A Non-splitting Theorem in the Enumeration degrees ⋆

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Abstract

We complete a study of the splitting/non-splitting properties of the enumeration degrees below $\mathbf{0_e'}$ by proving an analog of Harrington's non-splitting theorem for the Σ_2^0 enumeration degrees. We show how non-splitting techniques known from the study of the c.e. Turing degrees can be adapted to the enumeration degrees.

Key words: Enumeration reducibility, Σ_2^0 e-degrees, Non-splitting

1 Introduction

In an upper semi-lattice $\langle D, <, \cup \rangle$ we say that a pair of elements $\mathbf{u}$ and $\mathbf{v}$ form a splitting of the element $\mathbf{a}$ if $\mathbf{u} < \mathbf{a}$ and $\mathbf{v} < \mathbf{a}$ but $\mathbf{u} \cup \mathbf{v} = \mathbf{a}$. Sacks [11] showed that every computably enumerable (*c.e.*) degree $> \mathbf{0}$ has a c.e. splitting and [12] that the combutably enumerable degrees are dense. It had been commonly believed (see [1]) that these two results can be combined. Lachlan [9] showed that this is not the case by proving the existence of a c.e. $\mathbf{a} > \mathbf{0}$ which has no c.e. splitting above some proper c.e. predecessor. His proof introduced the $0'''$-priority method and for the first time made use of a tree of strategies. This technique is significantly more complicated than any other known at the time and the article came to be known as " The monster paper". Harrington's work presented as hand-written notes [8] led to a better understanding of the technique. He improved the result by showing that one could take $\mathbf{a} = \mathbf{0}'$. The technique has been widely used thereafter and has had a number of consequences for definability and elementary equivalence in the Turing degrees below $\mathbf{0}'$.

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Naturally we turn our attention to the richer semi-lattice of the enumeration degrees. Intuitively we say that a set A is *enumeration reducible* to a set B, denoted as $A \leq_e B$, if there is an effective procedure to enumerate A given any enumeration of B(a formal definition shall be given in Section 2). By identifying sets that are reducible to each other we obtain a degree structure, the structure of the enumeration degrees $\langle \mathcal{D}_e, \leq \rangle$. It is an upper semi-lattice with jump operator and least element $\mathbf{0}_e$, the collection of all computably enumerable sets. An important substructure of $\mathcal{D}_e$ is given by the Σ^0_2 e-degrees. Cooper [5] proved that the Σ^0_2 e-degrees are the e-degrees below $\mathbf{0}'_e$.

One of the main motives for studying the structure of the enumeration degrees is given by Rogers' embedding ι, an order theoretic embedding of the Turing degrees into the enumeration degrees which preserves least upper bound and the jump operator. Rogers' ι embeds the c.e. Turing degrees exactly onto the Π^0_1 enumeration degrees. Thus all structural properties of the Turing degrees including Lachlan's and Harrington's non-splitting theorems can be transferred to the Π^0_1 enumeration degrees, a proper subclass of the Σ^0_2-enumeration degrees. Cooper and Soskova [7] generalized Harrington's theorem further by showing the existence of a Π^0_1 enumeration degree $\mathbf{a} < \mathbf{0'_e}$ such that no pair of a Π^0_1 e-degree $\mathbf{c} \geq \mathbf{a}$ and a Σ^0_2 e-degree $\mathbf{d} \geq \mathbf{a}$ form a non-trivial splitting of $\mathbf{0'_e}$. This was a step towards adapting Harrington's method for the far more complicated world of the Σ^0_2 enumeration degrees. It was already known that the restriction of the first degree $\mathbf{c}$ to the class of the Π^0_1 enumeration degrees in the pairs considered was essential as Arslanov and Sorbi [3] had shown that there is a Δ^0_2-splitting of $\mathbf{0'_e}$ above every Δ^0_2 enumeration degree.

A further step towards adapting the non-splitting techniques from the c.e. Turing degrees to the case of the Σ^0_2 enumeration degrees is given by Arslanov, Cooper, Kalimullin and Soskova [2]. There an analog of Lachlan's non-splitting theorem is proved, namely that there is a pair of a Π^0_1 enumeration degree $\mathbf{a}$ and a 3-c.e. enumeration degree $\mathbf{b} < \mathbf{a}$ such that $\mathbf{a}$ does not split above $\mathbf{b}$.

A question that remains to be answered in order to complete the splitting/non-splitting study of the e-degrees below $\mathbf{0'_e}$ is wether or not $\mathbf{0'_e}$ can be split above every Σ^0_2 enumeration degree. In this article we give a negative answer to the question above and provide an analog of Harrington's non-splitting theorem for the Σ^0_2 enumeration degrees, thus completing the final step of the transformation of the non-splitting techniques from the c.e. Turing degrees into the enumeration degrees.

Theorem 1 *There is a Σ^0_2-enumeration degree $\mathbf{a} < \mathbf{0'_e}$ such that $\mathbf{0'_e}$ cannot be split in the enumeration degrees above the degree $\mathbf{a}$.*

Notation and terminology below is based on that of [6] and [13].

2 Requirements and Strategies

We shall start by giving a formal definition of enumeration reducibility and then move on to establish the requirements and basic strategies for the proof of the main theorem.

Definition 2 *(1) A set A is enumeration reducible ($\leq_e$) to a set B if there is a c.e. set Φ such that:*

$$n \in A \Leftrightarrow \exists u(\langle n, u\rangle \in \Phi \wedge D_u \subseteq B),$$

where D_u denotes the finite set with code u under the standard coding of finite sets. We will refer to the c.e. set Φ as an enumeration operator and its elements will be called axioms.

(2) A set A is enumeration equivalent ($\equiv_e$) to a set B if $A \leq_e B$ and $B \leq_e A$. The equivalence class of A under the relation $\equiv_e$ is the enumeration degree $d_e(A)$ of A.

The structure of the enumeration degrees $\langle \mathcal{D}_e, \leq\rangle$ is the class of all e-degrees with relation $\leq$ defined by $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.

In this article we shall be only concerned with the local structure of the Σ^0_2 enumeration degrees. An enumeration degree is Σ^0_2 if it contains a Σ^0_2 set. The greatest Σ^0_2 e-degree is $\mathbf{0'_e}$, the degree of $\overline{K}$. Any set in the degree $\mathbf{0'_e}$ will be called complete as it can reduce any other Σ^0_2 set.

We will denote enumeration operators by capital Greek letters $\Phi, \Theta \dots$. Notation in the following exposition will be unfortunately quite complicated. An expression Exp might be considered in relation to a certain stage s denoted by $Exp[s]$, in relation to a particular requirement with index i, denoted by Exp_i and in relation to a particular element n denoted by Exp_n. We shall try to keep things as clear as possible, omitting indices where they are clear and using Latin letters s, t for stages, n, m for elements and i, j, k, l for indices of requirements or strategies.

2.1 Requirements

We assume a standard listing of all enumeration operators $\{\Psi_i\}_{i<\omega}$ and of all triples $\{(\Theta, U, V)_i\}_{i<\omega}$ of enumeration operators Θ, Σ^0_2 sets U and V. We shall denote the elements of the i-th such triple by Θ_i, U_i and V_i respectively. We will construct a Σ^0_2 set A whose enumeration degree $\mathbf{a}$ will be the one required in Theorem 1 and an auxiliary Π^0_1 set E to satisfy the following list of requirements:

(1) The degree $\mathbf{a}$ should be strictly less than that of $\mathbf{0'_e}$. It will be enough to construct the set A as Σ^0_2-incomplete. We shall use the set E to witness the

incompleteness of A.

$$\mathcal{N}_i : E \neq \Psi_i^A$$

(2) Any pair of Σ^0_2 enumeration degrees $\mathbf{u}$ and $\mathbf{v}$ above $\mathbf{a}$ should not form a splitting of $\mathbf{0}'_{\mathbf{e}}$. The second group of requirements ensures that either $\mathbf{u} \cup \mathbf{v}$ is incomplete or at least one of the degrees $\mathbf{u}$ or $\mathbf{v}$ is already complete:

$$\mathcal{P}_i : E = \Theta_i^{U_i, V_i} \Rightarrow (\exists \Gamma_i, \Lambda_i)[\overline{K} = \Gamma_i^{U_i, A} \vee \overline{K} = \Lambda_i^{V_i, A}]$$

where $\Gamma_i^{U_i,A}$, for example, denotes an e-operator enumerating relative to the data enumerated from two sources U_i and A.

The requirements shall be given the following priority ordering:

$$\mathcal{N}_0 < \mathcal{P}_0 < \mathcal{N}_1 < \mathcal{P}_1 < \dots$$

Requirements in earlier positions have higher priority. Each particular requirement can be satisfied in more than one way. We connect to each such way an outcome. The choice of the correct way to satisfy a certain requirement depends on the outcomes of higher priority requirements. Therefore we represent the set of all possible sequences of outcomes as a *tree of strategies*. Each node α on the tree is labelled by a requirement R and the node α will be referred to as an R-strategy. The children of α correspond to each of α's possible outcomes. So, although each of those nodes might be labelled by the same requirement, each may have a different approach to satisfying its requirement depending on what it "believes" to be the outcome of α. The set of all possible outcomes for each requirement will be linearly ordered ($<_L$ defined below) and the nodes of the tree of strategies will be ordered by the induced lexicographical ordering $\leq$. The construction is by stages; in each stage s we construct a set $A[s]$ approximating A and a string $\delta[s]$ of length s in the tree of strategies. The initial segments $\delta \subseteq \delta[s]$ are the nodes of the tree visited during stage s of the construction; they are the strategies that might act to satisfy their requirements. The intent is that there will be a true path, a leftmost path of nodes visited infinitely often, such that all nodes along the true path are able to satisfy their requirements. If the node β is visited on stage s, we say that s is a β-true stage.

2.2 Basic Strategies

We shall describe the basic strategies for both types of requirements, the problems that we need to overcome in order to implement them and the conflicts that might arise when we combine them.

An $\mathcal{N}$-requirement, say $\mathcal{N}_i$, could be satisfied by a simple Friedberg-Muchnik strategy, which we shall denote in our further discussions by FM. Select a witness x and wait for $x \in \Psi_i^A$. If this never happens then the requirement will be satisfied and we denote this outcome by w. Otherwise extract x from E while restraining each $y \in A \upharpoonright use(\Psi_i, A, x)$ (the use function $use(\Psi, A, x)$ is defined in the usual way by $use(\Psi, A, x) = \mu y[x \in \Psi^{A \upharpoonright y}]$). The requirement is again satisfied with outcome denoted by f.

We are given three options to satisfy a single $\mathcal{P}$-requirement, say $\mathcal{P}_i$. The first and simplest one is to provide some proof that $\Theta_i^{U_i,V_i} \neq E$. The other two options are to construct enumeration operators Γ_i or Λ_i proving that at least one of the sets U_i or V_i is already too powerful and can reduce $\overline{K}$ by itself without the help of the other.

Recall that the length of agreement between two sets A and B, denoted by $l(A, B)$, is the length of the initial segment on which the sets A and B agree. The intent is that we monitor the length of agreement $l(\Theta_i^{U_i,V_i}, E)[s]$ on each stage s of the construction. A bounded length of agreement should turn out to be sufficient proof for the inequality between the two sets. Further actions only need to be made on expansionary stages, stages on which the length of agreement attains a greater value than it has had on previous stages. Initially we will use a $(\mathcal{P}_i, \Gamma_i)$-strategy designed to monitor the length of agreement and if there are infinitely many expansionary stages to construct an enumeration operator Γ_i which will reduce the set $\overline{K}$ to the sets U_i and A. A bounded length of agreement shall be represented by the outcome l and an unbounded length of agreement by the outcome e. We progressively try to rectify Γ_i at each stage s by ensuring that $n \in \overline{K}[s] \Leftrightarrow n \in \Gamma_i^{U_i,A}[s]$ for each n below $l(\Theta_i^{U_i,V_i}, E)[s]$. We will do this by defining markers $u_i(n)$ and $\gamma_i(n)$ and enumerating axioms of the form $\langle n, U_i[s] \upharpoonright u_i(n), \{\gamma_i(n)\}\rangle$ for elements $n \in \overline{K}[s]$. If at a later stage n leaves the set $\overline{K}$ then Γ_i can be rectified via an extraction of the marker $\gamma_i(n)$ from A.

Difficulties with this strategy arise from the fact that we are dealing with Σ_2^0 sets U_i and V_i. Consider a Σ_2^0 set U with a Σ_2^0 approximation $\{U[s]\}_{s<\omega}$. If $n \in U$ then $n \in U[s]$ on all but finitely many stages s. If $n \notin U$ then the only thing we know is that $n \notin U[s]$ on infinitely many stages s. So we could easily have that there are two elements $n_1, n_2 \notin U$ such that $\{n_1, n_2\} \cap U[s] \neq \emptyset$ on all stages s. This property of the Σ_2^0 approximations could have as a consequence that:

(1) The length of agreement $l(\Theta_i^{U_i,V_i}, E)[s]$ measured on each stage s is bounded while the sets E and $\Theta_i^{U_i,V_i}$ are equal.

(2) We might not be able to approximate any initial segment of the set U_i and so we will not be able to use initial segments in the definition of the axioms for the operator Γ_i.

To deal with these difficulties we shall define special approximations to the sets U_i, V_i and $U_i \oplus V_i$ in Section 3.

The second difficulty arises when we consider how to combine the strategies of the two different types. Consider one $\mathcal{N}$-requirement below one $\mathcal{P}$-requirement. $(\mathcal{P}, \Gamma)$ is constructing an operator Γ using markers $u(n)$ and $\gamma(n)$ for the axiom of elements n on expansionary stages for the sets $\Theta^{U,V}$ and E. The A-restraint for $\mathcal{N}$ following the extraction of x from E conflicts with the need to rectify the operator Γ. We try to resolve this by using a modified strategy $(\mathcal{N}, \Gamma)$. It will choose a threshold d and try to achieve $\gamma(n) > use(\Psi, A, x)$ for all $n \geqslant d$ at a stage previous to the imposition of the restraint. We will need to use a modified version of the use-function.

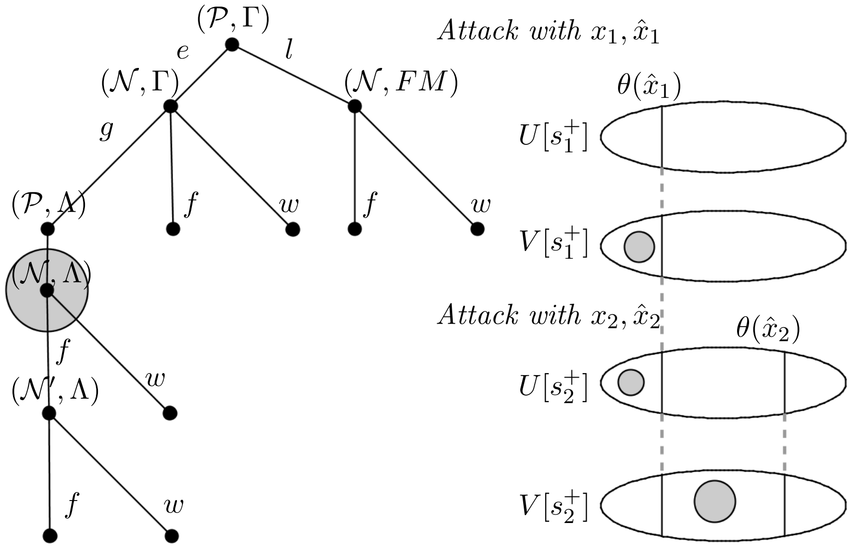
Definition 3 *Let Φ be an enumeration operator and A a set. The* generalised *use-function φ is defined as follows:*

$$\varphi(x) = \max\left\{use(\Phi, A, y) | (y \leq x) \wedge (y \in \Phi^A)\right\}.$$

$(\mathcal{N}, \Gamma)$ tries to maintain $\theta(x) < u(d)$ in the hope that after we extract x from E each return of $l(E, \Theta^{U,V})$ will produce an extraction from $U \upharpoonright \theta(x)$ which can be used to avoid an A-extraction in moving $\gamma(d)$.

$(\mathcal{N}, \Gamma)$ will have an extra outcome g which shall be visited in the event that some such attempt to satisfy $\mathcal{N}$ ends with a $V \upharpoonright \theta(x)$-change. Then we must implement a backup $\mathcal{P}$-strategy, $(\mathcal{P}, \Lambda)$, which is designed to allow lower priority $\mathcal{N}$-requirements to work below the Γ-activity and to construct an operator Λ reducing $\overline{K}$ to V and A, using the $V \upharpoonright \theta(x)$-changes to move λ-markers. Below $(\mathcal{P}, \Lambda)$ is a backup strategy $(\mathcal{N}, \Lambda)$ designed to take advantage of the improved strategy for $\mathcal{P}$. Both strategies $(\mathcal{N}, \Gamma)$ and $(\mathcal{N}, \Lambda)$ will attack simultaneously on stage s_1 by extracting their witnesses x_1 and $\hat{x}_1$ from E ensuring that at least one of them will succeed in providing the necessary U- or V-change on the next expansionary stage s_1^+. Here $\hat{x}_1 < x_1$, thus any change in U or V below $\theta(\hat{x}_1)$ will be a change in U or V below $\theta(x_1)$.

If $(\mathcal{N}, \Lambda)$ turns out successful then $(\mathcal{N}, \Gamma)$ will clear the working space for the backup strategies by extracting the current markers of its threshold forcefully, we shall refer to this as *capricious destruction*, choose a new witness x_2 and start a new cycle timing its next attack with the next $\mathcal{N}$-strategy $(\mathcal{N}', \Lambda)$ below $(\mathcal{P}, \Lambda)$ with witness $\hat{x}_2 > \hat{x}_1$ on stage s_2. The success of $(\mathcal{N}, \Lambda)$ depends on the change in the set $V \upharpoonright \theta(\hat{x}_1)$. Unfortunately there is no guarantee that this change will remain permanent as we are dealing with Σ_2^0 sets. So it could happen that after the next attack on stage s_2^+ there is a further V-change below $\theta(\hat{x}_2)$ at an element greater than $\theta(\hat{x}_1)$ making us visit the backup strategies, but the old V-change below $\theta(\hat{x}_1)$ has moved to the set U, i.e. the change we observed on stage s_1^+ in V has disappeared $(V \upharpoonright \theta(\hat{x}_1)[s_1] = V \upharpoonright \theta(\hat{x}_1)[s_2^+])$. This will result in an irreparable injury to the strategy $(\mathcal{N}, \Lambda)$. The figure below illustrates this situation, the grey circles represent the changes in he sets U an V as they appear after each attack.



Fortunately we are constructing a Σ_2^0 set as well and are thus allowed to extract its elements any finite number of times without consequence to its characteristic function. $(\mathcal{N}, \Gamma)$ will keep track of its old witnesses. If the change associated with the old witness x_1 moves to the set U, $(\mathcal{N}, \Gamma)$ will restore A as it was during the attack with x_1 at stage s_1 and use the U-change for success. Only after changes in V for each of the old witnesses have been observed will the backup strategies be visited.

In Section 4 we shall implement each of these strategies in detail, listing all of their parameters and outcomes. Section 5 will also contain more explanations about the strategies and their design. Finally in Sections 6 and 7 we shall consider all requirements and give the complete construction and proof of Theorem 1.

3 The Approximations

Consider the requirement $\mathcal{P}$ (We have omitted the index for simplicity). $(\mathcal{P}, \Gamma)$ shall approximate the sets U, V and Θ on every stage on which it is active. We shall choose special approximations to these sets. To ensure an unbounded length of agreement in the case of equality between the set $\Theta^{U,V}$ and E we need a good approximation $(U \oplus V)[s] = B[s]$ to the set $U \oplus V = B$ as defined in [10] i.e. one that has the following properties:

G1 $\forall n \exists s (B \upharpoonright n \subseteq B[s] \subseteq B)$, such stages s are called good stages.

G2 $\forall n \exists s \forall t > s (B[t] \subseteq B \Rightarrow B \upharpoonright n \subseteq B[t])$.

On the other hand, to use initial segments in our axioms for the constructed operators Γ and Λ, the approximations of the sets U and V that are derived from $\{B[s]\}$ by setting $U[s] = \{n | 2n \in B[s]\}$ and $V[s] = \{n | 2n+1 \in B[s]\}$ should be good Σ_2^0 approximations, i.e should have properties $G1$, $G2$ and Σ_2^0:

Σ_2^0 $\forall n(n \in U \Rightarrow \exists s \forall t > s(n \in U[t]))$

First we will choose a more convenient representation of the sets U and V. The Σ_2^0 sets are exactly the ones *c.e.* in K and can be listed by $\{W_e^K | e < \omega\}$. The index i of the $\mathcal{P}_i$-requirements will correspond to a triple (e, a, j), where $U = W_e^K$, $V = W_a^K$ and $\Theta = W_j$.

We approximate K via a better approximation as defined in [10]. A better approximation to the set K is computable sequence of finite binary functions $\kappa[s]$ such that:

B1 $\forall n \exists s_{1,n}(\chi_K \upharpoonright n \subseteq \kappa[s_{1,n}] \subseteq \chi_K)$.

B2 $\forall n \exists s_{2,n} \forall t > s_{2,n}(\{x | \kappa[t](x) = 1\} \subseteq K \Rightarrow \chi_K \upharpoonright n \subseteq \kappa[t])$.

Consider $K[s]$ to be the standard approximating sequence to the c.e. set K. And let $b(s) = \mu m[m \in K[s] \backslash K[s-1]]$, $b(s) = s$ if $K[s] = K[s-1]$. It is not hard to see that $\{\kappa[s]\}$, where $\kappa[0] = \emptyset$ and if $s > 0$

$$\kappa[s](x) = \begin{cases} 1 & \text{if } x \in K[s], \\ 0 & \text{if } x \notin K[s] \text{ and } x < b(s) \\ \text{not defined otherwise.} \end{cases}$$

is a better approximating sequence to K. Furthermore as for all t we have that $\{x | \kappa[t](x) = 1\} \subseteq K$, the second property of a better approximating sequence can be improved:

B2 $\forall n \exists s_{2,n} \forall t > s_{2,n}(\chi_K \upharpoonright n \subseteq \kappa[t])$.

Now we can approximate U via $U[s] = W_e^\kappa[s]$, V via $V[s] = W_a^\kappa[s]$ and B via $B[s] = U[s] \oplus V[s]$.

Proposition 4 *$U[s]$ is a good Σ_2^0 approximation to U. And $V[s]$ is a good Σ_2^0 approximation to V.*

PROOF. We will prove the proposition for $U[s]$. We first note that if $\kappa[s] \subseteq \chi_K$ then $U[s] \subseteq U$. For each n there is an m and an s such that $U \upharpoonright n = (W_e[s])^{\chi_K \upharpoonright m}$. So if $t > \max(s_{2,m}, s)$, where $s_{2,m}$ is the stage from $B2$ for m then $\chi_K \upharpoonright m \subseteq \kappa[t]$ and hence $U \upharpoonright n = (W_e[s])^{\chi_K \upharpoonright m} \subseteq (W_e[t])^{\kappa[t]} = U[t]$. This proves *G2* and the fact that the approximation is Σ_2^0. For *G1* consider $t > \max(s_{1,m}, s)$ to be a stage such that $\kappa[t] \subseteq \chi_K$ then $U[s] \subseteq U$ and $U \upharpoonright n \subseteq U[t]$. □

Proposition 5 *$B[s]$ is a good approximation to B.*

PROOF. G1: Fix n. Choose s' to be the stage from the second property of a good approximation to U for $n/2$ and s'' to be the stage from the second property of a good approximation to V for $n/2$. Then let $s > \max(s', s'')$ be a stage such that $\kappa[s] \subseteq \chi_K$. Then $U[s] \subseteq U$ and $V[s] \subseteq V$, hence $B[s] \subseteq B$. On the other hand $s > s'$ hence $U \upharpoonright n/2 \subseteq U[s]$ and $s > s''$ hence $V \upharpoonright n/2 \subseteq V[s]$. Thus $B \upharpoonright n \subseteq B[s]$.

G2: Proved easily as well using the stages from property *G2* of the good approximations to U and V. $\square$

As a consequence of the properties of a good approximation we have that if $\Theta^{U,V} = E$ then there will be infinitely many expansionary stages, as $lim_{s \text{ is a good stage}} \Theta^{U,V}[s] = \Theta^{U,V}$. Furthermore for each marker $u(n)$ there will be infinitely many stages s on which $U[s] \upharpoonright u(n) = U \upharpoonright u(n)$. This allows us to carry out the original design of the $\mathcal{P}$-strategy. Of course we should keep in mind that the expansionary stages are not necessarily the good stages and that if $\Theta^{U,V} \neq E$, we could still have infinitely many expansionary stages.

4 The Basic Modules for one $\mathcal{P}$- and one $\mathcal{N}$-Requirement

4.1 The Main Strategies

The set A is going to be constructed as a Σ^0_2 set in the following way. At each stage A will be initially approximated by $\mathbb{N}$ and ultimately by the resulting set after all extractions by strategies visited on this stage. Then $n \in A$ iff there is a stage s such that $\forall t > s(n \in A[t])$. This will ensure that essentially only the strategies along the true path will be responsible for the elements extracted from A. This is an important feature of the construction that distinguishes it from Harrington's original proof of the non-splitting theorem in the Turing degrees.

We will describe the modules for each of the strategies and list the parameters that will be related to them. In each of our descriptions of a particular strategy we shall have the context of the tree in mind. The strategy shall be assigned to a particular node δ on the tree (a formal definition of the tree of strategies will be given in Section 6.1), the current stage will be denoted by s and the previous δ-true stage by $s^-(s^- = s$ if δ has been initialized since the last stage on which it was visited). All parameters will inherit their values from s^- unless otherwise specified. For this reason we will sometimes omit the indices that specify the stage if the stage is clear.

4.1.1 The $(\mathcal{P}, \Gamma)$-strategy

We have already discussed the main idea for this strategy in Section 2.2. Here we will add details to it and give the formal module. Let us note again that the axioms in Γ are of the form $\langle n, U_n, \{m\}\rangle$. To every element on every stage s we will associate current markers $u(n)[s]$ and $\gamma(n)[s]$ and a corresponding current axiom

$\langle n, U[s] \upharpoonright u(n)[s], \{\gamma(n)[s]\}\rangle$. An axiom $\langle n, U_n, \{m\}\rangle$ is valid on stage s if $U_n \subseteq U[s]$ and $m \in A[s]$.

We will examine the current axiom in Γ for an element $n \in \overline{K}[s]$ if n is below the length of agreement between $E[s]$ and $\Theta^{U,V}[s]$, choosing a new axiom as current if the old one is invalid. In this way will be sure to catch the true approximation to the set $U \upharpoonright u(n)$ so that if $u(n)$ remains constant, so will the axiom for n after a certain stage due to the Σ^0_2-property of our approximations. If $n \notin \overline{K}$ then it will be enough to ensure that it does not appear in $\Gamma^{U,A}[s]$ on infinitely many stages s. We choose the expansionary stages for this purpose. Note that during the construction we may enumerate a number of axioms for a particular element. Any enumerated axiom might seem invalid at one stage but turn out to be valid on a later stage. On expansionary stages s for elements $n \notin \overline{K}[s]$ we shall make sure that there are no valid axioms by extracting the A-markers of any axiom that seems valid on stage s.

Each $\mathcal{P}$-strategy α shall be assigned a distinct infinite recursive set A_α from which it will choose the values of its A-markers. Whenever a strategy chooses a fresh marker it will be of value greater than any number appeared so far in the construction.

Suppose for definiteness that the $(\mathcal{P}, \Gamma)$-strategy we visit on stage s is α.

(1) If the stage is not expansionary then $o = l$, otherwise $o = e$.

(2) Choose $n < l(\Theta^{U,V}, E)[s]$ in turn ($n = 0, 1, \dots$) and perform following actions:
  - If $u(n) \uparrow$ then define it anew as $u(n) = u(n-1) + 1$ (if $n = 0$ then define $u(n) = 1$).
  - If $n \in \overline{K}[s]$
    - · If $\gamma(n) \uparrow$ then define it anew and enumerate the current axiom $\langle n, U[s] \upharpoonright u(n), \{\gamma(n)\}\rangle$ in Γ.
    - · If $\gamma(n) \downarrow$ but the current axiom for n is not valid then define the current marker $\gamma(n)$ anew and enumerate the new current axiom $\langle n, U[s] \upharpoonright u(n), \{\gamma(n)\}\rangle$ in Γ.
  - If $n \notin \overline{K}[s]$ but $n \in \Gamma^{(U,A)}[s]$ and the stage is expansionary then look through all the axioms defined for n , say $\langle n, U_n, m\rangle \in \Gamma[s]$, and extract m for all valid ones.

Note that if $n \notin \overline{K}$ then we will enumerate only finitely many axioms for n in Γ and hence extract only finitely many markers from A. Also note that this strategy will extract markers only on expansionary stages. Hence if the true outcome is l, the strategy will not modify the set A and $\mathcal{N}$ can be satisfied via the simple Friedberg-Muchnik strategy proposed initially $(\mathcal{N}, FM)$. Below outcome e we shall need the more sophisticated $(\mathcal{N}, \Gamma)$-strategy.

4.1.2 The $(\mathcal{N}, \Gamma)$-strategy

Suppose the node on which the $(\mathcal{N}, \Gamma)$-strategy acts is labelled by $\beta \supset \alpha$. We shall say that α is the active $\mathcal{P}$-strategy at β.

We have already mentioned one of the parameters associated with β the threshold d, a natural number that determines the beginning of the influence of β on the set A. Furthermore β is equipped with a list of witnesses that it has used so far in its attempts to satisfy $\mathcal{N}$ denoted by Wit. One of the witnesses is called the current witness, denoted by x, and plays a special role.

The main feature of the construction, the way we approximate A, clashes with the idea that a certain strategy progressively acts towards satisfying its requirement. Any influence it has tried to inflict on the set A by extracting some element from it will be lost unless the element is extracted again and again infinitely many times. This is why each strategy will keep track of all elements it has previously extracted in course of its work and extract these elements on every stage on which the strategy is active. So if a strategy remains inactive, to the left of the true path, it will not have any influence on the set A. If it is on the true path then it will restore its previous work at the beginning of every true stage and build onto that work during the stage. We will have three different groups of parameters responsible for elements extracted by β during its activities. The first will be the set of markers extracted for elements less than the threshold in O_d by the active $\mathcal{P}$-strategy. Note that the valid axioms whose markers are extracted at expansionary stages need not be the same on every stage. We need to provide some stability for β: if a marker that was extracted from A on a previous β-true stage but is not extracted on this one, β will extract it nevertheless and keep track of such elements in O_d. The second group, O_β, will consist of markers extracted during the activity of β. The third will be markers extracted due to *capricious destruction* after an attack that seems unsuccessful, kept in a parameter O_w for each witness w. These can be later reenumerated in A (i.e. not extracted from A on β-true stages) if the attack turns out to be successful.

We start β's activity by performing *Check* first to see wether the threshold is chosen correctly and wether any activity of the active $\mathcal{P}$-strategy for elements below the threshold has injured β's work sofar. If so we restart the module from *Initialization*, otherwise we continue the module from where we left it at the previous β-true stage s^-. If β has been initialized since the last stage on which it was visited or if it has never been visited then $s^- = s$ and β starts from *Initialization*.

At *Initialization* the values of the threshold and witness are determined after that the markers for all elements $n \geq d$ are reset so that $(\mathcal{N}, \Gamma)$ will have some control over the current axioms. The third part of the module, called *Honestification*, ensures that a change in U after an attack will in fact be useful. Then $(\mathcal{N}, \Gamma)$ waits for its witness to enter Ψ^A but always checks if Γ has remained honest. If $x \in \Psi^A$ and the operator is honest, $(\mathcal{N}, \Gamma)$ is ready to start the *Attack*. After the attack comes the evaluation of the *Result*, which will determine wether the backup strategies should be activated or the requirement $\mathcal{N}$ is satisfied for the moment.

- **Check**

(1) If $d \notin \overline{K}[s]$, i.e the threshold has just been extracted from $\overline{K}$, then find the least $n > d$, $n \in \overline{K}[s]$ and let that be the new value of the threshold. Empty Wit, cancel the current witness and start from *Initialization*, initializing all strategies

below β. Note that the set $\overline{K}$ is infinite, hence we shall eventually find the right threshold.

(2) Scan the elements $n \leq d$ such that $n \notin \overline{K}[s]$. If a marker m of n, $m \notin O_\beta \cup O_d$, has been extracted from A on this expansionary stage by α then we will enumerate it in O_d, empty Wit, cancel the current witness and start from *Initialization*, initializing all strategies below β. Note that this can happen finitely often as long as the threshold remains permanent, as there are finitely many axioms and hence markers that can be extracted from A for elements $n \leq d$, $n \notin \overline{K}$.

(3) Extract from A: $Out_\beta = O_\beta \cup O_d \cup \bigcup_{w \in Wit, w<x} O_w$.

- **Initialization**

(1) If a threshold has not yet been defined or is cancelled, choose a fresh threshold $d > l(\Theta^{U,V}, E)[s]$.

(2) If a witness has not yet been defined or is cancelled, choose a new witness $x \in E[s]$, $d < x$, bigger than any witness defined until now. Enumerate $x \in Wit$.

(3) Wait for a stage s such that $x < l(\Theta^{U,V}, E)[s]$. $(o = w)$

(4) Extract from A and enumerate in O_β all A_α-markers $m(n)$ of potentially applicable axioms for elements n such that $d \leq n < l(\Theta^{U,V}, E)[s]$. An axiom is potentially applicable, if its A_α-marker is not already extracted from A and enumerated in Out_β. Cancel the current markers for these elements.

(5) For every element $y \leq x$, $y \in E[s]$, enumerate in the list *Axioms* the current valid axiom from $\Theta[s]$, which was valid the longest, i.e. for each axiom for y $Ax_y \in \Theta[s]$ let $t_{Ax_y} = \mu r[\forall t(s \geq t \geq r \Rightarrow$ the axiom Ax was valid on stage $t)]$, then choose the axiom $\langle y, U_y, V_y\rangle \in \Theta[s]$ with least t_{Ax_y}. Here the definition of $\theta(x)$ at stage s will be modified again to capture the greatest element of precisely these axioms currently listed in the list *Axioms*. $(o = h)$

- **Honestification** Scan the list *Axioms*. If for any element $y \leq x$, $y \in E[s]$, the listed axiom was not valid on any stage t since the last β-true stage then update the list *Axioms*, let (o = h) and

(1) Extract and enumerate in O_β all A_α-markers $m(n)$ of potentially applicable axioms for elements n such that $d \leq n$, cancel the current markers for these elements and define $u(d) > \theta(x)$. This ensures the following property: for all elements $n \geq d$, $n \in \overline{K}[s]$ the U-parts of the axioms in Γ include the U-parts of all axioms listed in *Axioms* for elements $y \leq x$, $y \in E[s]$. If $n \notin \overline{K}[s]$ then all its A_α-markers will be extracted from A and enumerated in O_β so that no new extraction of a marker by the active $\mathcal{P}$-strategy α for these elements can surprise us.

Otherwise go to:

- **Waiting** Wait for a stage s such that $x \in \Psi^A[s]$ returning at each successive stage to *Honestification* (o = w).

- **Attack**

(1) If $x \in \Psi^A[s]$ and $u(d) > \theta(x)$ then extract x from E. The outcome is (o= g) starting a nonactive stage for the backup strategies. On this stage they cannot perform any actions except for attacking with their own witnesses. Define O_x to be the set of all A_α-markers of potentially applicable axioms for elements n

such that $d \leq n$ and $In_x = (A \upharpoonright use(\Psi, A, x))[s]$. At the next true stage go to *Result*.

- **Result** Let $\bar{x} \leq x$ be the least element that has been extracted from E during the stage of the attack. As this is an expansionary stage $\bar{x} \notin \Theta^{U,V}[s]$, hence all axioms for $\bar{x}$ in $\Theta[s]$ are not applicable, in particular the one enumerated in *Axioms*, say $\langle \bar{x}, U_{\bar{x}}, V_{\bar{x}} \rangle$. At least one element from $U_{\bar{x}}$ or $V_{\bar{x}}$ has been extracted from U or V respectively(i.e. is not in $U[s]$ or $V[s]$). We will attach to the witness x the necessary information about this attack, namely a parameter $Attack(x) = \langle \bar{x}, U_{\bar{x}}, V_{\bar{x}} \rangle$.

  If $V_{\bar{x}} \subseteq V[s]$ then the attack is successful. The A_α-markers of elements $n \geq d$ have been lifted above $use(\Psi, A, x)$ as all previously enumerated axioms for elements $n \geq d$ will not be valid. Hence if later on we want to ensure that $\Gamma^{U,A}(n) = 0$ we will only need to extract a marker that is already above the restraint.

  If the attack was unsuccessful then we had a change in V. The plan is to start the backup strategies and then try again with a new witness. In this case we will move the markers $\gamma(n)$ for $n \geq d$, $n \in \overline{K}[s]$, by extracting the current ones and defining the markers anew in order to provide a safe working space for the backup strategy. At any later stage when we activate the backup strategy we would like to have all changes in V for all unsuccessful witnesses that have already been used. As we already discussed in Section 2.3 the Σ^0_2 nature of the sets U and V can trick us to believe a certain witness is unsuccessful, where in fact after finitely many changes in V it turns out to be successful. We would like to be able to restore the old situation as it were during the attack with this old witness and use it to satisfy the requirement. This is where the parameter O_x comes into use. Every time we reach this step of the module we will stop and look back at what has happened with the previous witnesses recorded in the list Wit. If it turns out that we have a permanent U-change useful for some $w \in Wit$ then we can reenumerate the corresponding O_w in A and satisfy the requirement $\mathcal{N}$ with this witness. Otherwise as the stage is expansionary and hence $w \notin \Theta^{U,V}[s]$ we have the necessary change in V to rely on the backup strategy.

  Thus we scan all $w \in Wit$.

(1) Let $Attack(w) = \langle w', U_{w'}, V_{w'} \rangle$. If there was a change in $V_{w'}$ since this witness was last examined, i.e. there is a stage t such that t is bigger than the stage of the last attack and $V_{w'} \nsubseteq V[t]$ then extract O_w from A and go to the next witness.
(2) Otherwise w is successful, the outcome is $(o = f_w)$. We set the current witness to be w so that O_w is not extracted from A during *Check*. Return to *Result* at the next stage. We say that β restrains the elements In_x in A.
(3) If all witnesses are scanned and all are unsuccessful then cancel the last witness together with the current markers of the elements $n \in \overline{K}[s]$, $d \leq n$ and let the outcome be $o = g$ starting an active stage for the backup strategies. Return to *Initialization* at the next stage, choosing a new witness. Remove $\beta's$ restraint on A.

4.1.3 Analysis of Outcomes

We shall list the possible outcomes of the defined modules and determine a right boundary R below which successive strategies are allowed to work. The right boundary is relevant only for $\mathcal{N}$-strategies. It tells a strategy that it is safe to believe that the set A shall not change below R due to the activity of higher priority $\mathcal{N}$-strategies. The right boundary will move off to infinity as the stages grow. So for example the $(\mathcal{N}, FM)$ strategy working below R after selecting a witness x will (1) Wait for $x \in \Psi^A[s]$ with $use(\Psi, A, x) < R$ and then (2) extract x from E and restrain $(A \upharpoonright use(\Psi, A, x))[s]$ in A.

$(\mathcal{P}, \Gamma)$ has two possible outcomes:

(**l**) There is a stage after which $l(\Theta^{U,V}, E)[s]$ remains bounded by its previous expansionary value. Then $\mathcal{P}$ is trivially satisfied. In this case $\mathcal{N}$ will be satisfied by the strategy $(\mathcal{N}, FM)$ working below right boundary $R = \infty$.

(**e**) There are infinitely many expansionary stages. The $(\mathcal{N}, \Gamma)$-strategy β is activated.

The possible outcomes of $(\mathcal{N}, \Gamma)$ are:

(**w**) There is an infinite wait at *Waiting* for $\Psi^A(x) = 1$ for some witness x. Then $\mathcal{N}$ is satisfied because $E(x) = 1 \neq \Psi^A(x)$ and $(\mathcal{P}, \Gamma)$ remains intact. Successive strategies work below $R = \infty$.

($\mathbf{f_x}$) There is a stage after which some witness x with $Attack(x) = \langle \bar{x}, U_{\bar{x}}, V_{\bar{x}} \rangle$ never gets its $V_{\bar{x}}$-change. Then there is a permanent change in $U_{\bar{x}}$ and the markers of all witnesses are moved above $use(\Psi, A, x)$. At sufficiently large stages $\overline{K} \upharpoonright d$ has its final value. So there is no injury to the strategies below f_x. $\Psi^A(x) = 1 \neq E(x)$ and $\mathcal{N}$ is satisfied, leaving $(\mathcal{P}, \Gamma)$ intact. Successive strategies work below $R = \infty$.

(**h**) There are infinitely many occurrences of *Honestification* for some witness x precluding an occurrence of *Attack*. Then there is a permanent witness x which has unbounded $lim_{sup}\theta(x)$. This means that $\Theta^{U,V}(y) = 0$ for some $y \leq x$, $y \in E$, thus $\mathcal{P}$ is satisfied. In this case $\mathcal{N}$ is satisfied by a second instance of $(\mathcal{N}, FM)$ working below $R = \gamma(d)$.

(**g**) We implement the unsuccessful attack step infinitely often. As anticipated we must activate the backup strategies. They work below $R = x$.

4.2 The Backup Strategies

Notice that the outcome (g) is visited in two different cases: at the beginning of an attack and when the attack turns out to be unsuccessful. The first case starts a nonactive stage for the subtree below (g) allowing other $\mathcal{N}$-strategies to synchronize their attacks with the one performed by $(\mathcal{N}, \Gamma)$. The second case starts an active stage on which the strategies will do their usual work. Unless otherwise specified the described actions are only performed on active stages.

4.2.1 The $(\mathcal{P}, \Lambda)$-strategy

The $(\mathcal{P}, \Lambda)$-strategy is quite similar to the $(\mathcal{P}, \Gamma)$-strategy. The only difference is that it needs to be extra careful in order to catch the true approximations of the initial segments of V as it is only visited on expansionary stages, not necessarily true ones. It has only one outcome e.

(1) Choose $n < l(\Theta^{U,V}, E)[s]$ in turn $(n = 0, 1, \dots)$ and perform following actions:

- If $v(n) \uparrow$ then define it anew as $v(n) = v(n-1) + 1$.
- If $n \in \overline{K}[s]$
  - · If $\lambda(n) \uparrow$ then define it anew and enumerate the current axiom $\langle n, V[s] \upharpoonright v(n) + 1, \{\lambda(n)\}\rangle$ in Λ.
  - · If $\lambda(n) \downarrow$ but the current axiom was not valid on some stage t: $s^- < t \leq s$. Then define $\lambda(n)$ anew and choose out of all $V[p] \upharpoonright v(n)$ for $s^- < p \leq s$ the one that has been a subset of V the longest much like we chose the valid axiom for the witnesses in the list *Axioms* in Section 4.1.2, let that be V_n. Define the current axiom to be $\langle n, V_n, \{\lambda(n)\}\rangle$ and enumerate it in Λ.
- If $n \notin \overline{K}[s]$ but $n \in \Lambda^{V,A}[s]$ then extract from A all $A_{\alpha'}$-markers of axioms for n with V-part V_n such that $\forall t(s^- < t \leq s \Rightarrow V_n \subseteq V[t])$.

4.2.2 The $(\mathcal{N}, \Lambda)$-strategy

Let the $(\mathcal{N}, \Lambda)$-strategy be β'. The actions that $(\mathcal{N}, \Lambda)$ performs are similar to the ones performed by $(\mathcal{N}, \Gamma)$ but are directed at the active $\mathcal{P}$-strategy at β' which is now α'. The strategy β' extracts only $A_{\alpha'}$-markers used in the definition of the operator Λ.

- **Check**

(1) If $\widehat{d} \notin \overline{K}[s]$ then find the least $n > \widehat{d}$, $n \in \overline{K}[s]$ and let that be the new value of the threshold. Cancel the current witness and start from *Iinitialization*, initializing all strategies below β'.
(2) Scan the elements $n \leq \widehat{d}$ such that $n \notin \overline{K}[s]$. If an $A_{\alpha'}$-marker $m(n) \notin O_{\widehat{d}}$ has been extracted from A on this stage then enumerate it in $O_{\widehat{d}}$, cancel the current witness and start from *Initialization*, initializing all strategies below β'.
(3) Extract $Out_{\beta'} = O_{\beta'} \cup O_{\widehat{d}}$ from A.

- **Initialization**

(1) Choose a new threshold $\widehat{d}$, bigger than any defined until now such that $l(\Theta^{U,V}, E)[s] < \widehat{d}$.

(2) Choose a new witness $\widehat{x} \in E[s]$ such that $\widehat{d} < \widehat{x}$, bigger than any witness defined until now. Note that when $\widehat{x}$ is chosen β has just started an active backup stage and cancelled its own witness. The next witness that β will use will be defined after this stage and hence will be of value greater than $\widehat{x}$.
(3) Wait for a stage s such that $\widehat{x} < l(\Theta^{U,V}, E)[s]$. $(o = w)$
(4) Extract all $A_{\alpha'}$-markers $m(n)$ enumerating them in $O_{\beta'}$ for elements n such that $\widehat{d} \leq n$ and cancel the current markers.
(5) For every element $y \leq \widehat{x}$, $y \in E[s]$, enumerate in the list *Axioms* the current valid axiom from $\Theta[s]$, i.e. the one that has been valid longest. Define $v(\widehat{d}) > \theta(\widehat{x})$. $(o = h)$

- **Honestification** If for some $y \leq \widehat{x}, y \in E[s]$, the corresponding axiom in *Axioms* was not valid at any stage since the last β'-true stage then update the list and let (o = h) and then:

(1) Extract all $A_{\alpha'}$-markers $m(n)$, enumerating them in $O_{\beta'}$, for elements n such that $\widehat{d} \leq n < l(\Theta^{U,V}, E)[s]$, cancel the current markers and define $v(\widehat{d}) > \theta(\widehat{x})$.

Otherwise go to *Waiting*:

- **Waiting** Wait for a stage s such that $\widehat{x} \in \Psi^A[s]$ with $use(\Psi, A, \widehat{x})[s] < R$ returning at each successive step to *Honestification* (o = w). Once this happens go to *Attack*.

- **Attack**

(1) Wait for a nonactive stage $(o = w)$. This synchronizes the attacks of the two strategies.
(2) If Λ is not honest do nothing and return to *Honestification* at the next active stage. Otherwise extract $\widehat{x}$ from E. Let $In_{\widehat{x}} = A \upharpoonright use(\Psi, A, \widehat{x})$. $(o = w)$

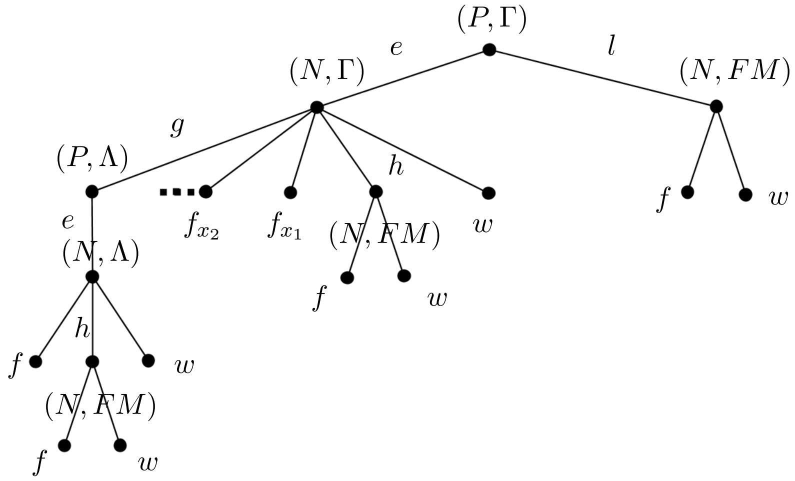
   The next stage at which this strategy will be accessible will be an unsuccessful attack for $(\mathcal{N}, \Gamma)$, hence if the strategy does not get initialized due to a $\overline{K} \upharpoonright \widehat{d}$-change, there will be a $V \upharpoonright \theta(\widehat{x})$-change. Indeed as $\widehat{x} < x$ the least element that has been extracted during the attack is $\bar{x} \leq \widehat{x}$. This outcome is visited on unsuccessful attacks hence $V \upharpoonright \theta(\bar{x})$ has changed but $\theta(\bar{x}) \leq \theta(\widehat{x})$, hence $V \upharpoonright \theta(\widehat{x})$ has changed as well. Hence at the next accessible stage we can simply go to *Result*:

- **Result** Successful attack. We say that β' is restraining $In_{\widehat{x}}$ in A. $(o = f_{\widehat{x}})$. Return to *Result* at the next stage. Note that every time this strategy is visited a corresponding $V \upharpoonright \theta(\widehat{x})$ change will be present so the successful attack is permanent.

4.2.3 Analysis of Outcomes

The possible outcomes of the $(\mathcal{N}, \Lambda)$-strategy are (**w**), ($\mathbf{f_x}$), and (**h**), exactly corresponding to the outcomes (**w**),($\mathbf{f_x}$) and (**h**) of $(\mathcal{N}, \Gamma)$ discussed in Section 4.1.3. In each of these outcomes we either have satisfied the requirement $\mathcal{P}$ and can implement $(\mathcal{N}, FM)$ to satisfy $\mathcal{N}$ or $\mathcal{N}$ is satisfied while the $(\mathcal{P}, \Lambda)$-strategy remains intact.

The tree of outcomes at this point looks as follows:

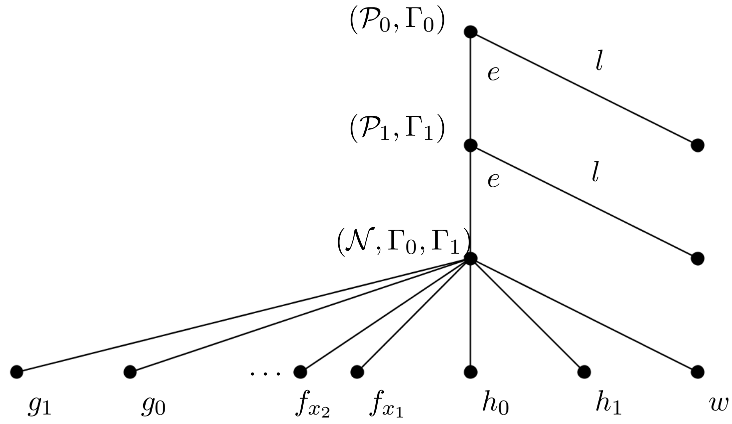


5 One $\mathcal{N}$-requirement below two $\mathcal{P}$-requirements

In this section we shall try to give the basic intuition about the case when one $\mathcal{N}$-requirement needs to handle two $\mathcal{P}$-requirements $\mathcal{P}_0$ and $\mathcal{P}_1$ of higher priority, leaving the formal definition of the various strategies for Section 6.2, where a general construction regarding all requirements will be given.

The $\mathcal{N}$-strategy now needs to respect two higher priority requirements, each constructing its own operator. During the course of the construction it might become obvious that a $\mathcal{P}$-requirement is satisfied or should switch to a Λ-strategy. Therefore we shall have more possibilities for the strategies. Each $\mathcal{P}$-requirement will have a $(\mathcal{P}_i, S_i)$-strategy with $S_i \in \{\Gamma_i, \Lambda_i\}$ and the $\mathcal{N}$-strategies will be $(\mathcal{N}, S_0, S_1)$, where $S_i \in \{\Gamma_i, \Lambda_i, FM_i\}$. $\mathcal{P}_0 < \mathcal{P}_1$ so should the the $\mathcal{P}_0$-strategy change it can afford to restart the $\mathcal{P}_1$-strategy. If the $\mathcal{P}_1$-strategy is changed though, we must make sure that this does not affect the strategy for $\mathcal{P}_0$.

The tree of strategies is quite a bit more complicated than in the first case we considered. The $(\mathcal{P}_i, S_i)$-strategies are exactly the ones used in Sections 4.1.1 and 4.2.1. We shall concentrate here on the subtree below both expansionary outcomes of the $\mathcal{P}$-strategies and discuss how the $(\mathcal{N}, \Gamma_0, \Gamma_1)$-strategy works as this is the most general case and captures all main ideas.



Let β be the $(\mathcal{N}, \Gamma_0, \Gamma_1)$-strategy with active $\mathcal{P}$-strategies α_0 and α_1. The module of β will be divided in the same submodules: *Check, Initialization, Honestification, Waiting, Attack* and *Result*. Most submodules and most parameters shall have two copies, one for each active $\mathcal{P}$-strategy.

Initialization There will be two thresholds $d_1 < d_0$ and one current witness x. Each new witness is enumerated in Wit_0. The first witness used is enumerated in Wit_1 as well. Any further witness will be enumerated in Wit_1 only if the attack performed with it will be timed with the backup strategies below outcome g_1, that is if after the previous attack we visited actively outcome g_1.

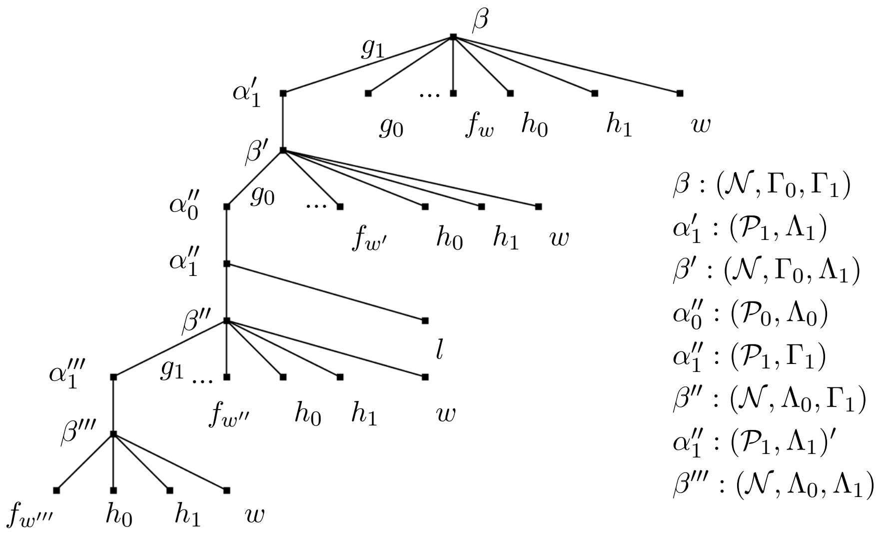
Honestification is performed first to Γ_0 with the list $Axioms_0$. If Γ_0 is not honest then β will clear both the A_{α_0}- and A_{α_1}-markers, providing safe working space for strategies below outcome h_0. This will destroy the strategy α_1, therefore below outcome h_0 we shall have a new copy of the $\mathcal{P}_1$-strategy $(\mathcal{P}_1, \Gamma_1)$ starting work from the beginning. If Γ_0 is honest then we will perform $Honestification_1$. In case Γ_1 is not honest only A_{α_1}-markers will be extracted. If this is the true outcome β shall not extract any A_{α_0}-markers and α_0 will remain intact and still be active for $\mathcal{N}$-strategies below outcome h_1.

Attack is performed once $x \in \Psi^A$ and both operators are honest. There are two sorts of backup strategies: the ones below outcome g_0 and the ones below outcome g_1. A nonactive stage shall be started for strategies below the outcome visited during the previous attack.

Result is performed first for Γ_0. If the attack is 0-unsuccessful then outcome g_0 is visited and capricious destruction is performed on both operators. Again below outcome g_0 we have a copy of the $(\mathcal{P}_1, \Gamma_1)$-strategy starting its work from the beginning. The outcome g_0 can be visited after many consecutive attacks so the witnesses used will be collected in Wit_0. Only after we see a successful witness $w \in Wit_0$ will we examine the result for the second operator Γ_1.

The witnesses in Wit_1 are then examined one by one. In this case we are not able to guarantee a V_1-change for each of the witnesses to the backup strategies below outcome g_1. Instead a witness w with $Attack(w) = \langle \bar{w}, U_{\bar{w},0}, V_{\bar{w},0}, U_{\bar{w},1}, V_{\bar{w},1} \rangle$, where $\bar{w}$ is the least witness extracted by some strategy during the attack with w, is considered 1-unsuccessful if there is a $V_{\bar{w},0}$-change or a $V_{\bar{w},1}$-change. If all witnesses are 1-unsuccessful the outcome g_1 is visited.

To explain how an attack works we will need to consider the backup strategies below outcome g_1 as well. We have β' which is the $(\mathcal{N}, \Gamma_0, \Lambda_1)$-strategy, then below it β'' which is the $(\mathcal{N}, \Lambda_0, \Gamma_1)$-strategy, finally β''' is the $(\mathcal{N}, \Lambda_0, \Lambda_1)$-strategy.



All strategies attack together with $w''' < w'' < w' < w$. We have a connection between their parameters: $(U_{\bar{w},0}, V_{\bar{w},0}) = (U_{\bar{w}',0}, V_{\bar{w}',0}) = (U_{\bar{w}'',0}, V_{\bar{w}'',0}) = (U_{\bar{w}''',0}, V_{\bar{w}''',0})$, then $(U_{\bar{w},1}, V_{\bar{w},1}) = (U_{\bar{w}',1}, V_{\bar{w}',1})$ and $(U_{\bar{w}'',1}, V_{\bar{w}'',1}) = (U_{\bar{w}''',1}, V_{\bar{w}''',1})$.

Suppose that after we evaluate the result of β it has outcome g_1. This means that there is a change in $V_{\bar{w},0}$ or $V_{\bar{w},1}$. Then β' evaluates the result of the attack. If the change was not in $V_{\bar{w},0} = V_{\bar{w}',0}$ then β' is successful as it has the desired $U_{\bar{w}',0}, V_{\bar{w}',1}$-change. Otherwise we have a $V_{\bar{w}',0} = V_{\bar{w}'',0}$-change and β' will have outcome g_0. Now β'' has its desired $V_{\bar{w}'',0}$-change. If there is a $U_{\bar{w}'',1}$-change then it is successful, otherwise there is a $V_{\bar{w}'',1} = V_{\bar{w}''',1}$-change. β'' will have outcome g_1 and the strategy β''' will be successful. Thus whatever the distribution of the changes at least one of the strategies along the tree will be successful.

Of course we need to provide safe working space for the backup strategies below outcome g_1. We can afford to capriciously destroy Γ_1 as we know that a backup $(\mathcal{P}_1, \Lambda_1)$-strategy will follow. On the next cycle β will need carry on its work by extracting the A_{α_0}-markers to prepare Γ_0 for the attack with the next witness. This puts α_0 in danger of being destroyed although no advancement on the satisfaction of $\mathcal{P}_0$ has been made. To prevent this a new value of the threshold d_0 will be chosen on every active visit of the outcome g_1, the set O_{d_0} will be emptied. So on each new cycle after an active g_1-visit β will move its activity regarding A_{α_0}, allowing α_0 to remain intact.

As a consequence we will need to rethink the $Check_0$ submodule. It should not be allowed the initialize all strategies below β should an A_{α_0}-marker of an element less than d_0 be extracted by the active $\mathcal{P}_0$-strategy and enumerated in O_{d_0}. If the true outcome is g_1 then the value of d_0 will grow unboundedly and we might initialize all strategies β infinitely often. $Check_0$ shall instead be only allowed to initialize strategies that believe the threshold d_0 is constant, that is all except for the ones below g_1.

The strategy β' working below outcome g_1 has the same active $\mathcal{P}_0$-strategy. It has threshold $d_0' < d_0$ and prepares its attack by extracting A_{α_0}-markers. This preparation is useful for β as it ensures that α_0 will not extract markers for elements $n \geq d_0'$ if the attack is 0-successful. If we neglect this preparation the following situation might happen: Suppose β and β' attack with w and w'. Then while we are evaluating β's $Result$ a new marker m for an element n such that $d_0' < n \leq d_0$ is extracted by α_0. $Check_0$ would like to restart β from initialization. In this case the witness w will be discarded and the attack with w' will be neglected. So the next time we visit β' we might not have the right permission for w'. On the other hand if we incorporate the preparation provided by β' an extraction by α_0 which is below $use(\Psi, A, w)$ will give us more information, namely that the witness w' is 1-unsuccessful, as a $U_{0,\bar{w}} = U_{0,\bar{w}'}$ change will ensure that no markers for elements $n > d_0'$ will be extracted by α_0. For this reason the parameter O_w will appear in two ways: $O_{w,own}$ will have the same definition as in the first case, it will include the markers that we extract during capricious destruction, $O_{w,else}$ will contain markers extracted by backup strategies during their preparation for an attack that will be performed together with β's attack with w. The set $O_{w,else}$ will only be extracted by β during $Honestification$ and $Waiting$. After the attack we must not extract it as it might interfere with the elements that previous witnesses need to keep in A for their own success.

Now we are ready to proceed to the main construction and the proof that it works.

6 All requirements

We will start by describing the tree of outcomes. Each $\mathcal{P}$-requirement has at least one node along each path in the tree of strategies. Each $\mathcal{N}$-requirement has many nodes along each path, the number depends on the number of $\mathcal{P}$-requirements of higher priority.

For every $\mathcal{P}$-requirement $\mathcal{P}_i$ we have two different strategies: $(\mathcal{P}_i, \Gamma_i)$ with outcomes $e <_L l$ and $(\mathcal{P}_i, \Lambda_i)$ with one outcome e.

For every $\mathcal{N}$-requirement $\mathcal{N}_i$, where $i > 0$, we have strategies of the form $(\mathcal{N}_i, S_0, \ldots, S_{i-1})$, where $S_j \in \{\Gamma_j, \Lambda_j, FM_j\}$. The requirement $\mathcal{N}_0$ has one strategy $(\mathcal{N}_0, FM)$. The outcomes are f_x for $x \in \omega$, w and for each $j < i$ if $S_j \in \{\Gamma_j, \Lambda_j\}$ there is an outcome h_j, if $S_j = \Gamma_j$, there is an outcome g_j. They are ordered according to the following rules:

(1) For all j_1 and j_2, $g_{j_1} <_L \dots f_n <_L f_{n-1} <_L \dots <_L f_0 <_L h_{j_2} <_L w$

(2) If $j_1 < j_2$ then $g_{j_2} <_L g_{j_1}$ and $h_{j_1} <_L h_{j_2}$.

Let $\mathbb{O}$ be the set of all possible outcomes and $\mathbb{S}$ be the set of all possible strategies.

6.1 The tree of strategies

The tree of outcomes is a computable function $T : dom(T) \subset \mathbb{O}^* \to \mathbb{S}$ which has the following properties:

(1) $T(\emptyset) = (\mathcal{N}_0, FM)$.

(2) If $T(\alpha) = S$ and O_S is the set of outcomes for the strategy S then for every $o \in O_S$, $\alpha\hat{\ }o \in D(T)$.

(3) If $S = (\mathcal{N}_i, S_0, S_1, \dots, S_{i-1})$ then

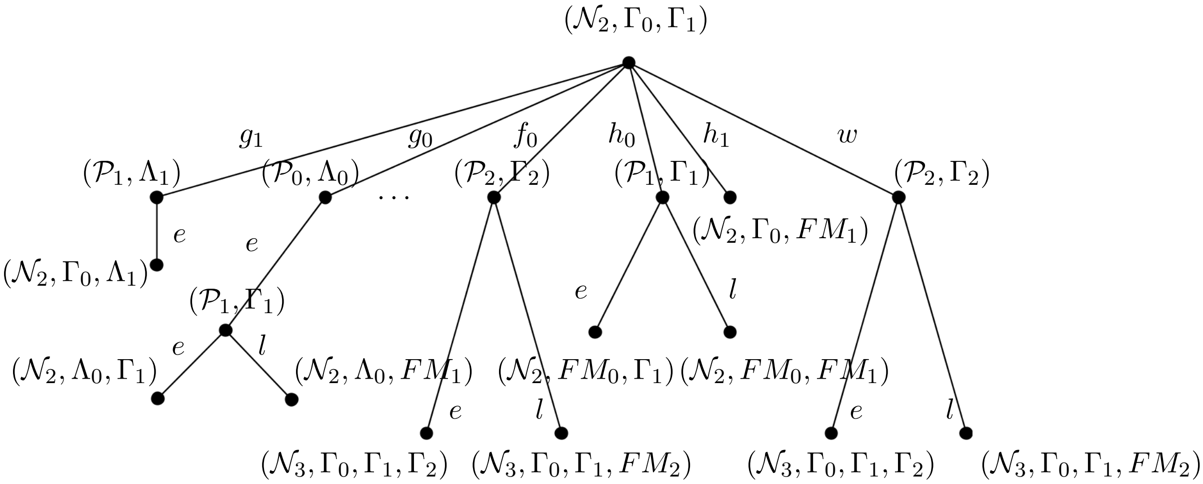
Below outcome $\mathbf{g_j}$: $T(\alpha\hat{\ }g_j) = (\mathcal{P}_j, \Lambda_j)$ and $T(\alpha\hat{\ }g_j\hat{\ }e) = (\mathcal{P}_{j+1}, \Gamma_{j+1})$, ... $T(\alpha\hat{\ }g_j\hat{\ }e\hat{\ }o_{j+1}\hat{\ }o_{i-2}) = (\mathcal{P}_{i-1}, \Gamma_{i-1})$, where $o_k \in \{e_k, l_k\}$ for $j+1 \le k \le i-2$. $T(\alpha\hat{\ }g_j\hat{\ }e\hat{\ }o_{j+1}\hat{\ }o_{i-1}) = (\mathcal{N}_i, S_0, \dots, \Lambda_j, S'_{j+1} \dots, S'_{i-1})$, where $S'_k = \Gamma_k$ if $o_k = e_k$ and $S'_k = FM_k$ if $o_k = l_k$ for every k such that $j < k < i$.

Below outcome $\mathbf{h_j}$: $T(\alpha\hat{\ }h_j) = (\mathcal{P}_{j+1}, \Gamma_{j+1})$, ... $T(\alpha\hat{\ }h_j\hat{\ }o_{j+1}\hat{\ }o_{i-2}) = (\mathcal{P}_{i-1}, \Gamma_{i-1})$, where $o_k \in \{e_k, l_k\}$ for $j+1 \le k \le i-2$.
$T(\alpha\hat{\ }h_j\hat{\ }o_{j+1}\hat{\ }o_{i-1}) = (\mathcal{N}_i, S_0, \dots, FM_j, S'_{j+1} \dots, S'_{i-1})$, where $S'_k = \Gamma_k$ if $o_k = e_k$ and $S'_k = FM_k$ if $o_k = l_k$ for every k such that $j < k < i$.

Below outcome $\mathbf{f_x}$: $T(\alpha\hat{\ }f_x) = (\mathcal{P}_i, \Gamma_i)$. Then $T(\alpha\hat{\ }f_x\hat{\ }e) = (\mathcal{N}_{i+1}, S_0, \dots, S_{i-1}, \Gamma_i)$, $T(\alpha\hat{\ }f_x\hat{\ }l) = (\mathcal{N}_{i+1}, S_0, \dots, S_{i-1}, FM_i)$.

Below outcome w: $T(\alpha\hat{\ }w) = (\mathcal{P}_i, \Gamma_i)$. Then $T(\alpha\hat{\ }w\hat{\ }e) = (\mathcal{N}_{i+1}, S_0, \dots, S_{i-1}, \Gamma_i)$, $T(\alpha\hat{\ }w\hat{\ }l) = (\mathcal{N}_{i+1}, S_0, \dots, S_{i-1}, FM_i)$.

The following picture illustrates these properties for $(\mathcal{N}_2, \Gamma_0, \Gamma_1)$.



6.2 The construction

On each stage s we shall construct a finite path through the tree of outcomes $\delta[s]$ of length s starting from the root. The nodes that are visited on stage s shall perform activities as described below. Their parameters will be modified. Each $\mathcal{N}$-node α shall have a right boundary R_α which will also be defined below. $R_\emptyset = \infty$. After the stage is completed, all nodes to the right of the constructed $\delta[s]$ will be initialized and their parameters will be cancelled or set to their initial value $\emptyset$.

An $\mathcal{N}$-strategy on node α works with respect to the active $\mathcal{P}$-strategies at α. It also synchronizes its work with some of the higher priority $\mathcal{N}$-strategies. It will be useful to define a notion of dependency between the different $\mathcal{N}$-strategies.

Definition 6 *A node α with $T(\alpha) = (\mathcal{N}_i, S_0, S_1, \dots, S_{i-1})$ depends on node $\beta \subset \alpha$, if $\alpha \supseteq \beta\hat{\ }g_j$ and $S_j = \Lambda_j$ for some j. The node α is independent if it is not dependent on any node $\beta \subset \alpha$.*

If α is dependent it might depend on many of its initial segments. The biggest(closest) node on which α depends will be called the instigator of α, denoted by $ins(\alpha)$. The strategy α must time its attacks with the attacks performed by $ins(\alpha)$, i.e. whenever α is ready to attack, it waits for an $ins(\alpha)$-nonactive stage and attacks on that stage. All the rest of the activity by α is performed only on active stages. We define a stage s to be *nonactive* if a strategy $\sigma \subset \delta[s]$ starts an attack on stage s. Stage s is also σ-nonactive. A stage is active if it is not nonactive. Note that if $\beta\hat{\ }g_j$ is on the true path then there will be infinitely many β-nonactive stages on which $\beta\hat{\ }g_j$ is visited. In fact every $\beta\hat{\ }g_j$-true active stage is followed by a $\beta\hat{\ }g_j$-nonactive stage before the next $\beta\hat{\ }g_j$-true active stage.

In our further discussions we shall denote with M_α, m_α, Z_α and z_α: Γ_α, γ_α, U_α and u_α respectively if α is a $(\mathcal{P}, \Gamma)$-strategy and Λ_α, λ_α, V_α and v_α respectively if α is a $(\mathcal{P}, \Lambda)$-strategy. We will denote by s^- the previous α-true stage and by o^- the outcome it had on that stage. If α has been initialized since its previous true stage or if it has never before been visited then $s^- = s$ and o^- is the most right outcome.

Suppose we have constructed $\delta[s] \upharpoonright n = \alpha$. If $n = s$ then the stage is finished and we move on to stage $s+1$. If $n < s$ then α is visited and the actions that α performs are as follows:

(I.) $T(\alpha) = (\mathcal{P}_i, \Gamma_i)$. This strategy is responsible for approximating the sets U_i, V_i and Θ_i. It will consider the next approximation only on active stages. On these we perform the actions as stated in the main module in Section 4.1.1. $\delta[s](n+1) = l$ at non-expansionary stages. At expansionary stages $\delta[s](n+1) = e$. At nonactive stages no actions are performed. The outcome is $o = o^-$.

(II.) $T(\alpha) = (\mathcal{P}_i, \Lambda_i)$. On active stages we perform the actions as stated in the main module in Section 4.2.1. $\delta[s](n+1) = e$. At nonactive stages no actions are performed, $\delta[s](n+1) = e$.

(III.) $T(\alpha) = (\mathcal{N}_i, S_0, \dots, S_{i-1})$ with active $\mathcal{P}$-nodes $\alpha_0, \dots \alpha_{i-1}$. On active stages we perform *Check* first. If it doesn't instruct us otherwise then we carry on with the module from where it was left at the previous α-true stage s^-(from *Initialization* if $s^- = s$). On nonactive stages α may only attack.

- **Check:** Let $Out_\alpha = \bigcup_{j<i} O_{d_j} \cup O_\alpha \bigcup_{w \in Wit_\alpha, w<x} O_{w,own} \cup O_{x,else}$. The strategy α performs $Check(j)$ for $j = i-1, i-2 \dots 1$.

  Check($i-1$) Scan all $n \leq d_{i-1}$. If an A_{α_k}-marker for n, $m_k(n) \notin Out_\alpha$, has been extracted from A by α_k, the active $\mathcal{P}_k$-strategy at α, for $k \leq i-1$ at a stage t: $s^- < t \leq s$ then we will enumerate it in $O_{d_{i-1}}$ and empty Wit and Wit_j, $j < i$, initialize all strategies below α and start from initialization.

  Check(j) Scan all $d_{j+1} < n \leq d_j$. If an A_{α_k}-marker for n, $m_k(n) \notin Out_\alpha$, has been extracted from A by α_k for $k \leq j$, at a stage t: $s^- < t \leq s$ then we will enumerate it in O_{d_j}. Then all successors of α that assume that d_j does not change infinitely many times are initialized. These are strategies γ such that $\gamma \supseteq \alpha\hat{\ } g_k$ for $k \leq j$ or $\gamma \supseteq \alpha\hat{\ } o$ where $o \in \{h_l, f_x, w | l < i, x \in \omega\}$, hence all strategies below and to the right of outcome g_j. Then we will empty Wit_k for $k \leq j$ and leave only the current witness x in them.

  If α is evaluating *Result* and the last active g-outcome was g_l and $l < j$ then α continues from the Initialization step. Otherwise α continues to evaluate *Result*.

  If a threshold d_j is extracted from $\overline{K}[s]$ then it is shifted to the next possible value, i.e to the least $n > d_j, n \in \overline{K}[s]$. If this injures the order between thresholds then the other thresholds are shifted as well. In this case the strategy resets its work in the same way as described in $Check_j$ when an element enters O_{d_j}.

  Extract Out_α from A.

- **Initialization:** Each strategy $S_j \neq FM_j$ picks a threshold if it is not already defined. The different thresholds must be in the following order:

$$d_{i-1} < d_{i-2} < \dots < d_0.$$

Strategy S_j picks its threshold as a fresh number such that its marker has not yet been defined by the active $\mathcal{P}_j$-strategy at α. Then α picks a witness $x \in E$ again as a fresh number. Then it enumerates x in Wit_α.

If this is the first witness that α picks after it was initialized then x is enumerated in all Wit_j $j < i$ where $S_j = \Gamma_j$ and $O_{x,else} = \emptyset$.

If the previous active g-outcome was g_j on stage s^- then x is enumerated in Wit_k, for $k \leq j$, such that $S_k = \Gamma_k$. Then $O_{x,else}$ is the set of all A_{α_k}-markers m_k of potentially applicable axioms in the operator defined at the current active $\mathcal{P}_k$-strategy for elements $n < d_k$, for $k < j$, that were extracted from A on stage s^-.

If $l(\Theta_j^{U_j,V_j}, E)[s] \leq x$ for some $j < i$ then $\delta(n+1)[s] = w$, working below $R = R_\alpha$.

If $l(E, \Theta_j^{U_j,V_j})[s] > x$ for all $j < i$ then α extracts from A and enumerates in O_α all A_{α_j}-markers for all potentially applicable axioms for all elements $n \geq d_j$ from all active operators S_j. Then cancels all current j-markers for $n \geq d_j$ and defines $z_{\alpha_j}(d_j) > \theta_j(x)$.

For every element $y \leq x$, $y \in E[s]$, α enumerates in the list $Axioms_j$ the current valid axiom from $\Theta_j[s]$ that has been valid longest as defined in Section 4.1.2. The next stage will start from *Honestification.* $\delta[s](n+1) = w$, working below $R = R_\alpha$.

- **Honestification:** The strategy α performs $Honestification(0)$.

  Honestification(j):If $S_j = FM_j$ then $(o = w)$. Otherwise:

  (1) Scan the list $Axioms_j$. If for any element $y \leq x$, $y \in E[s]$, the listed axiom was not valid on any stage t since the last α-true stage then update the list $Axioms_j$, let (o $= h$) and go to (2) otherwise let (o $= w$)

  (2) Extract and enumerate in O_α all A_{α_j}-markers $m_j(n)$ of potentially applicable axioms for elements n such that $d_j \leq n < l(\Theta_j^{U_j,V_j}, E)[s]$. Cancel their current j-markers. For the elements $n \in \overline{K}[s]$ define $z_{\alpha_j}(n) > \theta_j(x)$.

  If the outcome of $Honestification(j)$ is w then α performs $Honestification(j+1)$ if $j+1 < i$ and goes to waiting if $j+1 = i$. If the outcome is h then α extracts all A_{α_k}-markers of potentially applicable axioms for elements $n \geq d_k$, enumerating them in O_α for all $k > j$. Then cancels their current A_{α_k}-markers. The outcome is $\delta[s](n+1) = h_j$ working below $R = \min(R_\alpha, m_{\alpha_j}(d_j))$. At the next stage α start from $Honestification$.

- **Waiting:** If all outcomes of all $Honestification_j$-modules are w, i.e all enumeration operators are honest then α checks if $x \in \Psi_i^A[s]$ with $use(\Psi, A, x) < R_\alpha$. If not then the outcome is $\delta[s](n+1) = w$, working below $R = R_\alpha$. At the next stage α returns to $Honestification$. If $x \in \Psi_i^A[s]$ with $use(\Psi, A, x) < R_\alpha$ then α goes to *Attack*.

- **Attack:** If α is dependent then it waits for an $ins(\alpha)$-nonactive stage. $\delta[s](n+1) = w$, working below $R = R_\alpha$.

  If the stage is $ins(\alpha)$-nonactive, $x \in \Psi^A[s]$, $use(\Psi, A, x) < R_\alpha$ and all operators are honest,(i.e. the axioms recorded in the lists $Axioms_j$, $j < i$, have remained valid on all stages since s^-) then α extracts x from E. Define $O_{x,own}$ to be the set of all potentially applicable axioms in the active $\mathcal{P}_j$-operators for elements $n \geq d_j$ and $j < i$. Let $t_x = s$ and $L_x = use(\Psi, A, x)$ and $In_x = A \upharpoonright L_x$.

This starts an α-nonactive stage for the strategies below the most recently visited outcome g_j (if none has been visited until now then g_0) working below the boundaries they worked before.

- **Result** Let $\bar{x}$ be the least element extracted from E during the attack. It has a corresponding entry $\langle \bar{x}, U_{\bar{x},j}, V_{\bar{x},j} \rangle$ in $Axioms_j$. Define $Attack(x) = \langle \bar{x}, U_{\bar{x},0}, V_{\bar{x},0}, \dots U_{\bar{x},i-1}, V_{\bar{x},i-1} \rangle$. We will denote by $Attack(x)[j]$ the pair $(U_{\bar{x},j}, V_{\bar{x},j})$. Redefine L_x to be the maximum of all L_y for all elements y that were extracted during the attack. Empty $O_{x,else}$ as it has done its job. The strategy α performs $Result(0)$.

  Result(j):If $S_j = FM_j$ or $S_j = \Lambda_j$ then go to $Result(j+1)$. Otherwise scan all witnesses w in Wit_j. Let $Attack(w)[k] = (U_{\bar{w},k}, V_{\bar{w},k})$ for $k \leq j$. If one of the following two conditions is true for any $k \leq j$:

  (1) $S_k = \Gamma_k$ and there was a change in $V_{\bar{w},k}$ since this witness was last examined, i.e. there is a stage t, such that t is bigger than the stage on which this witness was last examined such that $V_{\bar{w},k} \nsubseteq V_k[t]$.

  (2) An A_{α_k}-marker $m_k < L_w$ of an element $n < d_k[t_w]$ such hat $m_k \in A[t_w]$ was enumerated in O_{d_k} for $k < j$.

  Then extract $O_{w,own}$ from A and go to the next witness.

  Otherwise w is k-successful for all $k \leq j$ then go to $Result(j+1)$.

  If all witnesses are scanned then cancel the last witness cancel the current A_{α_k}-markers for elements $n \geq d_k$, $k > j$. Empty Wit_l and cancel d_l together with O_{d_l} for $l < j$. Return to $Initialization$ at the next stage, choosing a new witness and thresholds. The outcome is $\delta[s](n+1) = g_j$. boundary is $R = \min(x, R_\alpha)$.

  Result(i) We reach this result only in case we have found some witness w that is j-successful for all $j < i$. Then let the current witness be w. Restrain In_w in A. Let the outcome be $o = f_w$, working below $R = R_\alpha$. At the next stage go back to $Result_q$ where q is the greatest index of a Γ-strategy among $(S_0, \dots S_{i-1})$ and $q = i$ if the are no Γ-strategies.

7 The Proof

We shall now prove that the construction described in Section 6.2 works. We shall start by defining a true path in the tree of strategies. Using this path we shall then prove some basic properties of the construction. This will enable us to prove that all $\mathcal{P}$-requirements are satisfied. Finally we will turn our attention to the $\mathcal{N}$-requirements.

7.1 The true path

Lemma 7 *There is an infinite path f in our tree of strategies with the following properties:*

(1) $\forall n \exists^\infty s(f \upharpoonright n \subseteq \delta[s])$ - *the infinite visit property.*

(2) $\forall n \exists s_l(n) \forall s > s_l(n)(\delta[s] \not<_L f \upharpoonright n)$ - *the leftmost property.*

(3) $\forall n \exists s_i(n) \forall s > s_i(n)(f \upharpoonright n$ *is not initialized anymore*) - *the stability property.*

PROOF. We will define the true path with induction on n and prove that it has the properties needed. The case $n = 0$ is trivial: $f \upharpoonright 0 = \emptyset$ is visited on every stage of the construction and is never initialized, $s_l(0) = s_i(0) = 0$.

Suppose we have constructed $f \upharpoonright n = \alpha$ with the required properties. We shall define $f \upharpoonright (n+1)$:

If α is a $(\mathcal{P}_i, \Gamma)$-strategy then it has two possible outcomes $e <_L l$. If outcome e is visited infinitely often then let $f \upharpoonright (n+1) = \alpha\hat{\ }e$. It has the infinite visit property and being the most left possible outcome has the leftmost property with $s_l(n+1) = s_l(n)$. Otherwise there is a stage t after which whenever we visit α, we visit also $\alpha\hat{\ }l$. Then $f \upharpoonright (n+1) = \alpha\hat{\ }l$ is visited infinitely often and has the leftmost property with $s_l(n+1) = \max(s_l(n), t)$. α does not initialize its successors, hence $s_i(n+1) = \max(s_i(n), s_l(n+1))$.

If α is a $(\mathcal{P}_i, \Lambda_i)$-strategy then it has only one outcome $o = e$ visited on every α-true stage, hence $f \upharpoonright (n+1) = \alpha\hat{\ }e$ has the needed properties with $s_l(n+1) = s_l(n)$. α does not initialize its successors, hence $s_i(n+1) = \max(s_i(n), s_l(n+1))$.

Let α be an $(\mathcal{N}_i, S_0, \dots, S_{i-1})$-strategy, where $S \in \{\Gamma, \Lambda, FM\}$. After a stage $t > s_i(n)$, α has a permanent threshold $d_{i-1} \in \overline{K}$. If we assume otherwise this would mean that $\overline{K}$ is finite and hence computable which is not true. There are finitely many elements $n \in K$, $n < d_{i-1}$, with finitely many axioms defined for them in the corresponding operators $S_0, \dots S_{i-1}$, as once an element exits $\overline{K}$ no more axioms are enumerated for it in any operator. Hence there are finitely many markers, which can initialize all nodes below α each only once, on its entry in $O_{d_{i-1}}$, which is never again emptied after stage $s_i(n)$. Hence there is a stage $t_1 > t$ after which no more markers enter $O_{d_{i-1}}$. If α has an outcome g_{i-1} (and hence $S_{i-1} = \Gamma_{i-1}$) that is visited infinitely often, then let $f \upharpoonright (n+1) = \alpha\hat{\ }g_{i-1}$ with $s_l(n+1) = s_l(n)$ and $s_i(n+1) = \max(t_1, s_l(n+1))$. It is the leftmost possible outcome, hence it has all required properties.

In general suppose g_j is the leftmost outcome that is visited infinitely often. Then there is a stage $t > s_i(n)$ such that for all α-true stages $s > t$ no outcome g_k for $j < k < i$ is visited again. In this case the thresholds $d_{i-1} \dots d_j$ are never cancelled and the corresponding sets $O_{d_{i-1}} \dots O_{d_j}$ are never emptied after stage t.

Eventually the thresholds stop shifting, as $\overline{K}$ is infinite. There are finitely many elements $n < d_j$ such that $n \in K$ with finitely many axioms defined in each of the operators $S_0, \dots S_{i-1}$. There will be a stage $t_1 > t$ after which no new marker enters O_{d_k}, for $j \leq k < i$. After this stage the outcome g_j will not be initialized. Hence we can define $f \upharpoonright (n+1) = \alpha\hat{\ }g_j$ with $s_l(n+1) = t$ and $s_i(n+1) = \max(t_1, s_l(n+1))$.

If no g-outcome is visited infinitely many times then there is a stage $t > s_i(n)$ such that for all α-true stages $s > t$ no g-outcome g_k for $k < i$ is visited again. In this case none of the thresholds are ever cancelled again and none of the sets O_{d_j} are emptied after stage t. Similarly to the previous case we get a stage $t_1 > t$ such that no new markers enter any of the sets O_{d_j} and α does not initialize any of its successors during *Check*.

If the last time we visited a g-outcome it was on an active stage, if $Check$ restarts α after stage t or if we never visited any g-outcome then the only possible outcome accessible at stages $s > t$ are w and h_j for $j < i$.

If h_0 is visited infinitely often then let $f \upharpoonright (n+1) = \alpha\hat{\ }h_0$ with $s_l(n+1) = \max(s_i(n), t)$ and $s_i(n+1) = \max(t_1, s_l(n+1))$.

In general let h_j be the leftmost h-outcome visited infinitely often. Then after stage $t_2 > t$ no other h outcome is visited again and then we can define $f \upharpoonright (n+1) = \alpha\hat{\ }h_j$ with $s_l(n+1) = \max(s_i(n), t_2)$ and $s_i(n+1) = \max(t_2, s_l(n+1))$.

If none of the h-outcomes are visited infinitely often then there is a stage $t_2 > t$ after which h_j for $j < i$ is never visited again. Then $f \upharpoonright (n+1) = \alpha\hat{\ }w$ with $s_l(n+1) = \max(s_i(n), t_2)$ and $s_i(n+1) = \max(t_2, s_l(n+1))$.

Suppose the last time we visited a g-outcome it was on a non-active stage and α is not restarted after stage t. Then after stage t no more witnesses will be defined as in order to cancel a witness and choose a new one we pass through a g-outcome. Hence at stages $s > t$ $Wit[s] = Wit[t]$. The only accessible outcomes after stage t are finitely many: f_x for $x \in Wit_{i-1}$. Denote them by $f_{x_k} <_L \dots <_L f_{x_1}$.

Suppose outcome f_{x_p} is visited on a stage $s > t$. Then the only outcomes that can be accessible at later stages will be f_{x_q} with $q \geq p$. In order to reach outcomes w ,h or f_{x_r} with $r < p$ we need to pass through a g-outcome again, which we know does not happen. Then choose the biggest p such that there is a stage $t_1 > t$ on which we pass through outcome f_{x_p}. It follows that after this stage we will always pass through f_{x_p} whenever we visit α.

Hence $f \upharpoonright (n+1) = \alpha\hat{\ }f_{x_p}$ with $s_l(n+1) = \max(s_l(n), t_1)$ and $s_i(n+1) = s_l(n+1)$. □

Now that we have established the existence of the true path we can prove formally one more property of the true path, one that we have already claimed in the previous sections concerning the distribution of active and nonactive stages.

Proposition 8 *Suppose $\alpha\hat{\ }g_j \subseteq \beta \subset f$. Then β is visited on infinitely many active and on infinitely many α-nonactive stages.*

PROOF. We will prove this proposition with induction on the distance d between α and β.

If the distance is 1 then $\beta = \alpha\hat{\ }g_j$. The g-outcome that α has during an attack is determined by α's previous active g-outcome. β is visited infinitely many times, hence it is visited on infinitely many active stages and after each β is visited on an α non-active stage.

Suppose the distance is greater than 1. If there are no nodes σ such that $\alpha\hat{\ }g_j \subset \sigma\hat{\ }g_k \subseteq \beta$ then the same argument proves that β will be visited on an active stage followed by an α-nonactive stage, as on nonactive stages the strategies between α and β will have the same outcome as on the previous active stage. If there is such a σ then induction hypothesis gives us the lemma for α and σ: σ is visited on infinitely many active stages each followed by an α-nonactive visit. By the induction hypothesis again but now for σ and β the strategy β will be visited on infinitely many active stages each followed by a σ-nonactive visit. The only thing left to note is that any σ-nonactive stage is also α-nonactive.(although not every α-nonactive stage will be σ-nonactive). □

7.2 The $\mathcal{P}$-strategies

Proposition 9 *Suppose $\Theta_i^{U_i,V_i} = E$ and $\alpha \subset f$ is a $\mathcal{P}_i$-strategy.*

(1) *Suppose $T(\alpha) = (\mathcal{P}_i, \Gamma_i)$. And suppose that for some element $n \in \overline{K}$ the current U_α-marker and the A_α-marker are not changed by any other strategy after stage t. Then α will stop changing the current marker eventually and $n \in \Gamma_\alpha^{U_i,A}$.*

(2) *Suppose $\alpha = (\mathcal{P}_i, \Lambda_i)$. And suppose that for some element $n \in \overline{K}$ the current V_α-marker and the A_α-marker are not changed by any other strategy after stage t. Then α will stop changing the current markers eventually and $n \in \Lambda_\alpha^{V_i,A}$.*

PROOF. We shall omit the index α in the proof as we will be talking only about parameters that belong to α. We shall omit the index i as well as we will only be concerned with U_i,V_i and Θ_i.

(1) Suppose $u(n)$ remains the same after stage t. We will use what we know from Section 3, more precisely Proposition 4, about the approximation to the set U, namely that it is *Good* and Σ_2^0. As α is the strategy responsible for the approximations of the set all the rest of the stages that appear in the proof of (1) can be considered α-true. Let G denote the set of all good stages, then there will be a stage $t_1 > t$ such that:

Good: $(\forall s > t_1)(s \in G \Rightarrow U \restriction u(n) = U[s] \restriction u(n))$.

$\mathbf{\Sigma^0_2}$: $(\forall s > t_1)(U \restriction u(n) \subseteq U[s])$.

By Proposition 5 $\{U[s] \oplus V[s]\}$ is a good Σ^0_2 approximation to $U \oplus V$ and hence if $x \in \Theta^{U,V}$, there is a stage s such that $\forall s' > s(x \in \Theta^{U,V}[s'])$ and if $x \notin \Theta^{U,V}$ then on good stages $s' \in G(x \notin \Theta^{U,V}[s'])$. It follows that as $E = \Theta^{U,V}$ for any number x there will be a stage t_x such that on all good stages $s > t_x(l(\Theta^{U,V}, E)[s] > x)$.

So there will be a good stage $t_2 > \max(t_1, t_n)$ on which $n < l(\Theta^{U,V}, E)[t_2]$. On this stage we will examine the current axiom for n in Γ, say $\langle n, U_n, \{m\}\rangle$. If it is valid then $U_n \subseteq U[t_2] = U \restriction u(n)$. And hence at all stages $s > t_2$ $(U_n \subseteq U[s])$. If it isn't valid then we will enumerate a new axiom $\langle n, U[t_2] \restriction u(n), \{\gamma(n)\}\rangle$ and for this axiom we will have that at all stages $s > t_2(U[t_2] \restriction u(n) \subseteq U[s])$. In both cases the marker $\gamma(n)$ will not be moved at any later stage and the axiom remains valid forever, hence $n \in \Gamma^{U,A}$.

(2) Here the strategy α is not responsible for the approximations of the sets. Instead there is a $(\mathcal{P}_i, \Gamma_i)$-strategy $\beta \subset \alpha$ that approximates the sets. All stages considered for the rest of this proof are β-true. Suppose $v(n)$ remains constant after stage t. As in part (1) we can find a stage $t_1 > t$ such that:

Good: $(\forall s > t_1)(s \in G \Rightarrow V \restriction v(n) = V[s] \restriction v(n))$.

$\mathbf{\Sigma^0_2}$: $(\forall s > t_1)(V \restriction v(n) \subseteq V[s])$.

There will be a good stage $t_2 > t_1$ on which $n < l(\Theta^{U,V}, E)[s]$. On the next α-true stage $t_3 \geq t_2$ we will examine the current axiom for n in Λ, say $\langle n, V_n, \{m\}\rangle$. If the current axiom is valid, i.e it was valid on all stages since the last α-true stage t_3^- then $V_n \subseteq V[t_2] \restriction v(n) = V \restriction v(n)$. And hence at all stages $s > t_3(V_n \subseteq V[s])$. If it isn't valid then we will enumerate a new axiom $\langle n, V'_n, \{\lambda(n)\}\rangle$. We choose this V'_n as $V[t] \restriction v(n)$ for some $t : t_3^- < t \leq t_3$ so that it was valid longest, i.e. the one with the least t such that $V[t] \subseteq V[s]$ for all $s : t \leq s \leq t_3$. Obviously $V[t_2] \restriction v(n)$ would be among these choices. Hence $V'_n \subseteq V[t_2]$. In both cases the marker $\lambda(n)$ will not be moved at any later stage and the axiom remains valid forever, hence $n \in \Lambda^{V,A}$. □

Proposition 10 (1) *Let α be the biggest $(\mathcal{P}_i, \Gamma_i)$-strategy along the true path f. Suppose that the current A_α for some element n grows unboundedly. If $\Theta_i^{U_i,V_i} = E$ then there is an outcome g_i along the true path.*

(2) *Let $\alpha \subset f$ be the biggest $\mathcal{P}_i$-strategy. Suppose it builds an operator M_i. Suppose that the current A_α-marker for some element n grows unboundedly. Then $\Theta_i^{U_i,V_i} \neq E$.*

PROOF. (1) Assume for a contradiction that $\Theta_i^{U_i,V_i} = E$ and there is no g_i-outcome along the true path. Let n be the least element, whose current A_α-marker moves off to infinity. If $n \in K$ then there will be a stage at which n enters K. After that stage no more axioms for n are enumerated in Γ_α, hence the marker $\gamma_\alpha(n)$ will remain constant. Hence $n \in \overline{K}$.

On every stage s there are finitely many $\mathcal{N}$-strategies along that can move n's markers, namely the ones with threshold $d_i[s] \leq n$.

Every time a new $\mathcal{N}$-strategy is activated it chooses its threshold $d_i > l(\Theta_i^{U_i,V_i}, E)[s]$. Hence once the length of agreement $l(\Theta_i^{U_i,V_i}, E)[s]$ is above n, no newly activated $\mathcal{N}$-strategy or no strategy whose threshold d_i is cancelled and then rechosen will have influence on n . So out of the finitely many $\mathcal{N}$-strategies which have $d_i \leq n$ on any stage only the once that are active infinitely many times and do not get initialized after they have chosen this threshold can have a permanent effect on n, i.e. only the strategies along the true path. The ones to the right will be initialized and will rechoose their thresholds to be bigger than n, the ones to the left will not be accessible after a certain stage.

We assumed $\Theta_i^{U_i,V_i} = E$, hence there will not be an outcome h_i along the true path. Indeed if $\beta \supset \alpha$ has active P_i-strategy α and true outcome h_i then there is a permanent witness x_β so that $Axioms_{i,\beta}$ changes its entries infinitely often. $Axioms_i$ has finitely many entries, one for each $y \leq x, y \in E$. Hence the entry for at least one element $y \in E$ changes infinitely often, thus $y \notin \Theta_i^{U_i,V_i}$.

Assume that there is no outcome g_i along the true path. Then let $f \upharpoonright m$ be the biggest $\mathcal{N}$-strategy which has an active $\mathcal{P}_i$-strategy α and a permanent threshold $d_i \leq n$ after stage t_0. Let t_2 be a stage that is bigger than $\max(s_l(m+1), s_i(m+1), t_0)$ and such that all other $\mathcal{N}$-strategies along the true path and to the right of it have already changed the value of their threshold d_i to a value greater than n.

We claim that after stage t_2 no $\mathcal{N}$-strategy β will change the current i-markers of n. So suppose β is visited on stage $t > t_2$ and has outcome o. Suppose $\beta \subset f$ and has a permanent threshold $d_i[t] < n$. In all other cases it follows from the choice of stage t_2 that β will not change the i-markers of n. Note that according to the choice of $t_2 > s_i(m+1)$ the outcome o is equal to or to the right of the true outcome o_β of β. We shall examine the different possibilities for o_β. Outcome $o_\beta = g_j$ for $j > i$ would cancel d_i on every $\beta\hat{\ }o_\beta$-true stage contradicting the assumption that d_i is permanent. If $o_\beta = g_k$ or $o_\beta = h_k$, for $k < j$ then there will be a new $(\mathcal{P}_i, \Gamma_i)$-strategy along the true path, contradicting the assumption that α is the biggest one. If $o_\beta = f_x$ then it follows from Lemma 7 and the choice of $t_2 \geq s_i(m+1)$ that $o = f_{x'}$ where $x' \geq x$. If $o_\beta = w$ or $o_\beta = h_j$, for $j > i$ then $o = w$ or $o = h_k$ for $k > j$. In all three cases β will not move any i-markers on stage t.

Proposition 9 proves that in this case the strategy α will not move the markers either. Hence our assumption is wrong and there is an outcome g_i along the true path.

(2) Assume for a contradiction that $\Theta_i^{U_i,V_i} = E$. Let n be the least element whose A_α-marker moves off to infinity. If $M_i = \Gamma_i$ then according to the previous case

there will be an $\mathcal{N}$-strategy along the true path with true outcome g_i, followed by another $\mathcal{P}_i$-strategy, namely working with Λ_i. Hence $M_i = \Lambda_i$.

We will prove that after a certain stage t the current marker of n is not moved by any $\mathcal{N}$-strategy β. The ones that are not in α's subtree do not have access to the markers defined by α. There are only finitely many strategies with permanent threshold $d_i \leq n$. They are all on the true path. Let s be a stage bigger than $s_i(m+1)$, where $f \upharpoonright m$ is the greatest such $\mathcal{N}$-strategy and such that all nonpermanent thresholds are already bigger than n, $n < l(\Theta_i^{U_i,V_i}, E)[s]$ and all strategies to the right of $f \upharpoonright m$ are initialized. Note that after this stage, whenever we visit $\beta \supset \alpha$ such that $\beta \subseteq f \upharpoonright m$ then β can only have an outcome equal to or to the right of the true path.

Let $\beta \supset \alpha$ be an $\mathcal{N}$-strategy along the true path with true outcome o_β. Outcomes $o_\beta = g_j$ for $j > i$ would mean that $d_i > n$ and β does not influence n's marker after stage t. There is no outcome g_i. Outcomes $o_\beta = g_k$ and $o_\beta = h_k$, $k < i$ would activate a bigger $\mathcal{P}_i$-strategy. As in (1) the only possible true outcomes turn out to be $o_\beta = h_j$, $j > i$, outcomes $o_\beta = w$ and $o_\beta = f_x$. But we have seen that in this case the β does not move any i-markers after stage t.

If $n \in K$ then there will be a stage s at which n enters K and after which the $\lambda_\alpha(n)$ remains the same. Hence $n \notin K$ and Proposition 8 proves that in this case α will also stop moving the current marker.

We have reached a contradiction, hence $\Theta_i^{U_i,V_i} \neq E$. $\Box$

Corollary 11 *The $\mathcal{P}_i$-requirements are satisfied.*

PROOF. If $\Theta^{U_i,V_i} \neq E$ then $\mathcal{P}_i$ is trivially satisfied. Assume $\Theta^{U_i,V_i} = E$. Consider the biggest $\mathcal{P}_i$-node α on the true path. It follows from Proposition 10 that for all its elements all its current markers eventually settle down. Hence by Proposition 9 for any $n \in \overline{K}$ we have that $n \in \Gamma_i^{U_i,A}$ if α is constructing Γ_i and $n \in \Lambda_i^{V_i,A}$ if α is constructing Λ_i.

If $n \notin \overline{K}$. Then $n \notin \overline{K}[t]$ for all $t > s_0$. If $T(\alpha) = (\mathcal{P}_i, \Gamma_i)$ then $n \notin \Gamma_i^{U_i,A}[t]$ on all α-true expansionary stages $t > s_0$, thus $n \notin \Gamma_i^{U_i,A}$. If $T(\alpha) = (\mathcal{P}_i, \Lambda_i)$ then for each axiom $\langle n, V_n, m\rangle \in \Lambda_i$ there are infinitely many stages $t > s_0$ on which this axiom is not valid. Namely for each α-true stage $t > s_0$ with previous α-true stage t^- either there is a stage t_n, $t^- < t_n \leq t$ on which $V_n \not\subseteq V_i[t_n]$ or else on stage t we extract m from A. Thus $n \notin \Lambda_i^{V_i,A}$ $\Box$

7.3 The $\mathcal{N}$-strategies

Proposition 12 *Let $\alpha \subset f$ be an $\mathcal{N}$-strategy with $ins(\alpha) = \beta$. Suppose $\alpha \supset \beta\hat{\ } g_j$ and α attacks with a witness x on stage t together with an attack of β with x_1. Then $Attack(x)[k] = Attack(x_1)[k]$ for all $k \leq j$.*

PROOF. Let $T(\alpha) = (\mathcal{N}_i, S_0 \dots S_{i-1})$. It follows from the definition of an instigator that $S_j = \Lambda_j$ and both strategies α and β are dealing with the same approximations of the sets Θ_j, U_j, V_j controlled by the active $\mathcal{P}_j$-strategy at β. First we will prove that the active $\mathcal{P}_k$-strategies at α for $k < j$ are the active $\mathcal{P}_k$-strategies at β. Suppose this is not true. Then some $\mathcal{P}_k$ active strategy at β was destroyed by some $\mathcal{N}$-strategy σ such that $\beta\hat{\ }g_j \subseteq \sigma \subset \alpha$. If σ has a g-outcome then it would be the instigator of α. Hence it had outcome h_k where $k < j$. But then $\mathcal{P}_j$ starts from $(\mathcal{P}_j, \Gamma_j)$ below $\sigma\hat{\ }h_k$ and can only change back to Λ_j if a second strategy σ' such that $\sigma\hat{\ }h_k \subseteq \sigma' \subset \alpha$ has outcome g_j in which case σ' would be the instigator of α. Hence for all $k \leq j$ both α and β are dealing with the same approximations to the sets Θ_k, U_k and V_k.

By Proposition 8 α is visited on β-active stages, followed by β-nonactive stage. Stage t is a β-nonactive stage, let t^- be the previous β-active α-true stage. On this stage α had in its $Axioms_k^\alpha$ for $k \leq j$ a list of axioms for all elements $y \leq x$, which were valid the longest. After stage t^- the strategy β chooses its witness $x_1 > x$ and fills in the corresponding lists $Axioms_k^\beta$. For elements $y \leq x$ these are the same axioms that α recorded. If during $\beta's$ work, one of the list changes its entry for an element $y \leq x$ then on stage t the strategy α would not attack but go back to *Honestification* instead and wait for an active stage on which to modify its own lists. Hence the entry in all $Axioms_k^\beta$ for elements $y \leq x$ is the same as the entry $Axioms_k^\alpha$ for all $k \leq j$ and in particular the entries are the same for the least element extracted during the attack at stage t, say $\bar{x} \leq x < y$. Hence $Attack(x)[k] = Attack(x_1)[k]$ for all $k \leq j$. $\square$

The main aim now is to prove that if an $\mathcal{N}_i$-strategy α on the true path has outcome f_x for some x then the requirement $\mathcal{N}_i$ is satisfied as $x \in \Psi_i^A$. To ensure this we will need to establish that the set In_x that α is trying to restrain in A ends up indeed in A. Various strategies around α might try to prevent this from being true by extracting elements from A. We will first prove that a $\mathcal{P}$-strategy that is not active at α cannot extract any elements that α is trying to restrain in A. Then we shall prove that neither can any of the other $\mathcal{N}$-strategies. Finally we will establish this for the active $\mathcal{P}$-strategies at α.

Proposition 13 *Suppose we have an $\mathcal{N}_i$-strategy $\alpha = f \restriction n$ along the true path with active $\mathcal{P}_j$-strategies $\beta_j \subset \alpha$ for $j < i$ and true outcome $f(n+1) = g_j$ or h_j where $j < i$. Suppose $f \restriction (n+1)$ is visited on stage $s > s_i(n+1)$ with right boundary $R[s]$. Then if $m < R[s]$ is an A_{β_k}-marker where $k \geq j$ and m is extracted on stage $t > s$ by the active $\mathcal{P}_k$ strategy β_k then m is extracted from A on all $f \restriction (n+1)$-true stages $t \geq s$.*

PROOF. After stage $s_i(n+1)$ defined in Lemma 7 α has permanent thresholds d_k and permanent sets O_{d_k} for $k \geq j$ and $Out_\alpha[t] \supseteq \bigcup_{k \leq j} O_{d_j} \cup O_\alpha[s]$ on all $t \geq s$.

Suppose m is extracted by β_k where $k \geq j$ on stage $t > s$. Then m is an A_{β_k}-marker of an axiom for an element $e \notin \overline{K}$ such that $e > d_k$ as otherwise a new element would enter O_{d_k} contradicting our choice of stage s. If the marker m was defined after stage s then it is bigger than $R[s]$. If the marker is defined before stage s then so is the axiom Ax_m that it belongs to.

(1) If $f(n+1) = h_j$ then $Out_\alpha[s] \subseteq Out_\alpha[t]$ for all $t > s$. On stage s the axiom Ax_m is examined by α and if m is not already in Out_α then m is enumerated in Out_α at stage s. Hence $m \in Out_\alpha[t]$ for all $t \geq s$.

(2) If $f(n+1) = g_j$ then $Wit_j[s] \subseteq Wit_j[t]$ for all $t > s$ and the current witness $x[s]$ is in the set Wit_j. If $m \notin O_\alpha[s]$ on stage s then it is enumerated in $O_{x[s],own}$ and $O_\alpha[s] \cup O_{x,own}[s] \subseteq Out_\alpha[t]$ on all $\alpha\hat{\ }g_j$-true stages t. □

Proposition 14 *Suppose $\beta \subset f$ is visited on stages $s_1 > s_i(\beta)$ and $s_2 > s_1$. Suppose on stage s_1 β attacks and then restrains an element m in A until stage s_2. If the active $\mathcal{P}$-strategies at β do not extract m at stages t $s_1 < t \leq s_2$ then neither do the other strategies.*

PROOF. It follows that β is an $\mathcal{N}$-strategy that has outcome f_w on all β-true stages t, $s_1 < t < s_2$. The stage of the attack with w is $t_w \geq s_i(\beta)$. The set that β restrains in A is $In_w \subseteq A[t_w] \upharpoonright R_\beta[t_w]$ and $m < R_\beta[t_w]$. Suppose $\alpha \neq \beta$ extracts m on a stage t, $s_1 < t \leq s_2$. And let that be the least stage and α be the least strategy. We will prove that it is an active $\mathcal{P}$-strategy at β by examining the different possible cases for α.

- $\alpha <_L \beta$ is not possible, as α would not be accessible on stage t.
- $\alpha >_R \beta$, then on stage s_1 α is initialized. If α is a $\mathcal{P}$-strategy then all its markers would be defined after stage s_1 and would be greater than $R_\beta[s_1] > R_\beta[t_w] \geq m$. If α is an $\mathcal{N}$-strategy then it chooses its thresholds after stage s_1 as fresh numbers whose markers are not yet defined. The only markers $m' < R_\beta[s_1]$ that can enter $Out_\alpha[t]$ are the ones that enter $\alpha's$ O_{d_i} and have to be already extracted from A after stage s_1 by a smaller strategy, an active $\mathcal{P}$-strategy at α.
- $\alpha \supset \beta$, then α extracts markers only on active stages, hence if it is visited after stage s_1 then $\alpha \supseteq \beta\hat{\ }f_w$. Then α was initialized on the stage s_1. Similarly to the previous case it cannot be a $\mathcal{P}$-strategy and if $m \in Out_\alpha[t]$ then it must have been first extracted by an active $\mathcal{P}$ at α after s_1 which is smaller than α.
- $\alpha \subset \beta$. If α is a $\mathcal{P}_j$-strategy different from the active one at α then there is an $\mathcal{N}$-strategy $\sigma\hat{\ }o \subset \beta$ with $o \in \{h_j, g_j\}$ that destroys α. Proposition 13 proves that α does not extract m on stage t as otherwise $m < R_\beta[t_w] \leq R_\sigma[t_w]$ is extracted on all $\sigma\hat{\ }o$-true stages after and including t_w contradicting $m \in A[t_w]$.

  If α is an $\mathcal{N}$-strategy then we need to examine the possibilities for the true outcome of α:

  (1) $\beta \supseteq \alpha\hat{\ }w$ or $\beta \supseteq \alpha\hat{\ }f_x$. Then after stage $t_w \geq s_i(\beta)$ the strategy α has this outcome on all true stages, the set Out_α is constant. No new elements enter O_{α,d_j},

otherwise we initialize β. Wit is permanent as is the current witness. The strategy α does not enumerate more elements in O_α as it needs to have some h_j to do so.

(2) $\beta \supseteq \alpha\hat{}h_j$, then the elements that enter Out_α at stages $t > t_w$ are markers m_k $k \geq j$ for axioms from the operators of the active $\mathcal{P}$-strategies at α that are potentially applicable at stage t for elements bigger than d_k, hence markers defined after stage t_w. Indeed all markers defined before stage t_w that ever get extracted by α would already be in $Out_\alpha[t_w]$ but $m \in A[t_w]$.

(3) $\beta \supseteq \alpha\hat{}g_j$. Then α had an active outcome g_j on the last active stage t_w^- before the attack with w on stage t_w. The marker m was not extracted by α on stage t_w^- and after stage t_w^- α does not enumerate elements $m' < R_\beta[t_w] = R_\beta[t_w^-]$ in O_α or in $O_{x,own}$ for witnesses x defined after stage t_w^-.

On stage s_1 the strategy α attack again. If α extracts an element $m < R_\beta[t_w]$ at a stage $t > s_1$ there are two possibilities. The first one is that $m < d_k$ and m enters O_{d_k}. If $k < j$ then it is first extracted by the active $\mathcal{P}_k$-strategy at α after stage s_1 which is smaller. If $k \geq j$ then this would initialize β.

The second possibility is that $m \in O_{x,else}$ on stage t for some witness x of α defined after stage s_1 and after stage s_1' on which α had an active g-outcome after the attack on s_1. Then m was extracted from $A[s_1']$ and $s_1 < s_1' < t$. But we assumed that t is the first stage on which m is extracted from A, hence this is not possible. □

Lemma 15 *Let $\alpha = f \upharpoonright n$ be the last $\mathcal{N}_i$-node along the true path. Then α is successful.*

PROOF. Suppose $T(\alpha) = (\mathcal{N}_i, S_0 \ldots, S_{i-1})$ and let β_j, $j < i$, be the active $\mathcal{P}_j$-nodes at α, where $\mathcal{P}_j$ is undefined if $S_j = FM_j$. We know that no other node can interfere with α and injure its restraint except for α itself and β_j. As α is the last $\mathcal{N}_i$-node on the true path, it must have outcome w or outcome f_x for some x. Every other outcome is followed by another copy of an $\mathcal{N}_i$-strategy.

If the outcome is w then on all α-true stages $t > s_i(n)$ defined in Lemma 7, α has a permanent witness x and $x \notin \Psi^A[t]$ with $use(\Psi, A, x)[t] < R_\alpha[t]$. It is straight forward to prove that $lim_t R_\alpha[t] = \infty$ for all $\mathcal{N}$-nodes on the true path by induction on their length. Hence $x \notin \Psi^A$ and on the other hand $x \in E$. Thus the requirement is satisfied.

Suppose the outcome is f_x and let $Attack(x) = \langle \bar{x}, U_{\bar{x},0} V_{\bar{x},0} \ldots U_{\bar{x},i-1}, V_{\bar{x},i-1} \rangle$. Then once we visit $\alpha\hat{}f_x$ after stage $s_i(n+1)$ α will permanently restraint In_x in A. We will prove that the active $\mathcal{P}$-strategies at α do not extract markers from In_x after stage $s_i(\alpha\hat{}f_x)$, the last stage of the attack, and by Proposition 14 no other strategy will, hence $x \in \Psi^A$.

First we will establish that markers extracted by the active $\mathcal{P}$-strategies after $s_i(\alpha\hat{}f_x)$ cannot belong to elements $n < d_j[t_x]$ for all $j < i$. Here t_x is the stage of the attack with witness x. Let q be the greatest index such that $S_q = \Gamma_q$. Then

$x \in Wit_q$. After stage $s_i(\alpha)$ the thresholds $d_{i-1}, \dots d_q$ are not cancelled. If an element enters O_{d_j} or the value of d_j is shifted where, we initialize $\alpha\hat{\ }f_x$. Hence this does not happen after stage $s_i(\alpha\hat{\ }f_x)$. Now lets look at $j < q$. Every time we visit α we start from $Result_q$, examine all witnesses in Wit_q and reach x. Note that once we've reached x, then for all $w < x$ we have established one of the two properties that make us move to the next witness automatically until an active g-outcome is visited, so in this case forever. And our assumption tells us that we will never establish either of the two properties for x. For all Γ_k, $k \leq q$ there is no $V_{\bar{x},k}$ change and for all S_k, $k < q$ there is no $m \in O_{d_k}$ such that $m < L_x$ and $m \in A[t_x]$. Otherwise we would move to the left of f_x. Hence the only markers restrained in A that might be extracted by the active $\mathcal{P}$-strategies after $s_i(\alpha\hat{\ }f_x)$ need to belong to elements greater than $d_j[t_x]$ for all j.

If there are no $\Gamma_k = S_k$ for any $k < i$ then no thresholds are ever cancelled, if they are shifted or an elements enters O_{d_j} for $j < i$ then f_x is initialized. So this does not happen after stage $s_i(\alpha\hat{\ }f_x)$.

Thus, suppose β_j extracts $m < L_x$ such that $m \in In_x \subseteq A[t_x]$ at stage $t > s_i(\alpha\hat{\ }f_x)$. Then m is a marker of an axiom $\langle n, Z_n, \{m\}\rangle$ for some $n > d_j[t_x]$ which is valid on stage t and was defined at stage $t_0 < t_x$. The marker m was in $A[t_x]$ hence the axiom was potentially applicable on stage t_x.

If $S_j = \Gamma_j$ ($T(\beta_j) = (\mathcal{P}_j, \Gamma_j)$) then $j \leq q$ and $Z_n \supseteq U_{\bar{x},j}$. Indeed the axiom is potentially applicable hence the stage t_0 on which it was defined is after the last $Honestification$, so $\max(Z_n) = u_j(n)[t_0] > \theta(x)[t_0] = \theta(x)[t_x]$ and $U_{\bar{x},j} \subseteq U_j \upharpoonright u_j(n)[t_0] = Z_n$ or else α would perform another $Honestification$ after stage t_0 before it attacks on stage t_x. The axiom $\langle n, Z_n, \{m\}\rangle$ is valid on stage t so $U_{\bar{x},j} \subseteq U_j[t]$ and t is expansionary (as markers are extracted only on expansionary stages) so $x \notin \Theta_j^{U_j,V_j}[t]$. Hence $V_{\bar{x},j} \nsubseteq V_j[t]$. But then on the next α-true stage one of the conditions for the unsuccessfulness of x would be valid and α would have outcome to the left of f_x contradicting our assumptions.

The only case left to consider is $S_j = \Lambda_j$. We shall deal with all Λ-strategies at once. Suppose that the Λ-strategies at α are $S_{j_0}, S_{j_1}, \dots S_{j_r}$, with $j_0 < j_1 < \dots < j_r$. Then there are strategies $\alpha_0, \dots \alpha_r$ such that $\alpha_0\hat{\ }g_{j_0} = \beta_{j_0} \subset \dots \subset \alpha_r\hat{\ }g_{j_r} = \beta_{j_r} \subset \alpha$. Then $ins(\alpha) = \alpha_r$, $ins(\alpha_r) = \alpha_{r-1} \dots, ins(\alpha_1) = \alpha_0$.

When α attacks at stage t_x, it times its attack with all of the listed strategies: α_0 which attacked with x_0, ..., α_r, which attacked with x_r. By Proposition 12 $Attack(x)[k] = Attack(x_0)[k]$ for all $k < j_0$, ... $Attack(x)[k] = Attack(x_r)[k]$ for all $k < j_r$. On the previous α-active stage t_x^- the strategy α_0 had outcome g_{j_0}, α_1 had outcome g_{j_1}, ..., α_r had outcome g_{j_r}. And so $Out_\alpha[t_x^-] \subseteq O_{x_r,else} \dots \subseteq O_{x_0,else}$.

We claim that every time α_p has outcome g_{j_p} after stage $t_{x_p} = t_x$ there is a $V_{\bar{x},j_p}$-change for all $p \leq r$. So when we take $j = j_p$, we have a $V_{\bar{x},j}$-change on all β_j-true stages after t_x. Now we have that the axiom $\langle n, Z_n, \{m\}\rangle$, potentially applicable on stage t_x has the property that $V_{\bar{x},j} \subseteq Z_n$ and so β_j will not extract m on any stage after t_x.

Suppose the claim is true for $k < p$ and α_p has outcome g_{j_p} on stage $t > t_{x_p}$. One reason for this outcome would be the desired $V_{\bar{x},j_p}$-change. The other possible reasons for α_p to have this outcome are for some $k < j_p$:

- $S_k = \Gamma_k$ and there was a change in $V_{\bar{x}_p,k} = V_{\bar{x},k}$ since this witness was last examined, i.e. there is a stage t' such that t' is bigger than the stage of the last attack such that $V_{\bar{x},k} \not\subseteq V_k[t']$. But then when we visit α on the next α-true stage after t it would have an outcome to the left of f_x, so this reason is not possible.

- A marker $m_k < L_{x_p}$ of an element $n < d_{k,\alpha_p}[t_{x_p}]$ such that $m_k \in A[t_{x_p}]$ was enumerated in O_{d_k} of α_p.

  Recall that the active $\mathcal{P}_k$-strategy at α_p and α is the same as $k < j_p$. We already established that $n > d_{k,\alpha}[t_x = t_{x_p}]$. Also the marker m_k was defined before stage t_x and even t_x^- as otherwise it would be greater than L_{x_p}. The marker was not extracted by α on stage t_x^- or else it would be in $O_{x_p,else}$ and not in $A[t_{x_p}]$. So on stage t_x the corresponding axiom $\langle n, Z_n, \{m_k\}\rangle$ was potentially applicable at α and $Z_{x,k} \subseteq Z_n$. The marker m was extracted by the active $\mathcal{P}_k$-strategy on a stage t' after the attack, so $Z_{x,k}$ was a subset of $Z_k[t']$ on an expansionary stage t'. Now if $S_k = \Gamma_k$ this would result in a $V_{\bar{x},k}$-change on stage t' and α would once again have an outcome to the left of f_x on the next true stage, contradicting our assumptions. If $S_k = \Lambda_k$ this would result in no $V_{\bar{x},k} = V_{\bar{x}_p,k}$-change on a β_k-true stage t' contradicting the induction hypothesis.

This concludes the proof of the claim, this lemma and the theorem. $\square$

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