then $\mathcal{P}_1$ is satisfied and α_1' will have outcome l forever. Otherwise the $\mathcal{N}$-requirement will be satisfied by β''' or β''.
- $H_0(x) \subseteq A \setminus \{x\}$: The witness x will be sent back to β' and the axiom for x in Γ_0 will be restored. If $G_0(x) \subseteq A \setminus \{x\}$ then the requirement $\mathcal{P}_0$ will be satisfied and α_0 will have outcome l. If $G_0(x) \nsubseteq A \setminus \{x\}$ then either $H_1(x) \nsubseteq A \setminus \{x\}$ and β' is successful or the witness x is sent back to β and the axiom for x in Γ_1 is restored. If $G_1(x) \subseteq A \setminus \{x\}$ then $\mathcal{P}_1$ is satisfied and α_1 will have outcome l forever, otherwise $G_1(x) \nsubseteq A \setminus \{x\}$ and β is successful.

Thus in every case we have made progress on the satisfaction of requirements as at least one of the considered strategies α_0, α_1, β, β', α_1', β'' or β''' is successful.

We shall put all these ideas in techniques together to define the general construction.

All Requirements

For every requirement we have different possible strategies along the tree. For every $\mathcal{P}$-requirement $\mathcal{P}_i$ we have two different strategies: $(\mathcal{P}_i, \Gamma_i)$ with outcomes $e <_L l$ and $(\mathcal{P}_i, \Lambda_i)$ with one outcome e. For every $\mathcal{N}$-requirement $\mathcal{N}_i$ we have strategies of the form $(\mathcal{N}_i, S_0, \dots, S_i)$, where $S_j \in \{\Gamma_j, \Lambda_j, FM_j\}$. We will call S_j the j-method of this strategy. The possible outcomes of an $(\mathcal{N}_i, S_0, \dots, S_i)$-strategy are

$$g <_L f <_L w <_L l_0 \cdots <_L l_i,$$

although not every strategy shall have all of these outcomes. Before we can make the outcomes precise we shall introduce the notion of dependence between $\mathcal{N}$-strategies:

Definition 4.1 *If α is a node in the tree of strategies labelled by an $(\mathcal{N}_i, S_0, \ldots, S_i)$-strategy then let β be the largest node in the tree with $\beta\hat{}g \subset \alpha$. If there is no such node then we say that α is independent. Otherwise we say that α depends on β. We denote β by $ins(\alpha)$ and call it the instigator of α.*

A dependent strategy α will receive its witnesses from its instigator. The strategy $ins(\alpha)\hat{}g$ will be a $(\mathcal{P}, \Lambda_k)$-strategy for some $k \leq i$. We shall introduce a further parameter related to α, $k(\alpha)$ and its value will be the index of the requirement that $ins(\alpha)\hat{}g$ is working on. In this case $k(\alpha) = k$. If α is independent then $k(\alpha) = -1$. The methods that α works with will be divided into the following groups:

- If $S_j = FM_j$ we shall call it an *invisible* method.
- If $S_j \neq FM_j$ and $j < k$ then it is an *old visible* method.
- If $S_j \neq FM_j$ and $j \geq k$ then it is a *new visible* method.

The strategy α shall then have outcome g only if there is some $j \leq i$ such that $S_j = \Gamma_j$ and an outcome l_j for every new visible method $S_j = \Gamma_j$. Let $\mathbb{O}$ be the set of all possible outcomes and $\mathbb{S}$ be the set of all possible strategies.

The tree of strategies

The tree of strategies is a computable function $T : D(T) \subset \mathbb{O}^{<\omega} \to \mathbb{S}$ which has the following properties:

1. If $T(\alpha) = S$ and O_S is the set of outcomes for the strategy S then for every $o \in O_S$, $\alpha\hat{}o \in D(T)$.

2. The root of the tree is labelled by $(\mathcal{P}_0, \Gamma_0)$. The node e is labelled by $(\mathcal{N}_0, \Gamma_0)$ and the node l is labelled by $(\mathcal{N}_0, FM_0)$.

3. If $T(\alpha) = (\mathcal{N}_i, S_0, S_1, \ldots, S_i)$.

Below outcome g: $T(\alpha\hat{}g) = (\mathcal{P}_k, \Lambda_k)$, where $k \leq i$ is the largest index such that $S_k = \Gamma_k$. The next levels of the subtree with root $\alpha\hat{}g$ are assigned to $(\mathcal{P}_j, \Gamma_j)$-strategies for every j, $k < j \leq i$ such that S_j is visible. After this follows a level of $\mathcal{N}$-strategies $\beta = \alpha\hat{}g\hat{}e \ldots \hat{}o_j \ldots \hat{}o_i$, where $j > k$ and $o_j = \emptyset$ if $S_j = FM_j$, with the structure $(\mathcal{N}_i, S_0, \ldots, \Lambda_k, S'_{k+1} \ldots S'_i)$. For $j > k$ if $S_j = FM_j$ or $o_j = l$ then $S'_j = FM_j$ and otherwise $S'_j = \Gamma_j$.

Below outcomes $\mathbf{f}, \mathbf{w}$: $T(\alpha\hat{\ }o) = (\mathcal{P}_{i+1}, \Gamma_{i+1})$, where $o \in \{f, w\}$. $T(\alpha\hat{\ }o\hat{\ }e) = (\mathcal{N}_i, S_0, S_1, \dots, S_i, \Gamma_{i+1})$ and $T(\alpha\hat{\ }o\hat{\ }l) = (\mathcal{N}_i, S_0, S_1, \dots, S_i, FM_{i+1})$

Below outcome $\mathbf{l_k}$: The first levels of the subtree with root $\alpha\hat{\ }l_k$ are assigned to $(\mathcal{P}_j, \Gamma_j)$-strategies for every j, $k < j \leq i$ such that S_j is visible. After this follows a level of $\mathcal{N}$-strategies $\beta = \alpha\hat{\ }l_k \dots \hat{\ }o_j \dots \hat{\ }o_i$, where $j > k$ and $o_j = \emptyset$ if $S_j = FM_j$, with the structure $(\mathcal{N}_i, S_0, \dots, \Lambda_k, S'_k, \dots, S'_i)$. For $j > k$ if $S_j = FM_j$ or $o_j = l$ then $S'_j = FM_j$ and otherwise $S'_j = \Gamma_j$.

The construction

At each stage s we shall construct a finite path through the tree of outcomes $\delta[s]$ of length s starting from the root. The nodes that are visited at stage s shall perform activities as described below and modify their parameters. Each $\mathcal{N}$-node α shall have a right boundary R_α which will also be defined below. At all stages s the $\mathcal{N}$-strategies on the first level of the tree have $R_l[s] = R_e[s] = \infty$. After the stage is completed all $\sigma > \delta[s]$ will be initialized, their parameters including all their witnesses will be cancelled or set to their initial value $\emptyset$. Whenever we cancel a witness $x \in A[s]$ of a strategy σ we additionally enumerate an axiom $\langle x, \emptyset \rangle$ in every operator constructed by strategies $\delta \leq \sigma$. If $\omega(\gamma_j(x)) \in A[s]$ for any j then we will also enumerate the axiom $\langle \omega(\gamma_j(x)), \emptyset \rangle$ in these operators.

Suppose we have constructed $\delta[s] \restriction n = \alpha$. If $n = s$ then the stage is finished and we move on to stage $s + 1$. If $n < s$ then α is visited and the actions that α performs are as follows:

(I.) $T(\alpha) = (\mathcal{P}_i, \Gamma_i)$.

1. Scan all witnesses $x \notin A[s]$ for which there is an axiom in Γ_i starting from the least.
2. If $x \in \Gamma_i^{\Psi_i^A, B}[s]$ then let the outcome be $o = l$.
3. If all witnesses are scanned and none has produced an outcome $o = l$ then let the outcome be $o = e$.

(II.) $T(\alpha) = (\mathcal{P}_i, \Lambda_i)$.

1. Scan all sent witnesses $x \notin A[s]$ for which there is an axiom in Λ_i starting from the least.
2. If $x \in \Lambda_i^{\Theta_i^A, B}[s]$ with least valid axiom $\langle x, T_x \oplus B_x \rangle$ then define $L_i(x) = use(\Theta_i, A, T_x)[s]$. Restrain A on $L_i(x)$ and return x. End this stage.

3. If all witnesses are scanned and none are returned then let the outcome be e.

(III.) $T(\alpha) = (\mathcal{N}_i, S_0, \ldots, S_i)$ with defined $k(\alpha)$, right boundary $R_\alpha[s]$ and possibly undefined $ins(\alpha)$. We will denote by s^- the previous α-true stage. If α has been initialized since its previous true stage or if it has never before been visited then $s^- = s$. The strategy starts at *Setup* if $s^- = s$, otherwise it goes to the step indicated at s^-. Unless otherwise stated $R_{\alpha\hat{}o}[s] = R_\alpha[s]$.

- **Setup:** If $ins(\alpha)\downarrow$ then wait for a witness x together with its marker $\lambda_{k(\alpha)}(x)$ to be assigned by $ins(\alpha)$. End this stage if there is no assigned witness and return to this step at the next stage. If $ins(\alpha)\uparrow$ choose a new witness x as a fresh number and enumerate it into $A[s]$. Once the witness is defined, for every $j \geq \max(k(\alpha), 0)$ such that S_j is visible perform *Setup*(j) starting from the least such j. Note that if $k(\alpha) \geq 0$ then $S_{k(\alpha)} = \Lambda_{k(\alpha)}$ and if $j > k(\alpha)$ then $S_j = \Gamma_j$.

  Setup(j) for $j = k(\alpha) \geq 0$:

  Enumerate in $\Lambda_j[s]$ an axiom $\langle z, H_j(x) \oplus (B[s] \restriction \lambda_j(x) + 1) \cup U\rangle$, where

  - $z \in A[s]$, there is no valid axiom for z in $\Lambda_j[s]$ and z is x or a witness from a previous cycle of the strategy or z is a marker $\omega(\gamma_l(z'))$ for which there is no valid axiom in Λ_j and z' is x or a previous witness of the strategy.
  - U is the union of all finite sets D such that $\langle n, D\rangle \in \Lambda_j[s]$ is a valid axiom at stage s and $n < x$ is an uncancelled witness in $A[s]$.

  The axiom enumerated for x shall be called *the main axiom* for x in Λ_j. If $j < i$ go to *Setup*($j+1$). Otherwise let the outcome be $o = w$ and go to *Waiting* at the next stage.

  Setup(j) for $j > k(\alpha)$:

  1. If $x \notin \Xi_j^{\Psi_j^A, \Theta_j^A}[s]$ then let the outcome be $o = l_j$ and return to this step at the next stage. Otherwise go to the next step.
  2. Define $G_j(x)$, $H_j(x)$ as finite sets such that $G_j(x) \subseteq \Psi_j^A[s]$, $H_j(x) \subseteq \Theta_j^A[s]$ and $x \in \Xi_j^{H_j(x) \oplus G_j(x)}[s]$. Define $\gamma_j(x)$ and $\omega(\gamma_j(x))$ as fresh numbers. Enumerate $\gamma_j(x)$ in $B[s]$ and $\omega(\gamma_j(x))$ in $A[s]$. Define a new axiom
     $\langle \gamma_j(x), \{\omega(\gamma_j(x))\}\rangle$ in $\Omega[s]$.
     Enumerate in $\Gamma_j[s]$ an axiom $\langle z, G_j(x) \oplus (B[s] \restriction \gamma_j(x) + 1) \cup U\rangle$, where

- $z \in A[s]$, there is no valid axiom for z in $\Gamma_j[s]$ and z is either x, or a witness from a previous cycle of the strategy or $\omega(\gamma_l(z'))$, where $z' = x$ or z' is previous witness of the strategy.
- U is the collection of all finite sets D such that $\langle n, D\rangle \in \Gamma_j[s]$ is a valid axiom at stage s and $n < x$ is an uncancelled witness in $A[s]$.

The axiom enumerated for x shall be called *the main axiom* for x in Γ_j.

3. For all operators S_l, where $l < j$ with current axiom for x, say $\langle x, D_l\rangle$, enumerate the axiom $\langle \omega(\gamma_j(x)), D_l \cup \emptyset \oplus \{\gamma_j(x)\}\rangle$.

If $j < i$ then go to *Setup*$(j+1)$. Otherwise let the outcome be w and go to $Waiting$.

- **Waiting:** If $x \in \Phi_i^B[s]$ and the computation has use $u(\Phi_i, B, x)[s] < R_\alpha[s]$ then go to *Attack*. Otherwise let the outcome be $o = w$ and return to $Waiting$ at the next stage.

- **Attack:**

1. If α does not have an outcome g then extract x from $A[s]$. Go to *Result* 2. Otherwise let j be the largest index such that $\Gamma_j = S_j$ and go to the next step.
2. If there is a returned witness from a previous cycle $\bar{x}$ then go to *Result*. Otherwise go to the next step.
3. Define $R_{\alpha\hat{}g}[s] = \gamma_j(x)$. Extract $\gamma_j(x)$ from $B[s]$ and $\omega(\gamma_j(x))$ from $A[s]$. Define $\lambda_j(x) = \max(\gamma_j(x), use(\Phi_i, B, x)[s])$. Let s_a^- be the previous stage when α sent a witness. *Send* x assigning it to the least strategy β such that $\alpha\hat{}g \subset \beta \subseteq \delta[s_a^-]$ which requires a witness. If this is the first witness then assign it to the least strategy $\beta \supset \alpha\hat{}g$ which requires a witness. Let the outcome be $o = g$. At the next stage start from *Setup*.

- **Result:**

1. Enumerate $\gamma_j(\bar{x})$ back in $B[s]$ and $\langle \omega(\gamma_j(\bar{x})), \emptyset\rangle$ in Ω. *Cancel* all witnesses $z \in A[s]$ of the strategy α. Restrain A on $L_j(\bar{x})$ defined by $\alpha\hat{}g$. Go to the next step.
2. Let the outcome be $o = f$, return to this step at the next stage.

The verification

We start the verification with some of the more easier properties of the construction. We note that the sets A and B are constructed as a 2-c.e. and a 3-c.e. set respectively. It is straightforward to prove also that $B \leq_e A$.

Lemma 4.1 *The set B is enumeration reducible to the set A.*

Proof. We shall prove that $\Omega^A = B$. Fix any number n. If n is not a B-marker of a witness then $n \notin B$ and there is no axiom in Ω for n, so $n \notin \Omega^A$. Suppose n is a marker of a witness x defined by a strategy α at stage s then α enumerates $n \in B[s]$, $\omega(n) \in A[s]$ and an axiom $\langle n, \{\omega(n)\}\rangle$ in $\Omega[s]$. If n is not extracted from B at any stage then neither is $\omega(n)$ and hence the axiom is valid $n \in B \cap \Omega^A$. If n is extracted at stage s_1 then so is $\omega(n)$ and the axiom will remain invalid at all further stages. If n is not reenumerated in B then no further axioms for n are enumerated in Ω and hence $n \notin B \cup \Omega^A$. Otherwise n is reenumerated in B at stage s_2 at which the axiom $\langle \omega(n), \emptyset\rangle$ is enumerated in Ω. As n does not get extracted more than once, $n \in B \cap \Omega^A$.

Another quite easy statement about the tree of strategies is that along each path there are finitely many $\mathcal{P}_i$- and $\mathcal{N}_i$-strategies for every i. We saw that this is the case for $i = 0, 1$ in the preliminary description of the strategies. The rest of the statement follows with an easy induction using the fact that the method for $\mathcal{P}_i$ can be restarted only if the method for $\mathcal{P}_j$, where $j < i$ changes, and after that it can change at most once to Λ_i or to FM_i. The $\mathcal{N}_i$-strategy is restarted only if one of the $\mathcal{P}_j$ methods for $j \leq i$ changes.

The rest of the properties of the construction are quite harder to prove. The main difficulty will be to examine the construction of a certain operator as now many strategies define a single operator in contrast to most previous constructions. Furthermore the axioms for a witness in a fixed operator are related to the axioms of previous witnesses. We shall have to study in detail the interactions between strategies before we can prove that the construction is successful.

Properties of the witnesses

We will first try to establish some properties of the witnesses and the axioms defined for them. The first one is that every witness travels a finite path in the tree of strategies.

Propostion 4.1 *Each witness can be assigned to finitely many strategies.*

Proof.

Suppose x is a witness defined by the $(\mathcal{N}_i, S_0, \ldots, S_i)$-strategy β. Then β is an independent strategy. Suppose that x is β's first witness. If it is sent by β at stage s then it will be assigned to the first $\mathcal{N}$-strategy β_1 extending $\beta\hat{\ }g$. This is also an $\mathcal{N}_i$-strategy and x will also be β_1's first witness. As there are only finitely many $\mathcal{N}_i$-strategies along each path in the tree, the witness x will be assigned to finitely many strategies.

Suppose that x is β's n-th witness. Consider the sequence$\{(\beta_k, i_k, n_k)\}$, where β_k is the k-th strategy to which x is assigned, i_k denotes the index of the $\mathcal{N}$-requirement that β_k works with and n_k denotes that x is β_k's n_k-th witness. We know already that the sequence is finite if for some k we have $n_k = 1$. We will prove that:

If $i_{k+1} = i_k$ then $n_{k+1} \leq n_k$ and if $i_{k+1} > i_k$ then $n_{k+1} < n_k$.

Thus for almost all k we have $i_k = i_{k+1}$ and as there are only finitely many $\mathcal{N}_i$-strategies for every i, the sequence is finite and the proposition follows.

The first part of this statement is quite obvious. The strategy β_{k+1} receives all its witnesses from β_k so $n_{k+1} \leq n_k$. Suppose that $i_{k+1} > i_k$. From the definition of the tree it follows that there is an $\mathcal{N}_{i_k}$-strategy σ such that $\beta_k \subset \sigma \subset \beta_{k+1}$. Then before the first witness is assigned to β_{k+1} one of β_k's witnesses must be assigned to σ, thus $n_{k+1} < n_k$. $\square$

Propostion 4.2 *Suppose β is an $\mathcal{N}$-strategy.*

1. If β sends its witness at stage s then the next witness assigned to β is defined after stage s.

2. If β is initialized at stage s_i and β is not independent then the next witness that β works with will be defined after the next β-true stage $s > s_i$.

3. Suppose β is not initialized after stage s_i and visited at infinitely many stages. If at stage $s > s_i$ the strategy does not have an assigned witness then it will eventually be assigned a witness.

Proof.

1. This is obviously true for independent strategies. Let $\beta_0\hat{\ }g \subset \beta_1\hat{\ }g \ldots \beta_k + 1 = \beta$ be the strategies such that β_0 is independent and $ins(\beta_{i+1}) = \beta_i$ for $i < k$. Every witness assigned to β is defined by β_0.

Suppose that β sends its witness at stage s. Then at stage s all of these strategies have outcome g and send their witnesses. Thus the next witness that β_0 uses is defined after stage s. At stage $s+1$ each strategy β_{i+1} does not have a defined witness. It will receive its witness from β_i at the next stage $t \geq s+1$ at which β_i has outcome g and sends its witness.

2. If β is initialized at stage s_i then a strategy $\sigma \subset \beta$ has outcome o such that $\sigma\hat{}o <_L \beta$. If at stage s_i a witness is assigned to β then it is cancelled at stage s_i. Before the next witness is assigned to β there must be a stage s at which β is visited. Then at stage s the instigator $ins(\beta)$ sends its witness and by step 1. of this proposition its next witness will be defined after stage s.

3. This is again obviously true for independent strategies. Let $ins(\beta) = \delta$. Then $\delta\hat{}g$ is visited infinitely often and not initialized after stage s. There are finitely many strategies α such that $\delta\hat{}g \subset \alpha\hat{}o \subseteq \beta$ and for every such strategy $o \neq g$. Suppose at stage s the strategy α is the least such strategy that also has no witness. The strategy β is visited at stage $s_1 \geq s$. At the next $\delta\hat{}g$-true stage $s_2 > s_1$ if α still has no witness then the witness that δ sends at stage s_2 will be assigned to α. As β is not initialized at stages $t \geq s_i$ this will remain α's permanent witness. As there are finitely many such strategies α they will each be assigned a permanent witness eventually. After this a witness will finally be assigned to β. □

These two properties have a very important consequence which tells us a bit about the true path. It shows that the outcomes e and l of a $\mathcal{P}$-strategy are finitary. Thus the only infinitary outcome in this construction is the outcome g.

Propostion 4.3 *Let α be a $(\mathcal{P}_i, \Gamma_i)$-strategy initialized at stage s_1 and not initialized at stages t such that $s_1 < t < s_2$. If α has outcome l at a least stage s such that $s_1 \leq s < s_2$ then α has outcome l at all true stages t, $s < t < s_2$.*

Proof. Suppose this is true for higher priority strategies than α. Any strategy $\sigma \subset \alpha$ has outcome g at stage s or does not change its outcome at stages t, $s < t < s_2$. This follows from the induction hypothesis for $\mathcal{P}$-strategies. For $\mathcal{N}$-strategies with outcome $o \neq g$ it follows from the construction: σ is not initialized at stages $s < t < s_2$ so if it changes its outcome to o' at stage t then $o' <_L o$ and α would be initialized. Furthermore all of these strategies have a permanent witness for which they do not act by extracting elements at stages t, $s < t < s_2$. Strategies that have outcome g send their witnesses at stage s. A witness sent by σ is assigned to a strategy which was visited during σ's previous attack, thus is not assigned to a strategy extending $\alpha\hat{}l$. At stages t, $s < t < s_2$ accessible strategies have witnesses defined after stage s. This follows from Proposition 4.2 and the fact that all strategies $\delta \geq \alpha\hat{}l$ are in initial state at stage s. These witnesses together with their A- and B-markers are therefore larger then any number that has appeared in the construction until and including at stage s. At stage s the strategy α sees a valid axiom in Γ_i for a witness $x \notin A[s]$. This axiom remains valid at all further stages $t < s_2$ and whenever α is visited it will have outcome l. □

The next two properties will give us rules about the cancellation of a witness.

Propostion 4.4 *Suppose x is a witness that is defined at stage s_0 and sent or extracted at sub-stage s. If z is defined at substage t_0 with $s_0 < t_0 < s$ it is cancelled at the latest at stage s.*

Proof. Note that x is not cancelled until and at substage s. Let β_0 denote the strategy which defines x and δ_0 the strategy which defines z.

If $\beta_0 < \delta_0$ then $\beta_0\hat{\ }f <_L \delta_0$ as strategies below outcome $\beta_0\hat{\ }g$ do not define witnesses, rather they receive them from β_0 and strategies below outcome f are not accessible until x is extracted. Then δ_0 together with all its successors is initialized at stage s. The witness z, if not already cancelled, is assigned at stage s to a strategy extending δ_0 and hence is cancelled.

If $\delta_0 < \beta_0$ then similarly $\delta_0\hat{\ }g <_L \beta_0$. The witness z is defined at stage $t_0 > s_0$ so δ_0 is either in initial state at stage t_0 or at the previous δ_0-true stage t, $s_0 < t < t_0$, the strategy δ_0 sends its previous witness having outcome g. In all cases the strategy β_0 is in initial state at stage t_0 and x is cancelled contrary to assumption.

Finally suppose that $\delta_0 = \beta_0$. Let $\beta_0, \dots, \beta_k$ be all strategies to which x is assigned until stage s at stages $s_0 < s_1 < \dots < s_k \leq s$ respectively. Then $t_0 > s_1$. At stage $s \geq t_0$ the witness x is extracted or sent by β_k thus every strategy β_i, $i < k$ has outcome g at stage s. It follows that z is sent by β at stage t_1 such that $s_1 < t_0 < t_1 \leq s$ and assigned to a strategy δ_1.

Again we have three cases. If $\beta_1 < \delta_1$ then δ_1 is initialized at stage s, z is cancelled. If $\delta_1 <_L \beta_1$ then β_1 is in initial state at stage t_1 and x cancelled contrary to assumption. The final case is $\beta_1 = \delta_1$. Then $s_2 < t_1$. The same argument for $i = 1, 2, \dots, k-1$ proves that $\beta_i \leq \delta_i$ and if $\delta_i \neq \beta_i$ then z is cancelled at stage s, where δ_i denotes the i-th strategy to which z is assigned. If $\delta_i = \beta_i$ then $t_i > s_{i+1}$, where t_i denotes the stage at which z is assigned to β_i. Now as β_k extracts or sends x at stage s the witness z is sent by β_{k-1} at a stage t_k such that $s_k < t_k \leq s$. At stage t_k the strategy β_k does not require a witness. Thus if z is not cancelled already by stage s it is assigned to a strategy $\delta_k >_L \beta_k\hat{\ }f$ and hence z is cancelled at stage s at which β_k has outcome f or g.
□

Propostion 4.5 *If x is a witness with marker $m_j(x)$, where m_j is either γ_j or λ_j, defined at stage s_0 and a marker $\gamma_l(z) < m_j(x)$ of a different witness $z \neq x$ is extracted from B at stage $s > s_0$ then x is cancelled.*

Proof. Any B-marker defined after stage s_0 is greater than $m_j(x)$. Suppose that the marker $\gamma_l(z)$ is defined at stage $t_0 \leq s_0$ and extracted by δ at stage s. Suppose that x is assigned to β at stage s.

If $\delta\hat{\ }g <_L \beta$ then β is initialized at stage s and x is cancelled.

If $\beta <_L \delta$ then δ is initialized at the last β-true stage $t < s$. The marker $m_j(x)$ must be defined before stage t, hence $s_0 < t$ otherwise it will be defined after stage s. The witness z must be defined after stage t by Proposition 4.2 hence $t < t_0$. Thus $s_0 < t < t_0$ contradicting the assumptions.

If $\beta\hat{\ }o \subset \delta$ we shall examine the different possibilities for o. If $o = g$ then at stage s the strategy β has outcome g, sends its witness and does not have a witness when δ is visited. In all other cases δ is in initial state when x is assigned to β. The marker $m_j(x)$ must be defined before the next δ-true stage t. Then the witness z is defined at $t_0 > t$ if δ is not independent by Proposition 4.2 or at stage $t_0 \geq t$ if δ is independent. Thus the marker $m_j(x)$ is defined before the marker $\gamma_l(z)$ contrary to assumption.

Finally suppose $\delta\hat{\ }g \subset \beta$. Any witness assigned to β must first be sent by δ. It follows that $z > x$ and δ has already sent the witness x at a previous stage $\delta\hat{\ }g$-true stage. By Proposition 4.2 the witness z is defined after the last $\delta\hat{\ }g$-true stage $t < s$ and this is the last stage when strategies to which x is assigned might be accessible to define the markers of x. Thus $s_0 \leq t < t_0$. □

Properties of the axioms

This section reveals some properties of the axioms in the constructed operators. Our main goal will be to prove that if a $\mathcal{P}$-strategy has outcome l at all but finitely many stages then the corresponding $\mathcal{P}$-requirement is satisfied. We shall need to investigate the axioms that are enumerated in an operator for elements x which are extracted from A. We shall prove three properties for the axioms. First we will show a connection between a witness x and a witness z such that an axiom for x is enumerated in an operator using the main axiom for z. This rather technical property will enable us to prove that the only axiom that can be valid for a witness $x \notin A[s]$ at an operator S_i is the main axiom for x in S_i. Finally we shall show that if the main axiom for a witness $x \notin A[s]$ is valid in S_i then $\Xi_i^{\Psi_i^A, \Theta_i^A} \neq A$.

Propostion 4.6 *Let α be a $(\mathcal{P}_i, S_i)$-strategy and x be a witness which is not cancelled until stage s and for which there is an axiom in the operator constructed by α. Suppose that δ invalidates the main axiom for x. Then every further* *axiom for x related to a different witness z remains valid at all stages $t \leq s$ or is invalidated by the same strategy δ, to which z is sent eventually.*

Proof. Suppose x is assigned to strategies $\beta_0 \subset \beta_1 \subset \beta_k$ at stages $s_0 < s_1 < \cdots < s_k \leq s$, where β_0 is the strategy which enumerates the main axiom for x in S_i at stage s_0. At stage s_0 all strategies $\sigma >_L \beta_0\hat{}g$ are in initial state and will work with witnesses defined after stage s_0. Strategies below $\beta_0\hat{}g$ are not accessible until stage s_1. At stage s_i the witness x is assigned to β_i strategies σ such that $\beta_{i-1} \subset \sigma \subset \beta_{i+1}$ have a defined witness which does not change and do not extract any numbers from A or B at stages $s_i \leq t \leq s_k$ or else x would be cancelled before stage s_k. Strategies $\sigma >_L \beta_i$ are in initial state at stage s_i and work with witnesses defined after stage s_i. Thus the only strategies that can invalidate the axiom for x are among $\beta_0, \ldots, \beta_k$.

If $\delta = \beta_k$ then it must extract x as otherwise x would be sent to a further strategy. Thus no new axioms will be enumerated in S_i.

Suppose $\delta = \beta_i$, $i < k$. Then δ has outcome g extracting a B-marker of x at stage t_0. At the next β_0-true stage t_1 the strategy β_0 defines a new axiom for x using its new current witness z. If this witness is never sent then the axiom remains valid at all stages $t \leq s$ as the only accessible strategies are in initial state at stage t_1. If this witness is sent it is assigned to the least strategy visited at stage t_0 which requires a witness. By the argument above this must be β_1. If β_1 does not send z then the axiom for z remains valid at all further stages otherwise β_1 sends z and it is assigned to β_2.

Thus eventually z will reach δ at stage t_2 with a valid main axiom in S_i. At all stages t with $t_1 < t \leq t_2$ there is a valid axiom for x in S_i - the one that uses main axiom for z, thus β_0 does not enumerate any further axioms for x. If the axiom for z is not invalidated by δ or it is invalidated at the same stage at which x extracted then no more axioms will be enumerated in S_i for x. Otherwise δ invalidates the axiom for z at stage t_3 and at the next β_0-true we have a very similar situation as at stage t_1: at stage t_3 all strategies $\beta_0, \ldots, \delta\hat{}g$ were visited and there is no valid axiom for x. The strategy β_0 will define a witness z' and enumerate an axiom for x and z in S_i using the main axiom for z'. If this axiom is invalidated then the witness z' must be sent to δ and δ invalidates it. □

Corollary 4.1 *Let x be any witness extracted from A at stage s and α be a $(\mathcal{P}_i, S_i)$-strategy such that there is an axiom for x in S_i. The only axiom in S_i that can be valid at a further stage $t > s$ is the main axiom for x.*

Proof. Suppose that there is a different axiom for x valid at stage $t > s$ and it uses the main axiom for $z > x$ defined before stage s. It follows from the proof of proposition 4.6 that this witness z is sent to the same strategy δ that invalidates the main axiom for x. Otherwise x could not be extracted at stage s. This strategy has greatest Γ-method with index $k \leq i$ and always extracts a B-marker $\gamma_k(y)$ when it sends its witness y. Before x is extracted it must send z at stage s_1 invalidating the axiom for z. If this axiom is valid at stage $t > s$ then z must be returned by $\delta\hat{}g$, constructing the operator Λ_k after stage s. We will prove that this is impossible.

At stage s_1 the witness z is assigned to the least strategy which requires a witness. Suppose δ_1 is the strategy to which x was assigned after it was sent by δ. Consider a strategy σ such that $\delta \subset \sigma\hat{}o \subseteq \delta_1$. Then $o \neq g$ as otherwise x would be assigned to σ. Furthermore σ works with the same operator Λ_k as this method can change only below a further g-outcome. Until x is extracted σ has the same outcome o or else x would be cancelled. Thus z is assigned to a strategy $\delta_1' \supseteq \delta_1$. And by the same argument both δ_1 and δ_1' construct the same operator Λ_k.

If $\delta_1' \neq \delta_1$ then at stage s_1 the strategy δ_1 has outcome $o \neq g, f$ and it has this outcome until δ_1' is cancelled. At all such stages there is a valid axiom for x in Λ_k defined by δ_1 which does not change and it is included in any axiom for z that δ_1' defines. The element z is cancelled at stage s at which δ_1 has outcome g or f.

If $\delta_1' = \delta_1$ then both x and z are witnesses for of δ_1. Every axiom enumerated in Λ_k for z either includes an axiom for x or otherwise the same axiom is enumerated for x and all axioms for z are enumerated before stage s as z is cancelled at stage s by Proposition 4.4.

Thus in both cases if z can be returned by $\delta\hat{}g$ at stage s_z then there is a valid axiom for both x and z in Λ_k. If we assume that $s_z \leq s$ then x could not be extracted at stage s as $\delta\hat{}g$ ends stage s_z prematurely and δ would have outcome f at all stages $t > s_z$ until it is initialized. Thus $s < s_z$, the witness x is already extracted from $A[s_z]$ and $\delta\hat{}g$ will return x instead of z. □

Propostion 4.7 *Let α be a $(\mathcal{P}_i, \Gamma_i)$-strategy and let $\beta \supseteq \alpha\hat{}e$ be a strategy such that $S_i = \Gamma_i$ and this is the largest Γ-method at β. Suppose a witness x is returned to β at stage s and β restrains A on $L_i(x)$. If this restraint is injured at stage $s_1 > s$ then there is no valid axiom for x in Γ_i at all stages $t > s_1$ or else $\Xi_i^{\Psi_i^A, \Theta_i^A} \neq A$.*

Proof. Suppose the lemma is true inductively for witnesses $z < x$.

If α is initialized at stage s_1 then there will be no valid axiom for x in Γ_i at any further stage. Suppose that α is not initialized at stages t, $s \leq t \leq s_1$.

Any strategy that at stage s is in initial state or does not have an assigned witness will not injure the restraint by Proposition 4.2. The restraint is therefore injured by a strategy $\delta_1 \supseteq \alpha\hat{}e$ such that $\delta_1 \leq \beta$. In order for this strategy to be accessible there must be a strategy $\delta \supseteq \delta_1$ such that $\alpha\hat{}e \subset \delta\hat{}o \subset \beta$, $o \neq g$, and which has outcome g at stage s_1.

The strategy δ has the same witness $y < x$ and the same outcome o at all stages at which it is visited from the stage s_0 at which x is assigned to β until and including at stage s. Furthermore it works with the same operator Γ_i and the main axiom for y is not yet invalidated. The main axiom for x includes a valid axiom for every one of δ's witnesses $z \leq y$ and every B-marker defined for such a witness before stage s_0. Any further B-marker for a witness of δ is defined after stage s and the corresponding A-marker respects the restraint.

At stage s_1 the strategy δ_1 injures the restraint on A. Therefore it must extract from A a witness $z \leq y$ defined before stage s_0 or an A-marker $\omega(\gamma_l(z))$ together with $\gamma_l(z)$ for a witness $z \leq y$ both defined before stage s_0. If $z \in A$ then δ_1 extracts $\gamma_l(z)$ which invalidates all axioms for x and this marker is never reenumerated in B.

If $z \notin A$ and there is a valid axiom for z in Γ_i then by Corollary 4.1 this is the main axiom for z and by the induction hypothesis $H_i(z) \subseteq \Theta_i(A)$ hence $z \in \Xi_i^{\Psi_i^A, \Theta_i^A}$. Otherwise there is no valid axiom for z and hence no valid axiom for x. $\square$

Satisfaction of the requirements

We define the true path h to be the leftmost path in the tree such that the strategies along it are visited at infinitely many stages. As in two cases of the construction a strategy can end a stage prematurely we will need to prove that the so defined path is infinite. Once we have established that this is true we can prove that all $\mathcal{N}$- and $\mathcal{P}$-requirements are satisfied.

Lemma 4.2 *There is an infinite path h in the tree of strategies with the following properties:*

1. $(\forall n)(\exists^\infty s)[h \upharpoonright n \subseteq \delta[s]]$.
2. $(\forall n)(\exists s_l(n))(\forall s > s_l(n))[\delta[s] \geq h \upharpoonright n]$, *i.e. $h \upharpoonright n$ is not initialized after stage $s_l(n)$.*

Proof. We prove the statement with induction on n. The case $n = 0$ is trivial: $h \upharpoonright 0 = \emptyset$ is visited at every stage of the construction and is never initialized, $s_l(0) = 0$.

Suppose the statement is true for $h \restriction n = \alpha$. If α is a $(\mathcal{P}_i, \Gamma_i)$-strategy by Proposition 4.3 either α has outcome e at every α-true stage in which case $h(n+1) = e$ and $s_l(n+1) = s_l(n)$, or there is a stage $s > s_l(n)$ such that α has outcome l at every true stage $t > s$, so $h(n+1) = l$ and $s_l(n+1) = s$.

If $\alpha = \beta\hat{\ }g$ is a $(\mathcal{P}_i, \Lambda_i)$-strategy then α does not returns a witness after stage $s_l(n)$. Otherwise β will have outcome f at almost all true stages contradicting the assumption that α is visited at infinitely many stages. Thus α has outcome e at every true stage $t \geq s_l(n)$ and $h(n+1) = e$, $s_l(n+1) = s_l(n)$.

If α is an $(\mathcal{N}_i, S_0, \dots, S_i)$ then we have the following cases:

- α has outcome g at infinitely many stages. Then $h(n+1) = g$, $s_l(n+1) = s_l(n)$.
- There is a stage $s > s_l(n)$ at which α receives back a witness. Then α has outcome f at all further stages, $h(n+1) = f$, $s_l(n+1) = s$.
- There is a stage s at which α attacks for the last time. By Proposition 4.2 α will be assigned a new witness x at a stage $s_1 > s$. If α enters *Setup*(j) at stage $s_2 > s$ and never completes it then α has outcome l_j at all stages $t > s_2$, $h(n+1) = l_j$, $s_l(n+1) = s$. Otherwise there is a stage s_3 at which α enters *Waiting* and then α has outcome w at all stages $t > s_3$, $h(n+1) = w$, $s_l(n+1) = s$.

□

Lemma 4.3 *Every $\mathcal{N}$-requirement is satisfied.*

Proof. Let β be the last $\mathcal{N}_i$-strategy along the true path. Then $\beta\hat{\ }w \subset h$ or $\beta\hat{\ }f \subset h$ as along all paths below every other outcome of β there is another $\mathcal{N}_i$-strategy. By Lemma 4.2 the strategy β has a permanent witness x at stages $t \geq s_l(|\beta|+1)$. If $\beta\hat{\ }w \subset h$ then $x \in A$ and at every true stage $t > s_l(|\beta|+1)$ if $x \in \Phi_i^B[t]$ then $use(\Phi_i, B, x)[t] > R_\beta[t]$. If β is independent then $R_\beta[t] = \infty$. Otherwise at every stage t the right boundary is defined by $ins(\beta) = \alpha$. If α has witness z at stage t then $R_\beta[t] = \gamma_{k(\beta)}(z)$. The next witness that α uses is defined after stage t and its B-markers are of value greater than $R_\beta[t]$. Thus $\lim_t R_\beta[t] = \infty$ and $x \notin \Phi_i^B$.

Suppose $\beta\hat{\ }f \subset h$. If β has an outcome g the witness x is returned by $\beta\hat{\ }g = \alpha$ which is a $(\mathcal{P}_j, \Lambda_j)$-strategy at stage $s = s_l(|\beta|+1)$. When β sent this witness at stage $s_0 < s$ we had $x \in \Phi_i^B[s_0]$. The strategy then defined the marker $\lambda_j(x) \geq use(\Phi_i, B, x)[s_0]$. As x is not cancelled at any stage by Proposition 4.5

no B-marker $b < \lambda_j(x)$ for a different witness $z \neq x$ is extracted at any stage $t \geq s_0$.

At stage s_0 the main axiom for x, say $\langle x, A_x \oplus B_x \rangle$ is enumerated in the operator Λ_j constructed at α and $B[s_0] \upharpoonright \lambda_j(x) \setminus \{\gamma_j(x)\} \subseteq B_x$. The strategy α returns this witness at stage s as it is the least $x \in \Lambda_j^{\Theta_j^A, B} \setminus A[s]$. By Corollary 4.1 the only axiom that can be valid at stage s is the main axiom for x in Λ_j. So $B[s_0] \upharpoonright \lambda_j(x) \setminus \{\gamma_j(x)\} \subseteq B[s]$, no more markers for x are extracted at any stage $t > s$, and at stage $s_l(|\beta|+1)$ the strategy β enumerates $\gamma_j(x)$ back in the set B. So $x \in \Phi_i^B[t]$ at all stages $t \geq s_l(|\beta|+1)$ and hence $x \in \Phi_i^B \setminus A$.

Suppose β does not have an outcome g. Then at stage $s_l(n+1) = s$ the strategy sees $x \in \Phi_i^B[s]$ and extracts x from the set A. Let $u = use(\Phi_i, B, x)[s]$. Strategies $\sigma\hat{\ }o \subset \beta$ with $o \neq g$ do not extract any markers from the set B. Strategies $\sigma\hat{\ }g \subset \beta$ have just sent their witness and by Proposition 4.2 will not extract any markers that are less than u. Strategies $\delta \geq \beta\hat{\ }f$ are in initial state at stage s and by the same proposition will not extract markers of value less than u. Thus $B[s] \upharpoonright u \subseteq B[t]$ at all $t \geq s$ and hence $x \in \Phi_i^B \setminus A$. □

Lemma 4.4 *Every $\mathcal{P}$-requirement is satisfied.*

Proof. Let α be the last $(\mathcal{P}_i, S_i)$-strategy along the true path.

If $\alpha\hat{\ }l \subseteq h$ then α is a $(\mathcal{P}_i, \Gamma_i)$-strategy. Let $x \notin A$ be the witness such that $x \in \Gamma_i^{\Psi^A, B}$. There is a least strategy $\beta \supseteq \alpha\hat{\ }e$ such that x is assigned to and whose greatest Γ-method is Γ_i. Before x is extracted from A the marker $\gamma_i(x)$ is extracted from B. As $x \in \Gamma_i^{\Psi^A, B}$ then by Corollary 4.1 the main axiom for x in Γ_i is valid and hence $\gamma_i(x)$ is enumerated back in B by β on a stage s at which β restrained $H_i(x)$ in Θ_i^A. By Proposition 4.7 if this restraint is injured then $\Xi_i^{\Psi_i^A, \Theta_i^A} \neq A$. If this restraint is not injured then $G_i(x) \oplus H_i(x) \subset \Psi_i^A \oplus \Theta_i^A$ and again $\Xi_i^{\Psi_i^A, \Theta_i^A} \neq A$ as $x \in \Xi_i^{\Psi_i^A, \Theta_i^A} \setminus A$.

Suppose α is a $(\mathcal{P}_i, \Gamma_i)$-strategy such that there is an $\mathcal{N}$-strategy β working with i-th method Γ_i and $\beta\hat{\ }l_i \subset h$. Then β has a permanent witness x such that $x \in A \setminus \Xi_i^{\Psi_i^A, \Theta_i^A}[t]$ at all β-true stages $t > s_l(|\beta|+1)$. The requirement is satisfied by $A \neq \Xi_i^{\Psi_i^A, \Theta_i^A}$.

For all other cases denote by U the set Ψ_i^A if $S_i = \Gamma_i$ and Θ_i^A if $S_i = \Lambda_i$. We will prove that for all elements n enumerated in A at stages $t > s_l(n)$ we have $S_i^{U,B}(n) = A(n)$. Thus $A \leq_e U \oplus B$ and the requirement $\mathcal{P}_i$ is satisfied.

Let $n \notin A$ be a witness. If n is extracted at stage s_n then at all α-true stages $t > \max(s_l(n), s_n)$ we have $n \notin S_i^{U,B}[t]$. Otherwise if $S_i = \Gamma_i$ then by Proposition 4.3 the strategy α would have true outcome l and if $S_i = \Lambda_i$ the witness n would be returned by α which is impossible as we saw in the proof of Lemma 4.2. Thus $n \notin S_i^{U,B}$.

Let $n \notin A$ be an A-marker $\omega(\gamma_l(z))$. Every axiom for n in S_i is of the form $\langle n, D \cup \{\gamma_l(z)\}\rangle$ and there is similar axiom $\langle z, D\rangle$ for z in S_i. As $n \notin A$ the marker $\gamma_l(z)$ is extracted from B. If an axiom for n is valid at a further stage then $\gamma_l(z)$ is reenumerated in B and hence $z \notin A$. By the argument above there is no valid axiom for z and hence for n in S_i at any α-true stage.

If $n \in A$ and n is cancelled then there is valid axiom $\langle n, \emptyset\rangle \in S_i$. Thus $A(n) = S_i^{U,B}(n)$. Suppose n is a witness that is never cancelled. We will prove that there is a valid axiom for n in S_i. Let $\beta_0, \dots, \beta_k$ be all strategies to which n gets assigned in the course of the construction. As n is not cancelled $h \not<_L \beta_k$. Furthermore $\beta_k \supseteq \alpha\hat{\ }e$. Otherwise β_k would not be visited after stage $s_l(|\alpha|)$ and hence the witness x must be assigned to β_k before or at this stage. We are however dealing with witnesses that are defined after stage $s_l(|\alpha|)$.

Consider the least strategy $\beta_j \supseteq \alpha\hat{\ }e$. First we observe that $\beta_j \subset h$. If we assume otherwise then there is a strategy σ such that $\alpha\hat{\ }e \subset \sigma\hat{\ }o_1 \subset h$ and $\beta_j \supseteq \sigma\hat{\ }o_2$ and $o_2 <_L o_1$. Then $o_2 = g$ or else β_j is initialized before stage $s_l(|\sigma|)$ and not accessible after this stage and x is cancelled. But if $o_2 = g$ then β_j receives n from σ, so $\sigma = \beta_{j-1}$ and this contradicts our choice of β_j as the least strategy below $\alpha\hat{\ }e$.

The i-method of β_j is hence new and is S_i, as no strategy σ along the true path has outcome l_i and there is no strategy between α and β_j has outcome g, the only cases when the i-method changes. Thus β_j will enumerate axioms for n at all β_j-true stages at which there is no valid axiom in S_i.

If the main axiom $\langle n, D\rangle$ for n enumerated by β_j is never invalidated then $n \in S_i^{U,B}$. For every A-marker of n that is never extracted and is defined by stage $s_l(|\beta_j|)$, the strategy β_j enumerates an axiom in S_i using the current axiom for n. If a further A-marker $m = \omega(\gamma_k(n))$ for n is defined after this stage by a strategy β then $\beta \supseteq \beta_j$ and β has the same method S_l as β_j for $l \leq i$ otherwise the main axiom for n would be invalidated. As β can define a marker only for a new method, $k > i$ and β enumerates a new axiom for m of the form $\langle m, D \cup \emptyset \oplus \{\gamma_k\}\rangle$ in S_i. If $m \in A$ then $\gamma_k(n) \in B$ and this axiom is valid at all further stages.

Suppose that the main axiom for n in S_i is invalidated by δ at stage $s_0 > s_l(|\beta_j|)$. By Proposition 4.6 this is done by a strategy β_l, $l > j$. At the next true stage β_j enumerates an axiom for x using the main axiom for its current witness z. If this axiom is invalidated at all, it is invalidated by β_l. Now as β_l extracts a B-marker for a method with index less than i. It follows that $\beta_l\hat{\ }g$ is not on the true path, as otherwise there would be a further $\mathcal{P}_i$-strategy along the true path. Let s be the last $\beta_l\hat{\ }g$-true stage. Then the axiom for n enumerated at the first β_j-true stage after s will remain valid forever. Any A-marker of n, $m = \omega(\gamma_l(n)) \in A$ must be defined before stage s. Then if there is no valid axiom for m at the first β-true stage after s then an axiom is enumerated for m during $Setup(i)$. The axiom for m in S_i valid at this stage will remains valid forever.

This concludes the proof of the lemma and the theorem. □

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