THE AUTOMORPHISM GROUP OF THE ENUMERATION DEGREES

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ABSTRACT. We investigate the extent to which Slaman and Woodin's framework for the analysis of the automorphism group of the structure of the Turing degrees can be transferred to analyze the automorphism group of the structure of the enumeration degrees.

1. INTRODUCTION

A goal of computability theory is to give a mathematical account of a structure, which arises as a formal way of classifying the relative computational strength of objects. The most studied example of such structures is that of the Turing degrees, $\mathcal{D}_T$, arising from the notion of Turing reducibility. To understand such a structure we study its complexity: how rich is it algebraically; how complicated is its theory; what relations are first order definable in it; does it have nontrivial automorphisms. In the study of $\mathcal{D}_T$ we find that all these questions are interrelated in a very strong way. The definability of the jump operator by Slaman and Shore [15] relies on a method used by Slaman and Woodin [17] to analyze the automorphism group of $\mathcal{D}_T$, $Aut(\mathcal{D}_T)$. This analysis reveals a strong connection between the definability properties of the structure of the Turing degrees and second order arithmetic, leading to Slaman and Woodin's famous *Biinterpretability conjecture*. Even though this analysis does not give a complete answer to the question of the existence of nontrivial automorphisms of $\mathcal{D}_T$, it sheds light on the properties of $Aut(\mathcal{D}_T)$: it is shown that the group is at most countable, in fact that all its members are arithmetically definable and every automorphism is completely determined by its action on a single element.

A different approach to the analysis of the structure of the Turing degrees is to study it within a richer context, a context which would hopefully reveal new hidden relationships. Such a context is the structure of the enumeration degrees, $\mathcal{D}_e$. This structure is as well an upper semi-lattice with jump operation, induced by a weaker form of relative computability: a set A is enumeration reducible to a set B if every enumeration of the set B can be effectively transformed into an enumeration of the set A. The Turing degrees as an upper semi-lattice with jump operation can be embedded in the enumeration degrees, and thus can be studied as a substructure of the enumeration degrees, the substructure $\mathcal{TOT}$ of the total enumeration degrees.

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Working in $\mathcal{D}_e$ is different. The enumeration degrees are not closed under complement. This makes coding and relativization significantly more complicated. On the other hand in $\mathcal{D}_e$ we find that certain definability results have more natural proofs. Kalimullin [8] discovered the existence of pairs of enumeration degrees, called $\mathcal{K}$-pairs, with structural properties reminiscent of structural properties of generic Turing degrees, and showed that the class of such pairs of enumeration degrees is first order definable in $\mathcal{D}_e$, by a simple Π_1 statement in the language of lattices. The existence of $\mathcal{K}$-pairs is unique to the structure of the enumeration degrees. In fact their existence in the structure $\mathcal{D}_T$ would contradict the precise property of the Turing degrees that allows Slaman and Shore to extract the definition of the Turing jump from the automorphism analysis. This shows that the method devised by Slaman and Woodin to investigate $Aut(\mathcal{D}_T)$ cannot be transferred directly to analyze $Aut(\mathcal{D}_e)$. Still, the first step has been made. In [16] Slaman and Woodin prove the Coding Theorem for the structure $\mathcal{D}_e$, and as a consequence show that the first order theory of $\mathcal{D}_e$ is computably isomorphic to that of second order arithmetic.

Kalimullin [8] showed that the first order definability of the enumeration jump follows nevertheless from the existence of $\mathcal{K}$-pairs. A consequence to this is the first order definability of the total enumeration degrees above $\mathbf{0}_e{}'$. Ganchev and Soskova [4] bring down the definition of $\mathcal{K}$-pairs to a local level, showing that the notion is still definable in the local structure of the enumeration degrees bounded by $\mathbf{0}_e{}'$. In [6] they combine $\mathcal{K}$-pairs and the Coding Theorem to show that the first order theory of the local structure is computably isomorphic to first order arithmetic. In [3] Ganchev and Soskova investigate maximal $\mathcal{K}$-pairs and show that within the local structure the total degrees are first order definable. The question of the global definability of $\mathcal{TOT}$, set first by Rogers [12], remains unanswered and seems to play a central role in this puzzle. Rozinas [13] showed that $\mathcal{TOT}$ is an automorphism base for $\mathcal{D}_e$. Thus if $\mathcal{TOT}$ were definable then a negative answer to the question of the rigidity of $\mathcal{D}_e$ would yield a negative answer to the same question for $\mathcal{D}_T$.

We will outline how to adapt Slaman and Woodin's framework to investigate the properties of $Aut(\mathcal{D}_e)$. We shall show that $Aut(\mathcal{D}_e)$ is at most countable, that its members are arithmetically definable and that $\mathcal{D}_e$ has an automorphism base consisting of a single element. This analysis in $\mathcal{D}_e$ brings us one step closer to the definability of $\mathcal{TOT}$, namely one parameter away from it.

2. Preliminaries

The notions and definitions that will be used in this article come from various parts of logic and we will not be able to give definitions and outline their basic properties. We hope to be able to give sufficient references, to books and articles, where these notions are explained in detail. A main reference for this article is the Slaman and Woodin's manuscript [17]. We will give definitions to all notions that are not defined in [17] and are used below.

Definition 1. *A set A is* enumeration reducible *($\leq_e$) to a set B if there is a c.e. set Γ such that:*

$$A = \Gamma(B) = \{n \mid \exists u(\langle n, u\rangle \in \Gamma \;\&\; D_u \subseteq B)\},$$

where D_u denotes the finite set with code u under the standard coding of finite sets. We will refer to the c.e. set Γ as an enumeration operator.

A set A is *enumeration equivalent* ($\equiv_e$) to a set B if $A \leq_e B$ and $B \leq_e A$. The equivalence class of A under the relation $\equiv_e$ is the enumeration degree $\mathbf{d}_e(A)$ of A. The structure of the enumeration degrees $\langle \mathcal{D}_e, \leq \rangle$ is the class of all enumeration degrees with relation $\leq$ defined by $\mathbf{d}_e(A) \leq \mathbf{d}_e(B)$ if and only if $A \leq_e B$. It has a least element $\mathbf{0}_e = \mathbf{d}_e(\emptyset)$, the set of all c.e. sets. We can define a least upper bound operation, by setting $\mathbf{d}_e(A) \vee \mathbf{d}_e(B) = \mathbf{d}_e(A \oplus B)$.

The enumeration jump of a set A, denoted by A', is defined by Cooper [1] as $K_A \oplus \overline{K_A}$, where $K_A = \{\langle e, x\rangle |\ x \in \Gamma_e(A)\}$. The enumeration jump of the enumeration degree of a set A is $\mathbf{d}_e(A)' = \mathbf{d}_e(A')$.

For more on the structure of the enumeration degrees we refer to Cooper [2].

2.1. **The substructure of the total enumeration degrees.**

Definition 2. *A set A is called* total *if A is enumeration equivalent to the graph of the total function χ_A, the characteristic function of A. Equivalently A is total if $A \equiv_e A \oplus \overline{A}$. An enumeration degree is called* total *if it contains a total set. The collection of all total degrees is denoted by $\mathcal{TOT}$.*

As noted above, the structure $\mathcal{TOT}$ is an isomorphic copy of the Turing degrees. The map ι, defined by

$$\iota(\mathbf{d}_T(A)) = \mathbf{d}_e(A \oplus \overline{A})$$

is an embedding of $\mathcal{D}_T$ in $\mathcal{D}_e$, which preserves the order, the least upper bound and the jump operation. To distinguish between total sets and non-total sets we will identify total functions with their graphs and use total functions to denote total sets.

The following theorems provide the strong uniform connections between total degrees and non-total degrees. The uniformity is usually not explicitly stated in the original articles, however it follows from the given proofs.

Theorem 1 (Soskov[19])**.** *For every natural number n there exists an enumeration operator Γ_{JIT_n} such that for every X, and total set Y with $X^n \leq_e Y$, $\Gamma_{JIT_n}(X \oplus Y)$ is (the graph of) a total function f such that $X \leq_e f$ and $f^n \equiv_e \emptyset^n \oplus f \equiv_e Y$ uniformly in X and Y.*

Note that Theorem 1 shows that the class of the total enumeration degrees above $\mathbf{0}_e{}'$ coincides with the range of the enumeration jump operator. Thus the definability of this class follows from the definability of the enumeration jump operator.

Theorem 2 (Rozinas[13])**.** *There exist enumeration operators Γ_{T_1} and Γ_{T_2}, such that for every set X, $\Gamma_{T_1}(X'') = f$ and $\Gamma_{T_2}(X'') = h$ are total functions such that $\mathbf{d}_e(X) = \mathbf{d}_e(f) \wedge \mathbf{d}_e(h)$.*

Proof. Let $\Gamma_{T_1}(X'') = \Gamma_{JIT_1}(X \oplus X') = f$. We know that $f' = X'$ and hence $f'' = X''$ and that $X \leq_e f$. To obtain Γ_{T_2} we apply Sorbi's Lemma 2.1 from [18], observing that the construction of a total function h such that $\mathbf{d}_e(X) = \mathbf{d}_e(f) \wedge \mathbf{d}_e(h)$ is uniformly computable from f'', which is uniformly equivalent to X''. We outline this proof below. Let $\mathcal{H} = \{\sigma \in 2^{<\omega} \mid \text{if } \sigma(\langle n, z\rangle) = 1 \text{ then } n \in X\}$. Note that $\mathcal{H}$ is computable from X and hence from $f'' = X''$. Now we construct h as the union of a monotone sequence $\{\sigma_s\}_{s<\omega} \subseteq \mathcal{H}$. Set $\sigma_0 = \emptyset$. At stage $s = 2\langle i, j\rangle$ we have two cases: if there is an extension $\tau \supseteq \sigma$ and an x, such that $\tau \in \mathcal{H}$, $x \in \Gamma_i(\tau)$ and $x \notin \Gamma_j(f)$, then we set $\sigma_{s+1} = \tau$. Otherwise we set $\sigma_{s+1} = \sigma_s$. f'' can carry out this step computably. From this action it follows that if $\Gamma_i(h) = \Gamma_j(f)$

then $\Gamma_j(f) = \{x \mid \exists \tau \supseteq \sigma_s(\tau \in \mathcal{H} \,\&\, x \in \Gamma_i(\tau))\} \leq_e X$. At an odd stage $s = 2n+1$ we make sure that $X \leq_e h$: if $n \in X$ we extend σ_s to σ_{s+1} such that for some least z, $\sigma_{s+1}(\langle n, z\rangle) = 1$; if $n \notin X$ then $\sigma_{s+1} = \sigma_s$. □

2.2. **$\mathcal{K}$-pairs and generic degrees.** In parallel to Rozinas' Theorem 2, which shows that every enumeration degree is the meet of two total degrees, we have a theorem which shows that every total enumeration degree is the join of two non-total degrees. This theorem relies on the following notion.

Definition 3 (Kalimullin [8])**.** [1] *A pair of sets $\{A, B\}$ is a $\mathcal{K}$-pair relative to a set U, or for short a $\mathcal{K}_U$-pair, if there is a set $W \leq_e U$, such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$. If U is c.e. then $\{A, B\}$ is called a $\mathcal{K}$-pair.*

The class of $\mathcal{K}$-pairs has been studied extensively in [8, 4, 6, 3]. We describe some of their more basic properties below and give some examples.

If A is enumeration reducible to U then for every B, $\{A, B\}$ is a $\mathcal{K}_U$-pair, a trivial $\mathcal{K}_U$-pair. If $\{A, B\}$ are a nontrivial $\mathcal{K}_U$-pair then A, B and U are related in the following way:

$$B \leq_e \overline{A} \oplus U \quad \text{and} \quad \overline{B} \leq_e A \oplus U'.$$

If we fix one component of a nontrivial $\mathcal{K}_U$-pair, A, and consider all sets, which form a $\mathcal{K}_U$-pair with A, then we obtain an ideal bounded by $\overline{A} \oplus U$. Thus for a nontrivial $\mathcal{K}_U$-pair $\{A, B\}$ we obtain the following characterization of the jumps of the components: noting that $K_B \equiv_e B$ and hence also forms a $\mathcal{K}_U$-pair with A, we obtain that $K_B \oplus \overline{K_B} = B' \leq_e A \oplus \overline{A} \oplus U'$. In particular if $U <_e A, B \leq_e U'$ then A and B are low over U, i.e. $A' = B' = U'$. Furthermore both A and B are of quasiminimal degree with respect to U, i.e. every total set reducible to either A or B is reducible to U. Both of these properties are reminiscent of properties of 1-generic sets and enumeration reducibility.

A more interesting example of a nontrivial $\mathcal{K}$-pair is $\{A, \overline{A}\}$, where A is semi-recursive. Recall that a set A is semi-recursive if there is a computable selector function $s_A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, such that for all natural numbers x, y if $\{x, y\} \cap A \neq \emptyset$ then $s_A(x, y) \in A$. Jockusch [7] introduced this notion and showed that every Turing degree contains a semi-recursive set, such that neither it, nor its complement is c.e. It follows that every nonzero total enumeration degree is the join of a pair of non-total, in fact quasiminimal enumeration degrees. Note that $\mathcal{K}$-pairs of the form $\{A, \overline{A}\}$ are maximal: $\overline{A}$ is the largest with respect to $\leq_e$ set that can form a $\mathcal{K}$-pair with A and vice versa. Thus every total degree is the least upper bound of a maximal $\mathcal{K}$-pair.

Kalimullin [8] showed that a pair of sets $\{A, B\}$ is a $\mathcal{K}$-pair relative to a set U if and only if their enumeration degrees $\mathbf{a}, \mathbf{b}$ and $\mathbf{u}$ satisfy the following very strong minimal-pair property:

$$\mathcal{K}_{\mathbf{u}}(\mathbf{a}, \mathbf{b}) \leftrightarrow \forall \mathbf{x}((\mathbf{u} \vee \mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{u} \vee \mathbf{b} \vee \mathbf{x}) = \mathbf{u} \vee \mathbf{x}).$$

This property makes $\mathcal{K}_U$-pairs very useful for coding relations which are contained in intervals $(\mathbf{u}, \mathbf{u}')$, an idea applied in [6], where a standard model of arithmetic is coded below $\mathbf{0}_e{}'$. In this article we will need to code a larger variety of relations, and thus need a more universally applicable method for coding - genericity.

[1] The original term for this notion is *U-e-ideal.*

We will use the genericity notion that comes from the study of Turing degrees, we will use images of Cohen generic sets under the standard embedding ι. To make it explicit that we are dealing with a total set, we will refer to it as a total function and mean its graph. Thus an n-generic with respect to B function g will be a total binary function with the property that for every $\Sigma^0_n(B)$ set of finite binary strings W, there is an initial segment $\sigma \subseteq g$, such that $\sigma \in W$ or no extension of σ is in W. The truth of every $\Sigma^0_n(B)$ statement about g is forced by some final initial segment of g. We refer to Chapter 2 of [17] for an extensive analysis of forcing in arithmetic. The following two properties are direct translations of well known statements in the context of Turing reducibility, through the order- and jump-preserving embedding ι.

Proposition 1. (1) *Let g be n-generic relative to a total set B. Then $(g \oplus B)^n \equiv_e g \oplus B^n$.*

(2) *For every n there exists an enumeration operator Γ_{G_n} such that for every total set X, $\Gamma_{G_n}(X^n)$ is the characteristic function of an n-generic relative to X set.*

Given a total function g, we can split it into countably many total functions $\{g_i\}_{i<n}$, where $g_i(n) = g(\langle i, n\rangle)$. If g is n-generic relative to B then for every $i \neq j$, g_i in n-generic relative to $B \oplus g_j$. This property in the context of Turing degrees allows us to use 1-genericity relative to a set B in order to construct antichains which can distinguish between elements Turing reducible to B. A similar property is true in the enumeration degrees, however the fact that enumeration reducibility is not closed with respect to complement requires us to raise the amount of genericity needed.

Proposition 2. *Let g be 2-generic relative to a total set B. Denote by g_i the function defined by $g_i(n) = g(\langle i, n\rangle)$.*

(1) *Let A_1 and A_2 be sets that are enumeration reducible to B. For every i and finite set F if $A_1 \oplus g_i \leq_e A_2 \oplus \bigoplus_{j\in F} g_j$ then $A_1 \leq_e A_2$ and $i \in F$.*

(2) *Let $A \leq_e B$ and $i \neq j$. Then $A \oplus g_i$ and $A \oplus g_j$ is a minimal pair above A.*

Proof. (1) Suppose towards a contradiction that $g_i = \Gamma(A_2 \oplus \bigoplus_{j\in F} g_j)$ and that $i \notin F$. We adopt the same notation for finite binary functions as for infinite binary functions: given a string σ, let $\sigma_i(n) = \sigma(\langle i, n\rangle)$. Consider the set $W = \{\sigma \mid \exists x(\sigma_i(x) = 0 \ \&\ x \in \Gamma(A_2 \oplus \bigoplus_{j\in F} \sigma_j))\}$. The set W is $\Sigma^0_1(B)$. By our choice of g and by our assumptions it follows that no initial segment of g is in the set W. Hence there is an initial segment $\sigma \subseteq g$, which has no extension in the set W. Since g is 2-generic it follows that $\{x \mid g_i(x) = 1\}$ is infinite. Let x be larger than the length of σ_i, such that $g_i(x) = 1$. By our assumption $x \in \Gamma(A_2 \oplus \bigoplus_{j\in F} g_j)$ and hence there is a final initial segment $\tau \subseteq g$ such that $\tau_i(x) = 1$ and $x \in \Gamma(A_2 \oplus \bigoplus_{j\in F} \tau_j)$. Furthermore $\sigma \subseteq \tau$ as τ is an initial segment of g with larger domain. Now consider the binary string τ^* defined by inverting τ at bit $\langle i, x\rangle$. Then $\tau^* \supseteq \sigma$. Furthermore as $i \notin F$ we have that $\bigoplus_{j\in F} \tau_j = \bigoplus_{j\in F} \tau^*_j$ and $\tau^*_i(x) = 0$, i.e. $\tau^* \in W$, contradicting our choice of σ.

Now suppose that $A_1 = \Gamma(A_2 \oplus \bigoplus_{j\in F} g_j)$. Consider the set $W = \{\sigma \mid \exists x(x \notin A_1 \ \&\ x \in \Gamma(A_2 \oplus \bigoplus_{j\in F} \sigma_j))\}$. The statement "$x \notin A_1$" is a Π^0_1 statement relative to B. Thus the set W is $\Sigma^0_2(B)$. By our choice of g and our assumption it follows that no initial segment of g is in the set W. As g is 2-generic relative to B, there is an initial segment $\sigma \subseteq g$, which has no extension in W. It follows that we can enumerate A_1 by $\{x \mid \exists\tau \supseteq \sigma(x \in \Gamma(A_2 \oplus \bigoplus_{j\in F} \tau_j))\}$, i.e. $A_1 \leq_e A_2$.

(2) Suppose that $\Gamma(A \oplus g_i) = \Lambda(A \oplus g_j)$. Consider the set $W = \{\sigma \mid \exists x(x \in \Gamma(A \oplus \sigma_i) \ \&\ \forall\tau \supseteq \sigma(x \notin \Lambda(A \oplus \tau_j)))\}$. The set W is once again $\Sigma^0_2(B)$. Hence by our assumptions no initial segment of g is in W. Hence there is an initial segment $\sigma \subseteq g$, which has no extension in W. Now let $U = \{x \mid \exists\tau \supseteq \sigma(x \in \Gamma(A \oplus \tau_i))\}$. We claim that $\Lambda(A \oplus g_j) = U \leq_e A$. Clearly $\Lambda(A \oplus g_j)$ is a subset of U. Towards a contradiction suppose that x is in $U \setminus \Lambda(A \oplus g_j)$. The statement " $x \notin \Lambda(A \oplus g_j)$" is a $\Pi^0_1(B)$ statement about g which is forced by some initial segment of g, say μ. Furthermore this statement depends only on μ_j. Consider a finite binary string τ, such that $\tau \supseteq \sigma$ and $x \in \Gamma(A \oplus \tau_i)$. Then from μ and τ we can create a new extension σ^* of σ, which has $\sigma^*_i = \tau_i$ and $\sigma^*_j = \mu_j$. This extension would be an element of W, contradicting the choice of σ.

□

3. Effective Coding and Decoding theorems

Slaman and Woodin [16] showed that every countable relation on enumeration degrees can be uniformly defined in $\mathcal{D}_e$ using parameters. We will need to know a precise bound on these parameters in terms of a *presentation* of the given relation. The following notion was introduced in [6].

Definition 4. *Let $\mathcal{R} \subseteq \mathcal{D}_e^n$ be a countable relation on enumeration degrees. We will say that a relation $\mathcal{R}$ is e-presented beneath a set A if there is a set $W \leq_e A$ such that $\mathcal{R} = \{(\mathbf{d}_e(W_{i_1}(A)), \dots, \mathbf{d}_e(W_{i_n}(A))) \mid (i_1, \dots, i_n) \in W\}$. W will be called a presentation of $\mathcal{R}$.*

Theorem 3 (Effective Coding Theorem)**.** *For every n there is a formula φ_n, such that for every countable relation on enumeration degrees $\mathcal{R} \subseteq \mathcal{D}_e^n$ which is e-presented beneath R there are parameters $\bar{\mathbf{p}}$ below $\mathbf{d}_e(R)''$ such that*

$$\mathcal{R} = \{(\mathbf{x}_1, \dots, \mathbf{x}_n) \mid \mathcal{D}_e \models \varphi_n(\mathbf{x}_1, \dots, \mathbf{x}_n, \bar{\mathbf{p}})\}.$$

Proof. We will assume that R is total. If not by Theorem 1 we can replace R by a total set above it with the same jump, noting that if $\mathcal{R}$ is e-presented beneath a set R then it is e-presented beneath any other set, enumeration above the original. The proof of the Coding Theorem usually goes through the following three steps:

Step 1: Coding antichains. Slaman and Woodin [16] show that antichains, which are uniformly enumeration reducible to a low set, can be coded by parameters below $\mathbf{0}_e{}'$. More precisely they show that for every antichain $\mathcal{A}$ uniformly bounded by a member of a low degree $\mathbf{a}$, there are parameters $\mathbf{p}_1$ and $\mathbf{p}_2$, such that $\mathbf{x} \in \mathcal{A}$ if and only if $\mathbf{x} \leq_e \mathbf{a}$ and $\mathbf{x}$ is minimal among the e-degrees with the property $\exists\mathbf{y}(\mathbf{y} \leq (\mathbf{x} \vee \mathbf{p}_1) \ \&\ \mathbf{y} \leq (\mathbf{x} \vee \mathbf{p}_2) \ \&\ \mathbf{y} \not\leq \mathbf{x})$. Hence relativizing this theorem, given any total set f we can code antichains, which are uniformly reducible to a set that is low with respect to f by parameters below f'.

Step 2: Finding a useful antichain. A useful antichain $\{g_i\}_{i<\omega}$ is one with the following property: for every $A, B \leq_e R$ and $k \in \mathbb{N}$, if $A \oplus g_i \leq_e B \oplus g_{j_1} \oplus \dots \oplus g_{j_k}$ then $A \leq_e B$ and $i \in \{j_1, \dots, j_k\}$. By Proposition 2 a 2-generic with respect to R

function g will provide us with a useful antichain. Furthermore by Proposition 1, such a g can be obtained so that $g \leq_e R''$ and $(g \oplus R)' \leq_e g \oplus R' \leq_e R''$. Thus by Step 1 we can code antichains uniformly reducible to $g \oplus R$ with parameters below R''.

Step 3: Representing $\mathcal{R}$ using antichains. For an arbitrary $i \in \mathbb{N}$ denote by $Div(i)$ the divisor and by $Rem(i)$ the remainder resulting from the division of i by n. We use the useful antichain $\mathcal{G} = \{\mathbf{g}_i\}$ obtained from a 2-generic with respect to R set g to code the following three kinds of antichains:

(1) We divide $\mathcal{G}$ into n antichains: for $1 \leq k \leq n$ denote by $\mathcal{G}(k)$ the antichain $\mathcal{G}(k) = \{\mathbf{g}_i \mid Rem(i) = k\}$. $\mathcal{G}(k)$ is an antichain, uniformly below g, hence definable by parameters below R''.

(2) We represent the k-th projection of $\mathcal{R}$, $\mathcal{R}(k)$, by an antichain: denote by $W(k)$ the set $\{i \mid \exists i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_n((i_1, \dots, i_{k-1}, i, i_{k+1}, \dots, i_n) \in W)\}$. Then by the properties of the 2-generic function g, the set $G(k) + R(k) = \{g_i \oplus W_j(R) \mid Div(i) = j \in W(k) \;\&\; Rem(i) = k\}$ is an antichain, uniformly bounded by $g \oplus R$. Hence the antichain $\mathcal{G}(k) + \mathcal{R}(k) = \{\mathbf{x} \oplus \mathbf{y} \mid \mathbf{x} \in \mathcal{G}(k) \;\&\; \mathbf{y} \in \mathcal{R}(k)\}$ is definable by parameters below R''.

(3) Finally consider the set $G_W = \{g_{i_1} \oplus \cdots \oplus g_{i_n} \mid Rem(i_k) = k$, for $1 \leq k \leq n$ and $(Div(i_1), \dots, Div(i_n)) \in W\}$. This is also easily seen to be an antichain uniformly below $g \oplus R$. Hence $\mathcal{G}_W = \{\mathbf{d}_e(Y) \mid Y \in G_W\}$ is also definable by parameters below R''.

The usefulness of the antichain $\mathcal{G}$ thus allows us to code the relation $\mathcal{R}$ using the three types of antichains defined above as follows: $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathcal{R} \leftrightarrow \exists \mathbf{y}_1 \dots \exists \mathbf{y}_n (\mathbf{x}_1 \vee \mathbf{y}_1 \in \mathcal{G}(1) + \mathcal{R}(1) \;\&\; \dots \;\&\; \mathbf{x}_n \vee \mathbf{y}_n \in \mathcal{G}(n) + \mathcal{R}(n) \;\&\; \mathbf{y}_1 \vee \cdots \vee \mathbf{y}_n \in \mathcal{G}_W)$. □

Theorem 4. *(Decoding Theorem) Let $\mathcal{R}$ be a countable relation on enumeration degrees, which is coded by parameters $\bar{\mathbf{p}}$. Let $\mathbf{d}_e(P)$ be an upper bound on these parameters. Then there is a presentation W of the relation $\mathcal{R}$, such that $W \leq_e P^5$.*

Proof. This proof follows Slaman and Woodin's proof of the Decoding Theorem [17]. We will calculate the complexity of the presentation of $\mathcal{R}$ given by the set W:

$$W = \{(i_1, \dots, i_n) \mid (\mathbf{d}_e(W_{i_1}(P)), \dots, \mathbf{d}_e(W_{i_n}(P))) \in \mathcal{R}\}.$$

By using the same trick as in the previous theorem, we may assume again that P is a total set. The relation $i \leq_e j$ defined by $W_i(P) \leq_e W_j(P)$ is $\Sigma^0_3(P)$. The rest of the proof is an easy calculation. We will use $i \oplus j$ to express $W_i(P) \oplus W_j(P)$. Note that the index k of an operator W_k, such that $W_k(P) = W_i(P) \oplus W_j(P)$ is obtained effectively from i and j. Consider the relation $AC(i, a, p_1, p_2)$, expressing the fact that $\mathbf{d}_e(W_i(P))$ is in the antichain coded by $\mathbf{d}_e(W_a(P)), \mathbf{d}_e(W_{p_1}(P))$ and $\mathbf{d}_e(W_{p_2}(P))$. We need to say that $i \leq_e a$ and $\exists y$ such that $y \leq_e i \oplus p_1$, $y \leq i \oplus p_2$ and $y \not\leq_e i$, a $\Sigma^0_4(P)$ statement. Then we need to say that i is minimal with this property, a $\Pi^0_4(P)$ statement. $AC(i, a, p_1, p_2)$ is a $\Delta^0_5(P)$ statement.

For each of the antichains $\mathcal{G}(k)$, $\mathcal{G}(k) + \mathcal{R}(k)$, where $1 \leq k \leq n$, and $\mathcal{G}_W$ we know the indices of the operators, which applied to P, produce representatives of coding parameters. Denote these by $\overline{p}_{\mathcal{G}(k)}$, $\overline{p}_{\mathcal{G}(k)+\mathcal{R}(k)}$ and $\overline{p}_{\mathcal{G}_W}$ respectively. Thus to express that $(i_1, \dots, i_n)$ is in the set W, we need to say that there exist $y_1 \dots, y_n$, such that the conjunction of the $AC(y_k, \overline{p}_{\mathcal{G}(k)})$, $AC(i_k \oplus y_k, \overline{p}_{G(K)+\mathcal{R}(k)})$, where $1 \leq k \leq n$, and $AC(y_1 \oplus \cdots \oplus y_n, \overline{p}_{\mathcal{G}_W})$ is true. Therefore W is $\Sigma^0_5(P)$. By

McEvoy [10] this yields $W \leq_e (P \oplus \overline{P})^5$. By our assumption that P is total, hence enumeration equivalent to $P \oplus \overline{P}$, $W \leq_e P^5$. □

The set obtained by the Decoding Theorem is the most complicated possible presentation of the relation $\mathcal{R}$. Notice that for a one-place relation $\mathcal{R}$ to be e-presented beneath a set A we only require that there is set $W \leq_e A$ which contains for every member of the relation $\mathcal{R}$ at least one index of an operator, which applied to A produces a set in the enumeration degree of that member. The set W obtained by the Decoding Theorem contains for every member of $\mathcal{R}$ all possible indices of operators which applied to A produce a set in the e-degree of that member.

We will use the Coding Theorem and Decoding Theorem to transfer information through an automorphism of $\mathcal{D}_e$. Here are two examples of applications of these theorems.

Example 1: In some cases we can find a more efficient bound on the coding parameters. This efficiency is given by optimizing Step 2 from the proof above. Instead of using a generic set to obtain a useful antichain we use an antichain which is itself a $\mathcal{K}$-system, a countable set of sets of natural numbers, such that every pair of distinct elements in it is a $\mathcal{K}$-pair. In [5] it is shown that a $\mathcal{K}$-system can be obtained uniformly below any nonzero Δ^0_2 set.

Let X be any set of natural numbers. Consider the structure $\langle \mathbb{N}, 0, s, +, *, X\rangle$, the standard model of arithmetic with one additional predicate for membership in the set X. In [6] it is shown that a standard model of arithmetic can be represented by a combination of countable relations of enumeration degrees below $\mathbf{0}_e{}'$, which can furthermore be coded by parameters below $\mathbf{0}_e{}'$. The domain of this standard model is an antichain, uniformly bounded by a low set A, which is half of a Δ^0_2 nontrivial $\mathcal{K}$-pair $\{A, B\}$. The pair $\{A \oplus X, B \oplus X\}$ is a $\mathcal{K}_X$-pair. We may assume that it is a nontrivial $\mathcal{K}_X$-pair or that $A \leq_e X$, as by symmetry we can use a model coded below A or a model coded below B. In both cases $(A \oplus X)' \equiv_e X'$. In the first case by the properties of jumps of nontrivial $\mathcal{K}_X$-pairs $X' \leq_e (A \oplus X)' \leq_e B \oplus \overline{B} \oplus X' \leq_e \emptyset' \oplus X' \equiv_e X'$. In the second case $A \oplus X \equiv_e X$.

Denote by $\hat{\mathbf{n}}$ the enumeration degree which serves as a representative of the natural number n in the coded model. Thus in order to represent $\langle \mathbb{N}, 0, s, +, *, X\rangle$, we only need to code additionally one more antichain: $\{\hat{\mathbf{n}} \mid n \in X\}$. This antichain is uniformly bounded by the set $A \oplus X$, a set which is low over X, so applying Step 1 of the Effective Coding Theorem, relative to a total set f, such that $X \leq_e f$ and $X' \equiv_e f'$, given by Theorem 1, we obtain parameters which code this antichain bounded by X'.

If on the other hand $\langle \mathbb{N}, 0, s, +, *, X\rangle$ is coded by parameters bounded by the degree of a set P then by the Decoding Theorem we have representations $W_0 = \{i \mid \mathbf{d}_e(W_i(P)) = \hat{\mathbf{0}}\}$, $W_s = \{(i,j) \mid \hat{s}(\mathbf{d}_e(W_i(P))) = \mathbf{d}_e(W_j(P))\}$ and $W_X = \{i \mid \mathbf{d}_e(W_i(P)) = \hat{\mathbf{n}} \ \&\ n \in X\}$ and they are all enumeration reducible to P^5. Then the set X is also enumeration reducible to P^5, namely $X = \{n \mid \exists \alpha \in \omega^{<\omega}(|\alpha| = n+1 \ \&\ \alpha(0) \in W_0 \ \&\ \forall k < n((\alpha(k), \alpha(k+1)) \in W_s) \ \&\ \alpha(n) \in W_X)\}$.

Example 2: In other cases we will be content with obtaining an arithmetical bound to the coding parameters. Let $\mathcal{C}$ be a countable set of enumeration degrees, e-presented beneath C via W. The aim of this example is to code a counting of $\mathcal{C}$, i.e. a structure of the form $\langle \mathbb{N}, 0, s, +, *, \mathcal{C}, \psi\rangle$ consisting of a standard model of arithmetic, the countable set of degrees $\mathcal{C}$ and some bijection $\psi : \mathbb{N} \to \mathcal{C}$ below

C^{n_c} for some fixed number n_c. To code such a structure again we have parameters below $\mathbf{0}_e{}'$ coding the same standard model of arithmetic as in *Example 1*, with domain an antichain, $\{\hat{\mathbf{n}}\}_{n<\omega}$, uniformly bounded by a low set A, which is half of a $\mathcal{K}$-pair. By the coding lemma, we can code $\mathcal{C}$ with parameters below C''. To define a counting of $\mathcal{C}$, which is e-presented arithmetically in C we need to define a function $f : \mathbb{N} \to W$, such that the range of f is a still a presentation of $\mathcal{C}$ and such that for all $i \neq j$, $W_{f(i)}(C) \not\equiv_e W_{f(j)}(C)$. Such a function can be defined arithmetically in C, in fact below C^3. Finally we code the two-place relation $\psi = \{(\mathbf{d}_e(W_{f(i)}(C)), \hat{\mathbf{i}}) \mid i < \omega\}$, which is e-presented beneath $A \oplus C^3$. By the coding lemma we can find parameters below $(A \oplus C^3)''$ which code this relation. As $A \oplus C^3$ is low over C^3, $(A \oplus C^3)'' \equiv_e C^5$. Thus there are parameters that are below C^5 and code this structure.

Suppose we have two such structures,

$$\langle \mathbb{N}_1, 0_1, s_1, +_1, *_1, \mathcal{C}_1, \psi_1\rangle \text{ and } \langle \mathbb{N}_2, 0_2, s_2, +_2, *_2, \mathcal{C}_2, \psi_2\rangle,$$

both coded by parameters below P. Then the relation $\mathcal{C}_1 \to \mathcal{C}_2 = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in \mathcal{C}_1 \ \&\ \mathbf{y} \in \mathcal{C}_2 \ \&\ \psi_1^{-1}(\mathbf{x}) = \psi_2^{-1}(\mathbf{y})\}$ is arithmetically presented relative to P. Let $W_{0_i}, W_{s_i}, W_{C_i}$ and W_{ψ_i} for $i = 1, 2$ be $\Sigma^0_5(P)$ sets representing the corresponding relations, coded by parameters below P. Then $\mathcal{C}_1 \to \mathcal{C}_2$ is presented by the set $\{(i, j) \mid \exists n \exists \alpha_1, \alpha_2 \in \omega^{<\omega}(|\alpha_1| = |\alpha_1| = n + 1 \ \&\ \alpha_1(0) \in W_{0_1} \ \&\ \alpha_2(0) \in W_{0_2} \ \&\ \forall k < n((\alpha_1(k), \alpha_1(k+1)) \in W_{s_1} \ \&\ (\alpha_2(k), \alpha_2(k+1)) \in W_{s_2}) \ \&\ (\alpha_1(n), i) \in W_{\psi_1} \ \&\ (\alpha_2(n), j) \in W_{\psi_2})\}$. Hence it is presented by parameters below P^5.

4. Jump ideals in $\mathcal{D}_e$

Definition 5. *A set of enumeration degrees $\mathcal{I} \subseteq \mathcal{D}_e$ is a jump ideal if it is downwards closed, closed under least upper bound and closed under the jump operation.*

The first order definition of the enumeration jump for nonzero degrees $\mathbf{u}$ proved in [3] is given by the formula $\varphi_{\mathcal{J}}(\mathbf{u}, \mathbf{u}')$ expressing that $\mathbf{u}'$ is the maximal element in the set $J_{\mathbf{u}} = \{\mathbf{v} \mid \exists \mathbf{a}, \mathbf{b}(\mathcal{K}(\mathbf{a}, \mathbf{b}) \ \&\ \mathbf{a} \leq \mathbf{u} \ \&\ \mathbf{v} = \mathbf{a} \vee \mathbf{b})\}$. This definition turns out to be absolute between jump ideals in $\mathcal{D}_e$.

Theorem 5. *Let $\mathcal{I} \subseteq \mathcal{D}_e$ be a jump ideal. For every element $\mathbf{u} \in \mathcal{I}$ we have the following equivalence:*

$$\mathcal{I} \models \varphi_{\mathcal{J}}(\mathbf{u}, \mathbf{u}') \leftrightarrow \mathcal{D}_e \models \varphi_{\mathcal{J}}(\mathbf{u}, \mathbf{u}').$$

Proof. First we observe that the formula defining $\mathcal{K}$-pairs is absolute. If $\{\mathbf{a}, \mathbf{b}\} \subseteq \mathcal{I}$ is a pair of enumeration degrees such that $\mathcal{D}_e \models \forall \mathbf{x}((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x})$, i.e. a $\mathcal{K}$-pair in $\mathcal{D}_e$, then as $\mathcal{I}$ is an ideal $\mathcal{I} \models \forall \mathbf{x}((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x})$ as well. Suppose that $\{\mathbf{a}, \mathbf{b}\} \subseteq \mathcal{I}$ is not a $\mathcal{K}$-pair. Kalimullin [8] has shown that there are witnesses $\mathbf{x}, \mathbf{y} \leq_e \mathbf{a}' \vee \mathbf{b}'$ for this fact, i.e. $\mathcal{D}_e \models \mathbf{y} \leq \mathbf{a} \vee \mathbf{x} \ \&\ \mathbf{y} \leq \mathbf{b} \vee \mathbf{x} \ \&\ \mathbf{y} \not\leq \mathbf{x}$. As $\mathcal{I}$ is a jump ideal and $\mathbf{a}, \mathbf{b} \in \mathcal{I}$, it follows that $\mathbf{x}, \mathbf{y} \in \mathcal{I}$ and $\mathcal{I} \models \mathbf{y} \leq \mathbf{a} \vee \mathbf{x} \ \&\ \mathbf{y} \leq \mathbf{b} \vee \mathbf{x} \ \&\ \mathbf{y} \not\leq \mathbf{x}$. Thus $\{\mathbf{a}, \mathbf{b}\} \subseteq \mathcal{I}$ is a $\mathcal{K}$-pair in $\mathcal{D}_e$ if and only if $\mathcal{I} \models \mathcal{K}(\mathbf{a}, \mathbf{b})$.

To complete the proof we observe that if $\mathbf{u} \in \mathcal{I}$ then $J_{\mathbf{u}} \subseteq \mathcal{I}$, as $J_{\mathbf{u}}$ is contained in the degrees bounded by $\mathbf{u}'$. The maximal element of the set $J_{\mathbf{u}}$ is the same in $\mathcal{I}$ and in $\mathcal{D}_e$. □

Theorem 6. *Let $\mathcal{I}$ and $\mathcal{J}$ be jump ideals in $\mathcal{D}_e$, such that $\mathcal{I} \subseteq \mathcal{J}$. Let $\rho : \mathcal{J} \to \mathcal{J}$ be an automorphism of $\mathcal{J}$. Then $\rho \restriction \mathcal{I}$ is an automorphism of $\mathcal{I}$. Furthermore there is a fixed number n_a, such that if $\mathcal{I}$ is countable and e-presented beneath I and $I \in \mathcal{J}$ then $\rho \restriction \mathcal{I}$ is e-presented beneath I^{n_a}.*

Proof. To prove the first statement we use *Example 1* from the previous section. Fix $\mathbf{x} \in \mathcal{J}$, $X \in \mathbf{x}$ and $Y \in \rho(\mathbf{x})$. The structure $\langle \mathbb{N}, 0, s, +, *, Y\rangle$ can be coded by parameters $\bar{\mathbf{p}}$ below $\rho(\mathbf{x})'$. By the definability of the jump in $\mathcal{J}$, $\rho(\mathbf{x})' = \rho(\mathbf{x}')$. As ρ is an automorphism of $\mathcal{J}$, $\rho^{-1}(\bar{\mathbf{p}})$ are parameters which code the same structure and $\rho^{-1}(\bar{\mathbf{p}}) \leq \mathbf{x}'$. It follows that $Y \leq (X')^5 = X^6$ and hence if $\mathbf{x} \in \mathcal{I}$ then $\rho(\mathbf{x}) = \mathbf{d}_e(Y) \in \mathcal{I}$. That $\rho^{-1}(\mathbf{x})$ is bounded by $\mathbf{x}^6$ is proved by the same argument, using the automorphism ρ^{-1}.

Now suppose that $\mathcal{I}$ is countable and e-presented beneath I. Then by our second example there is a counting of $\mathcal{I}$, $\langle \mathbb{N}_1, 0_1, s_1, +_1, *_1, \mathcal{I}, \psi_1\rangle$ which is coded by parameters $\bar{\mathbf{p}}$, bounded by I^{n_c} and hence members of $\mathcal{J}$. Then $\rho(\bar{\mathbf{p}})$ must code a counting of $\rho(\mathcal{I})$, $\langle \mathbb{N}_2, 0_2, s_2, +_2, *_2, \rho(\mathcal{I}), \psi_2\rangle$. Furthermore for every natural number n, $\rho(\hat{\mathbf{n}}_\mathbf{1}) = \hat{\mathbf{n}}_\mathbf{2}$, i.e. the element representing the natural number n in the first structure is mapped by ρ to the element representing n in the second structure. Hence if $\hat{\mathbf{n}}_\mathbf{1}$, is mapped by ψ_1 to $\mathbf{x}$, then $\hat{\mathbf{n}}_\mathbf{2}$ is mapped by ψ_2 to $\rho(\mathbf{x})$. Now $\rho(\bar{\mathbf{p}})$ is bounded by $\rho(\mathbf{d}_e(I^{n_c}))$. As we have already seen in the previous paragraph $\rho(\mathbf{d}_e(I^{n_c}))$ is bounded by $(I^{n_c})^6 = I^{n_c+6}$. Thus I^{n_c+6} is a bound on both $\bar{\mathbf{p}}$ and $\rho(\bar{\mathbf{p}})$. It follows that the relation $\mathcal{I} \to \rho(\mathcal{I}) = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in \mathcal{I} \ \& \ \mathbf{y} \in \rho(\mathcal{I}) \ \& \ \psi_1^{-1}(\mathbf{x}) = \psi_2^{-1}(\mathbf{y})\}$ is e-presented beneath $(I^{n_c+6})^5 = I^{n_c+11}$. But this relation is precisely the graph of $\rho \upharpoonright \mathcal{I}$. Thus $n_a = n_c + 11$. □

Definition 6. *(Slaman and Woodin* [17]*) Let $\mathcal{I} \subseteq \mathcal{D}_e$ be countable jump ideal. An automorphism $\rho : \mathcal{I} \to \mathcal{I}$ is called persistent if for every $\mathbf{x} \in \mathcal{D}_e$ there is a countable jump ideal $\mathcal{J}$ and an automorphism $\rho_1 : \mathcal{J} \to \mathcal{J}$ such that $\{\mathbf{x}\} \cup \mathcal{I} \subseteq \mathcal{J}$ and $\rho_1 \upharpoonright \mathcal{I} = \rho$.*

Let $\mathcal{I}$ be a countable jump ideal e-presented beneath I. By Theorem 6 every persistent automorphism ρ of I is arithmetically presented relative to I. Indeed, let ρ_1 be an extension of ρ to an automorphism of a countable jump ideal $\mathcal{J}$ containing I. Then $\rho_1 \upharpoonright \mathcal{I} = \rho$ is e-presented beneath I^{n_a}. Theorem 6 furthermore shows that every automorphism π of $\mathcal{D}_e$, restricted to a countable jump ideal $\mathcal{I}$, is a persistent automorphism of $\mathcal{I}$. We will be able to show eventually that the opposite is true as well, every persistent automorphism of a countable ideal can be extended to a global automorphism. The first step is given by the following.

Theorem 7. *Let $\mathcal{I} \subseteq \mathcal{J}$ be countable jump ideals in $\mathcal{D}_e$. Every persistent automorphism of $\mathcal{I}$ can be extended to a persistent automorphism of $\mathcal{J}$.*

Proof. The proof follows [17]. Towards a contradiction suppose that ρ is a persistent automorphism of $\mathcal{I}$, which cannot be extended to a persistent automorphism of $\mathcal{J}$. Fix J to be a set such that $\mathcal{J}$ is e-presented beneath J. By our assumption for every automorphism ρ_i of $\mathcal{J}$ which extends ρ and is arithmetically presented relative to J there is an $\mathbf{x}_i \in \mathcal{D}_e$ such that ρ_i does not extend to an automorphism of a countable jump ideal containing $\mathbf{x}_i$. There are countably many arithmetic presentations relative to J. Fix $\mathbf{x}$ to be an enumeration degree larger than all such $\mathbf{x}_i$. Then by the persistence of ρ we can extend it to an automorphism ρ_1 of a countable jump ideal containing $\mathbf{x} \vee \mathbf{d}_e(J)$. Then $\rho_1 \upharpoonright \mathcal{J}$ is an automorphism of $\mathcal{J}$, with an arithmetic in J presentation, which extends to an automorphism of a countable jump ideal containing all $\mathbf{x}_i$, contradicting our choice of $\mathbf{x}_i$. □

Theorem 8. *The statement "There exists a countable jump ideal $\mathcal{I} \subseteq \mathcal{D}_e$ and a persistent automorphism $\rho : \mathcal{I} \to \mathcal{I}$ which is not the identity on $\mathcal{I}$." is a Σ^1_2 statement.*

Proof. Suppose that I and R, i and r are given. To say that $W_i(I)$ is a presentation of a jump ideal $\mathcal{I}$ is a statement arithmetic in I. We need to say that $\mathcal{I}$ is downwards closed, closed under least upper bound and closed under jump. To express that the ideal is closed under jump, for example, we need to say that for every $x \in W_i(I)$ there exists a $y \in W_i(I)$ such that $(W_x(I))' \equiv_e W_y(I)$. The last statement is clearly arithmetic in I. Similarly to say that $W_r(R)$ is an automorphism ρ of $\mathcal{I}$, which is not the identity, is arithmetic in I and R.

To say that ρ is a persistent automorphism it is enough to say that for every J and every j, if $W_j(J)$ is a presentation of a jump ideal $\mathcal{J}$ beneath J, which contains $\mathcal{I}$ as a subset, there exists r_1, such that $W_{r_1}(J^{n_a})$ is a presentation of an automorphism ρ_1 of $\mathcal{J}$, which in turn extends ρ. Indeed if ρ is a persistent automorphism of $\mathcal{I}$, then for every J, ρ will extend to an automorphism ρ_1 of a jump ideal containing $\mathbf{d}_e(J)$. If $\mathcal{J}$ is a jump ideal e-presented beneath J then by Theorem 6 the restriction of ρ_1 to $\mathcal{J}$ is an automorphism of $\mathcal{J}$, e-presented beneath J^{n_a}. If on the other hand the statement is true of I and R then ρ is persistent, as for every $\mathbf{x}$ there is a countable jump ideal $\mathcal{J}$ containing $\mathbf{x}$ and $\mathcal{I}$, namely the closure of $\mathcal{I} \cup \{\mathbf{x}\}$ under $\leq$, $\vee$ and $'$. This countable ideal is e-presented beneath a set J, e.g. we can list representatives of the countably many elements of $\mathcal{J}$ and set J to be their uniform upper bound.

Pasting all of this together, the statement can be restated as $\exists \mathcal{I} \exists R \forall J$, followed by an arithmetic statement involving I, R and J. □

5. Generic Extensions

In this section we connect the notion of a persistent automorphism of a countable jump ideal in $\mathcal{D}_e$ with the notion of a global automorphism of $\mathcal{D}_e$. For this we will need to use notions from set theory and refer to Kunen [9] for definitions and notation. The statements in this section and their proofs are identical to the ones found in the original analysis by Slaman and Woodin [17]. We outline the main steps below.

We will be working with well-founded ω-models of the fragment of ZFC which includes only instances of replacement and comprehension where the defining formulas are Σ_1. We denote this fragment by T. The precise definition of T is not important, as long as it is finitely axiomatizable and sufficiently powerful to let us carry out the finitely many proofs involved in this analysis. If $\mathcal{M} = \langle M, \in^{\mathcal{M}} \rangle$ is a model of T then in it we can define $\mathbb{N}^{\mathcal{M}}$, the standard model of arithmetic in the sense of $\mathcal{M}$ and $\mathcal{D}_e^{\mathcal{M}}$, the enumeration degrees in the sense of $\mathcal{M}$. The model is well-founded if $\in^{\mathcal{M}}$ is a well founded binary relation (with no infinite descending sequences), and it is an ω-model if $\mathbb{N}^{\mathcal{M}}$ is isomorphic to the standard model of arithmetic. By the Mostowski Collapse Lemma (see [9]) any such model is isomorphic to a standard model $\langle \mathcal{M}', \in \rangle$, i.e. a model in which $\in^{\mathcal{M}'} = \in$. In this model $\mathbb{N}^{\mathcal{M}'} = \mathbb{N}$ and $\mathcal{D}_e^{\mathcal{M}'}$ is a jump ideal in $\mathcal{D}_e$.

Proposition 3. *Let $\mathcal{M} \subseteq \mathcal{N}$ be standard ω-models of T and $\mathcal{I} \in M$ and $\rho \in N$. If $\mathcal{M} \models$ ($\mathcal{I} \subseteq \mathcal{D}_e$ is a countable jump ideal) and $\mathcal{N} \models$ (ρ is a persistent automorphism of $\mathcal{I}$) then $\rho \in M$ and $\mathcal{M} \models$ (ρ is a persistent automorphism of $\mathcal{I}$).*

Proof idea: We work inside the model $\mathcal{N}$. All statements about $\mathcal{D}_e$ discussed in the previous paragraphs are true of $\mathcal{D}_e^{\mathcal{N}}$ in $\mathcal{N}$. $\mathcal{M}$, an inner model of $\mathcal{N}$, knows that $\mathcal{I}$ is a countable jump ideal, hence there is a set $I \in M$ such that $\mathcal{I}$ is e-presented beneath

I. Every persistent automorphism ρ of $\mathcal{I}$ is arithmetically presented relative to I. As $\mathcal{M}$ is a standard ω-model of T, any relation which is e-presented beneath an element of M is also a member of M. Finally if $\mathbf{x} \in \mathcal{D}_e^{\mathcal{M}}$ then in $\mathcal{M}$ there is a countable jump ideal $\mathcal{J} \supseteq \mathcal{I} \cup \{\mathbf{x}\}$. By the persistence of ρ in $\mathcal{N}$, ρ can be extended to an automorphism ρ_1 of $\mathcal{J}$. So we are in the initial position: $\mathcal{M} \models (\mathcal{J} \subseteq \mathcal{D}_e$ is a countable jump ideal), and ρ_1 is a persistent automorphism of $\mathcal{J}$ in $\mathcal{N}$. It follows that $\rho_1 \in \mathcal{M}$, thus in $\mathcal{M}$ as well ρ is persistent.

Definition 7 (Slaman and Woodin [17])**.** *Let $\mathcal{I} \subseteq \mathcal{D}_e$ be a jump ideal and $\rho : \mathcal{I} \to \mathcal{I}$ be an automorphism. ρ is generically persistent if there is a generic extension $V[G]$ of V in which $\mathcal{I}$ is countable and ρ is persistent.*

Proposition 4. *If ρ is a generically persistent automorphism of the jump ideal $\mathcal{I} \subseteq \mathcal{D}_e$, then ρ is persistent in every generic extension of V in which $\mathcal{I}$ is countable.*

Proof idea: By Theorem 8 the property "$W_r(R)$ is a presentation of a persistent automorphism of the jump ideal represented by I" is Π^1_1 and thus by Shoenfield's absoluteness lemma [14] absolute between well founded models of ZFC. Assume that there are two forcing partial orders P and P_1 in V and two conditions: $p \in P$, forcing $\mathcal{I}$ to be countable and ρ to be persistent and $p_1 \in P_1$, forcing $\mathcal{I}$ to be countable. Consider a generic extension $V[H]$, in which the powersets of both partial orders P and P_1 are countable. Then in $V[H]$ we can construct two generic filters: G, containing p, generic with respect to all of the countably many dense sets in P, and G_1 containing p_1, generic with respect to all of the countably many dense sets in P_1. Now $V[G]$ and $V[G_1]$ are inner models of $V[H]$. All of these models are standard ω-models of T. In $V[G]$ $\mathcal{I}$ is countable and ρ is persistent. Let I be such that $\mathcal{I}$ is e-presented beneath $\mathcal{I}$ and R be such that ρ is e-presented beneath R in $V[G]$. Then there is an r, such that $W_r(R)$ is a presentation of ρ in $V[G]$, hence in $V[H]$. In $V[G]$ it is true that $W_r(R)$ is a presentation of a persistent automorphism of the jump ideal represented by I, a statement which is absolute between $V[G]$ and $V[H]$, hence in $V[H]$ as well ρ is persistent. Now since $\mathcal{I}$ is countable in $V[G_1]$, by Proposition 3 we have that $\rho \in V[G_1]$ and $V[G_1] \models (\rho$ is persistent).

Theorem 9. *If π is an automorphism of $\mathcal{D}_e$, then π is generically persistent.*

Proof idea: Let λ be an ordinal such that $\pi \in V_\lambda$ and V_λ is a model of T. Consider a countable structure $\mathcal{H}$, elementary equivalent to V_λ with respect to the language containing a constant for π. Let $\mathcal{M}$ be its transitive collapse. Now $\mathcal{M}$ is a countable standard ω-model of T, such that $\mathcal{D}_e^{\mathcal{M}}$ is a countable jump ideal in $\mathcal{D}_e$ and $\pi^{\mathcal{M}}$ is the restriction of π to $\mathcal{D}_e^{\mathcal{M}}$. Furthermore $\mathcal{M}$ is elementary equivalent to V_λ. By the properties of forcing and truth, the statement (π is generically persistent) can be expressed by a closed statement in the language containing π as a constant. Hence $\pi^{\mathcal{M}}$ is generically persistent in $\mathcal{M}$ if and only if π is generically persistent in V_λ. Let $P \in \mathcal{M}$ be the partial order to generically add a counting of $\mathcal{D}_e^{\mathcal{M}}$ to $\mathcal{M}$ and fix a generic filter G with respect to P. $\mathcal{M}[G]$ is a model of ($\mathcal{D}_e^{\mathcal{M}}$ is countable). The restriction of π to a countable jump ideal is a persistent automorphism of that ideal. Hence by Proposition 3, $\mathcal{M}[G]$ must be a model of ($\pi^{\mathcal{M}}$ is persistent). By Proposition 4 $\pi^{\mathcal{M}}$ is generically persistent and hence π is generically persistent as well.

Theorem 10. *Suppose that $V[G]$ is a generic extension of V. Suppose that $\pi \in V[G]$ is an automorphism of the enumeration degrees in V. If π is generically* *persistent in $V[G]$ then π is constructible from the powerset of the natural numbers in V (i.e. $\pi \in L(\mathbb{R})$). In particular every automorphism of the enumeration degrees in constructible from the powerset of the natural numbers in V.*

Proof idea: Consider the forcing partial order $P = Coll(\mathcal{D}_e^V, \omega)$ to add a generic counting H of $\mathcal{D}_e^V$ to $V[G]$. This forcing partial order consists of finite functions from $\mathcal{D}_e^V$ to ω, ordered by inclusion and is therefore a member of $L(\mathbb{R})$. In $V[G][H]$ π is persistent. Hence by Proposition 3 π is a member of every inner model of $V[G][H]$ in which $\mathcal{D}_e$ is countable. In particular $\pi \in L(\mathbb{R})[H]$. Thus for every P-generic H, $\pi \in L(\mathbb{R})[H]$. It follows that $\pi \in L(\mathbb{R})$.

Theorem 11. *Let $\mathcal{I}$ be a countable jump ideal in $\mathcal{D}_e$ and ρ a persistent automorphism of $\mathcal{I}$. Then ρ can be extended to an automorphism of $\mathcal{D}_e$.*

Proof idea: Let G be a generic counting of $\mathcal{D}_e$. Then in $V[G]$, $\mathcal{D}_e^V$ is a countable jump ideal and by Theorem 7, ρ can be extended in $V[G]$ to a persistent automorphism π of $\mathcal{D}_e^V$. Then π is persistent in $V[G]$. Every persistent automorphism is generically persistent. By Theorem 10, π is an element of $L(\mathbb{R})$, in particular π is in V.

Corollary 1. *The existence of a nontrivial automorphism of $\mathcal{D}_e$ is equivalent to a Σ^1_2 statement and therefore absolute between well-founded models of ZFC.*

Proof. By Theorem 8 the property "There exists countable jump ideal $\mathcal{I} \subseteq \mathcal{D}_e$ and a persistent automorphism $\rho : \mathcal{I} \to \mathcal{I}$ which is not the identity on $\mathcal{I}$ " is a Σ^1_2 property, hence absolute between well founded models of ZFC. By Theorem 11 this property is equivalent to the existence of a nontrivial automorphism of $\mathcal{D}_e$. □

Corollary 2. *If π is a nontrivial automorphism of $\mathcal{D}_e$ and $V[G]$ is a generic extension of V then π can be extended to an automorphism of $\mathcal{D}_e^{V[G]}$ in $V[G]$.*

Proof idea: Extend $V[G]$ to $V[G][H]$ by adding a counting of $\mathcal{D}_e^{V[G]}$. In $V[G][H]$ extend π to a persistent automorphism π_1 of $\mathcal{D}_e^{V[G]}$. By Theorem 10, $\pi_1 \in L(\mathbb{R}^{V[G]})$, hence $\pi_1 \in V[G]$.

6. Arithmetically representing automorphisms of $\mathcal{D}_e$

In this section we will show that every automorphism of $\mathcal{D}_e$ has an arithmetic presentation. It follows that there are at most countably many distinct members of $Aut(\mathcal{D}_e)$. For this we will need the following theorem proved by Soskov and Ganchev [20], building on a result of Richter [11].

Theorem 12 (Soskov, Ganchev [20])**.** *Every automorphism of $\mathcal{D}_e$ is the identity on the cone above ${\mathbf{0}_e}^4$.*

Proof. Richter [11] showed that if $\mathbf{a}$ and $\mathbf{b}$ are Turing degrees such that the structures $(\{\mathbf{x} \mid \mathbf{a} \leq_T \mathbf{x}\}, \leq_T, ')$ and $(\{\mathbf{x} \mid \mathbf{b} \leq_T \mathbf{x}\}, \leq_T, ')$ of the Turing cones above $\mathbf{a}$ and $\mathbf{b}$ in the language which includes the jump operation are isomorphic, then $\mathbf{a}^2 \leq_T \mathbf{b}^3$.

Suppose that π is an automorphism of $\mathcal{D}_e$. The set of total enumeration degrees above ${\mathbf{0}_e}'$ is first order definable, hence π restricted to the total enumeration degrees in the cone above ${\mathbf{0}_e}'$ induces an automorphism of the Turing degrees above $\mathbf{0}'_T$. Let $\mathbf{x}$ be an enumeration degree and let $\mathbf{y} = \pi(\mathbf{x})$. Then $\pi(\mathbf{x}') = \mathbf{y}'$ and hence the

Turing cone above $\iota^{-1}(\mathbf{x}')$ is isomorphic to the Turing cone above $\iota^{-1}(\mathbf{y}')$. Applying Richter's result and noting that ι preserves the jump operation, we get $\mathbf{x}^3 \leq \mathbf{y}^4$.

Now suppose that $\mathbf{c}$ is a total enumeration degree above ${\mathbf{0}_e}^4$. Then by Theorem 1 there exists a total enumeration degree $\mathbf{f}$ such that $\mathbf{c} = \mathbf{f}^4 = \mathbf{f} \vee {\mathbf{0}_e}^4$. Then $\pi(\mathbf{c}) = \pi(\mathbf{f} \vee {\mathbf{0}_e}^4) = \pi(\mathbf{f}) \vee {\mathbf{0}_e}^4 \leq \pi(\mathbf{f})^3 \vee {\mathbf{0}_e}^4 \leq_e \mathbf{f}^4 \vee {\mathbf{0}_e}^4 = \mathbf{f}^4 = \mathbf{c}$. Now consider the automorphism π^{-1} and note that by the definability of the total enumeration degrees above ${\mathbf{0}_e}'$, $\pi(\mathbf{c})$ must also be total and above ${\mathbf{0}_e}^4$. A symmetric argument gives the reverse inequality, hence $\pi(\mathbf{c}) = \mathbf{c}$.

Finally if $\mathbf{x}$ is an arbitrary degree above ${\mathbf{0}_e}^4$, by Theorem 2 there are total enumeration degrees $\mathbf{f}$ and $\mathbf{g}$, such that $\mathbf{x} = \mathbf{f} \wedge \mathbf{g}$. Applying the argument from the previous paragraph, we get that $\pi(\mathbf{x}) = \pi(\mathbf{f} \wedge \mathbf{g}) = \pi(\mathbf{f}) \wedge \pi(\mathbf{g}) = \mathbf{f} \wedge \mathbf{g} = \mathbf{x}$. □

Theorem 13. *Let π be an automorphism of $\mathcal{D}_e$. There exists an enumeration operator Γ such that for every 8-generic f, $\pi(\mathbf{d}_e(f)) = \mathbf{d}_e(\Gamma(f \oplus \emptyset^4))$.*

Proof. Let P_{ω_1} be the partial order to add ω_1 many Cohen reals (the set of finite functions $p : \omega_1 \times \omega \to \{0, 1\}$). Let $\mathcal{G}$ be generic with respect to P_{ω_1}. We can view $\mathcal{G}$ as an ω_1-sequence of mutually Cohen generic sets of natural numbers. By Corollary 2 we can extend the automorphism π to $\pi^* : \mathcal{D}_e^{V[\mathcal{G}]} \to \mathcal{D}_e^{V[\mathcal{G}]}$ and by Theorem 10 π^* is constructible from the reals $\mathbb{R}^{V[\mathcal{G}]}$ in the model $V[\mathcal{G}]$. This means that the relation π^* can be defined by a formula φ using parameters: $\mathbb{R}^{V[\mathcal{G}]}$, a parameter $X \in \mathbb{R}^{V[\mathcal{G}]}$ and an ordinal γ. First we "eliminate" the parameter X by moving to a model which has X as an element. As $X \in V[\mathcal{G}]$ it has a name τ_X in V and it is forced by some condition p in P_{ω_1} that:

1. $\tau_X \in 2^\omega$.
2. The relation $\{v \mid L_\gamma(\mathbb{R}^{V[\mathcal{G}]}) \models \varphi(v, \tau_X)\}$ is an automorphism of $\mathcal{D}_e^{V[\mathcal{G}]}$, which extends π.

The condition p, being a finite function $p : \omega_1 \times \omega \to \{0, 1\}$, can only specify facts about finitely many of the elements in the ω_1-sequence of reals $\mathcal{G}$, i.e. there is a countable ordinal β such that p specifies facts only about the first β-many elements of $\mathcal{G}$. By the homogeneity of Cohen forcing (see [17]) we can represent $V[\mathcal{G}]$ as $V[\mathcal{G}_1][\mathcal{G}_2]$, where $\mathcal{G}_1$ is the countable sequence of the first β-many elements in $\mathcal{G}$ and $\mathcal{G}_2$ is the rest of the sequence, which in turn can be viewed as an ω_1-sequence of reals, and at the same time as an P_{ω_1} generic filter over $V[\mathcal{G}_1]$. X is now an element of $V[\mathcal{G}_1]$ and we can replace it in the formula φ by a constant. Now in $V[\mathcal{G}_1]$ the empty condition forces the statement "the relation $\{v \mid L_\gamma(\mathbb{R}^{V[\mathcal{G}]}) \models \varphi(v, X)\}$ is an automorphism of $\mathcal{D}_e^{V[\mathcal{G}]}$, which extends π".

Let g be (the characteristic function of) any Cohen generic over $V[\mathcal{G}_1]$ real, which is an element of $V[\mathcal{G}]$. By the homogeneity of Cohen forcing we can further represent $V[\mathcal{G}] = V[\mathcal{G}_1][g][\mathcal{G}_3]$, where $\mathcal{G}_3$ is P_{ω_1}-generic over $V[\mathcal{G}_1][g]$.

By Soskov and Ganchev's Theorem 12 π^* is the identity on the cone above ${\mathbf{0}_e}^4$. Thus $\pi^*(\mathbf{d}_e(g)) \leq \pi^*(\mathbf{d}_e(g) \oplus {\mathbf{0}_e}^4) = \mathbf{d}_e(g) \oplus {\mathbf{0}_e}^4$. Let $\Pi^*(g) \in \pi^*(\mathbf{d}_e(g))$. There is an enumeration operator Γ, such that $\Pi^*(g) = \Gamma(g \oplus \emptyset^4)$. Thus $\Gamma(g \oplus \emptyset^4) \in \pi^*(\mathbf{d}_e(g))$ is true and hence must be forced in $V[\mathcal{G}_1]$ by some condition $p \in P_1 \times P_{\omega_1}$. This condition cannot depend on its ω_1-part. This is because the value of $\Gamma(g \oplus \emptyset^4)$ depends only on g and is not affected by $\mathcal{G}_3$ and the term, which defines π^*, depends only on $\mathbb{R}^{V[\mathcal{G}]}$ which is invariable under finitely many changes in $\mathcal{G}$. Indeed, if we assume that there were a condition $q \in P_1 \times P_{\omega_1}$ with the same value as p on P_1, which forces a different value for $\pi^*(\mathbf{d}_e(g))$ then we can extend q to a generic filter

$\mathcal{G}^*$ which is the same as $\mathcal{G}$ except on the finitely many values determined by q. Then $V[\mathcal{G}] = V[\mathcal{G}^*]$, in particular $\mathbb{R}^{V[\mathcal{G}]} = \mathbb{R}^{V[\mathcal{G}^*]}$ and we would get a contradiction with the fact that the relation $\{v \mid L_\gamma(\mathbb{R}^{V[\mathcal{G}]}) \models \varphi(v, X)\}$ is a function, as two different pairs, both with first component $\mathbf{d}_e(g)$, satisfy φ in $L_\gamma(\mathbb{R}^{V[\mathcal{G}]})$. We can furthermore incorporate the finite condition p in the definition of Γ. If g is any generic over $V[\mathcal{G}_1]$ function, let g_p be the function obtained by appending g to the condition p. Then g_p is also generic over $V[\mathcal{G}_1]$, extends p and has the same enumeration degree as $\mathbf{d}_e(g)$. Thus $\pi^*(\mathbf{d}_e(g)) = \pi^*(\mathbf{d}_e(g_p))$ and hence $\Gamma(g_p \oplus \emptyset^4) \in \pi^*(\mathbf{d}_e(g))$. We can effectively redefine Γ, so that it has the property $\Gamma(f) = \Gamma(f_p)$, for every total function f. (Here f_p is defined from f and p in the same way as g_p was defined from g and p). Thus it is forced by the empty condition that $\Gamma(g \oplus \emptyset^4) \in \pi^*(\mathbf{d}_e(g))$ for every g, which is Cohen generic over $V[\mathcal{G}_1]$.

The final step is to show that this equation reflects back to all sufficiently generic sets in V. We use the following technical tool, introduced in [17].

Definition 8 (Slaman, Woodin [17])**.** *Let X and $Y = Y_{even} \oplus Y_{odd}$ be sets of natural numbers. We define total function $\mathbb{C}(X,Y)$ as follows.*

$$\mathbb{C}(X,Y)(n) = \begin{cases} Y_{even}(n-m) & \text{if } Y_{odd}(n) = 0 \text{ and } |Y_{odd} \restriction n| = m \\ X(m) & \text{if } Y_{odd}(n) = 1 \text{ and } n \text{ is the } m\text{-th element of } Y_{odd}. \end{cases}$$

In [17] it is shown that the genericity properties of the second argument of the functional $\mathbb{C}$ transfer to its output. In particular if Y is n-generic then so is $\mathbb{C}(X,Y)$ and if Y is Cohen generic over $V[\mathcal{G}_1]$ then so is $\mathbb{C}(X,Y)$. Furthermore if X and Y are total sets then the following equivalence is easily seen to hold $\mathbb{C}(X,Y) \oplus Y \equiv_e X \oplus Y$.

Let f be an 8-generic in $\mathcal{D}_e$ and g_1 and g_2 be generic over $V[\mathcal{G}_1]$, such that g_1 and g_2 and f are mutually 8-generic. Then by Proposition 2 the enumeration degrees $\mathbf{g}_1$ and $\mathbf{g}_2$ of g_1 and g_2 form a minimal pair above the enumeration degree $\mathbf{f}$ of f:

$$\mathbf{f} = (\mathbf{g}_1 \vee \mathbf{f}) \wedge (\mathbf{g}_2 \vee \mathbf{f}).$$

Let $\mathbf{c}(f, g_1) = \mathbf{d}_e(\mathbb{C}(f, g_1))$ and $\mathbf{c}(f, g_2) = \mathbf{d}_e(\mathbb{C}(f, g_2))$. Then we have

$$\mathbf{f} = (\mathbf{c}(f, g_1) \vee \mathbf{g}_1) \wedge (\mathbf{c}(f, g_2) \vee \mathbf{g}_2).$$

This structural property is preserved by π^*, thus

$$\pi^*(\mathbf{f}) = (\pi^*(\mathbf{c}(f, g_1)) \vee \pi^*(\mathbf{g}_1)) \wedge (\pi^*(\mathbf{c}(f, g_2)) \vee \pi^*(\mathbf{g}_2)).$$

Consider the following statement.

$$\Gamma(x \oplus \emptyset^4) \equiv_e (\Gamma(\mathbb{C}(x, y_1) \oplus \emptyset^4) \oplus \Gamma(y_1 \oplus \emptyset^4)) \wedge (\Gamma(\mathbb{C}(x, y_2) \oplus \emptyset^4) \oplus \Gamma(y_2 \oplus \emptyset^4)).$$

This is a Π^0_8 statement true of all triples of total functions x, y_1 and y_2, which are sufficiently mutually generic, e.g. mutually generic over $V[\mathcal{G}_1]$. It must therefore be forced by the empty condition in the partial order to add three Cohen generic reals. But a Π^0_8 statement forced by the empty condition is true of all mutually 8-generic total functions. As f, g_1 and g_2 are mutually 8-generic, we obtain the equality:

$$\Gamma(f \oplus \emptyset^4) \equiv_e (\Gamma(\mathbb{C}(f, g_1) \oplus \emptyset^4) \oplus \Gamma(g_1 \oplus \emptyset^4)) \wedge (\Gamma(\mathbb{C}(f, g_2) \oplus \emptyset^4) \oplus \Gamma(g_2 \oplus \emptyset^4)).$$

The right-hand side of the equation is a set, which is member of the enumeration degree $\pi^*(\mathbf{f})$, hence so is the left-hand side. As $f \in V$ and π^* is an extension of π, it follows that $\pi(\mathbf{f}) = \pi^*(\mathbf{f})$. Finally we obtain $\Gamma(f \oplus \emptyset^4) \in \pi(\mathbf{f})$ for every 8-generic f.

□

Corollary 3. *Let π be an automorphism of $\mathcal{D}_e$. There exists an arithmetic formula φ such that $\varphi(X,Y)$ is true if and only if $\pi(\mathbf{d}_e(X)) = \mathbf{d}_e(Y)$. There are therefore at most countably many automorphisms of $\mathcal{D}_e$.*

Proof. The proof is divided in three steps:

(1) By Theorem 13 there is an enumeration operator Γ, such that for every 8-generic real f, $\pi(\mathbf{d}_e(f))$ is represented by $\Gamma(f \oplus \emptyset^4)$. Thus if f is 8-generic then $\varphi(f,Y)$ is true if and only if Y is enumeration equivalent to $\Gamma(f \oplus \emptyset^4)$. This is an arithmetical relationship between f and Y. Using this preliminary result we move towards an arithmetical definition of φ, which works for all X, not just the 8-generic total functions.

(2) We expand the definition to include arbitrary total sets. Let $\mathbf{b}$ be a total enumeration degree and $B \in \mathbf{b}$ a total set. We apply Proposition 1 to obtain the characteristic function g of an 8-generic set relative to B uniformly from B^8. We split g into two mutually 8-generic sets relative to B, g_1 and g_2: for every natural number n, $g_1(n) = g(2n)$ and $g_2(n) = g(2n+1)$. Now we use the same trick:

$$\pi(\mathbf{b}) = (\pi(\mathbf{c}(B,g_1)) \vee \pi(\mathbf{g}_1)) \wedge (\pi(\mathbf{c}(B,g_2)) \vee \pi(\mathbf{g}_2)) =$$

$$(\mathbf{d}_e(\Gamma(\mathbb{C}(B,g_1) \oplus \emptyset^4)) \oplus \mathbf{d}_e(\Gamma(g_1 \oplus \emptyset^4))) \wedge (\mathbf{d}_e(\Gamma(\mathbb{C}(B,g_2) \oplus \emptyset^4)) \oplus \mathbf{d}_e(\Gamma(g_2 \oplus \emptyset^4))).$$

The last line gives an arithmetical relationship between B and any member of the enumeration degree $\pi(\mathbf{b})$.

(3) Finally let X be an arbitrary set of degree $\mathbf{x}$. By Theorem 2 there are enumeration operators Γ_{T_1} and Γ_{T_2}, such that $\Gamma_{T_1}(X'')$ and $\Gamma_{T_2}(X'')$ are total sets with greatest lower bound X. Then $\pi(\mathbf{d}_e(X))$ is the greatest lower bound of $\pi(\mathbf{d}_e(\Gamma_{T_1}(X'')))$ and $\pi(\mathbf{d}_e(\Gamma_{T_2}(X'')))$, representatives of which can be found arithmetically in $\Gamma_{T_1}(X'')$ and $\Gamma_{T_2}(X'')$ respectively and hence arithmetically in X.

□

Definition 9. *An automorphism base for a structure $\mathcal{S}$ with domain S is a subset of its domain $B \subseteq S$, such that for every pair of automorphisms $\pi_1 : S \to S$ and $\pi_2 : S \to S$, the following implication holds:*

$$(\forall \mathbf{x} \in B(\pi_1(\mathbf{x}) = \pi_2(\mathbf{x}))) \Rightarrow \pi_1 = \pi_2.$$

Equivalently $B \subseteq S$ is an automorphism base if for every automorphism $\pi : S \to S$, we have that:

$$(\forall \mathbf{x} \in B(\pi(\mathbf{x}) = \mathbf{x})) \Rightarrow \pi = id,$$

where id is the identity function on S. This can be seen by taking $\pi(\mathbf{x}) = \pi_2^{-1}(\pi_1(\mathbf{x}))$.

Corollary 4. *The structure of the enumeration degrees $\mathcal{D}_e$ has an automorphism base consisting of:*

1. *A single total degree* $\mathbf{g}$.
2. *A single quasiminimal degree* $\mathbf{a}$.
3. *The enumeration degrees below* ${\mathbf{0}_e}^8$.

Proof. (1) Fix an 8-generic total function g and let $\mathbf{g} = \mathbf{d}_e(g)$. Suppose that π is an automorphism of $\mathcal{D}_e$ and $\pi(\mathbf{g}) = \mathbf{g}$. Let Γ be the enumeration operator representing π on 8-generic reals, given by Theorem 13. It follows that $g \equiv_e \Gamma(g \oplus \emptyset^4)$. This is an arithmetical statement of complexity Σ^0_8. As g is 8-generic, the truth of this statement must be forced by some finite Cohen condition p. Now fix any other 8-generic function f. We define f_p by appending the characteristic function f to the finite condition p, to make p an initial segment of f_p. Now f_p is still 8-generic and of the same enumeration degree as f. The condition p is now extend by f_p hence the statement $f_p \equiv_e \Gamma(f_p \oplus \emptyset^4)$ is true. It follows that for all enumeration degrees $\mathbf{f}$ which contain an 8-generic set, $\pi(\mathbf{f}) = \mathbf{f}$. Since the 8-generic functions generate all total degrees, and hence all enumeration degrees, it follows that π is the identity. By Theorem 2 every enumeration degree is determined by the set of total degrees above it and by Proposition 2 every total degree is determined by the set of 2-generic degrees above it. Thus if $\pi(\mathbf{g}) = \mathbf{g}$ then $\pi = id$ and hence the enumeration degree $\mathbf{g}$ of any 8-generic g is an automorphism base.

(2) By Jockusch's result about semi-recursive sets, discussed in Section 2.2 every nonzero total degree is the least upper bound of a maximal $\mathcal{K}$-pair. Fix an 8-generic total function g of degree $\mathbf{g}$. Let $\{\mathbf{a}, \mathbf{b}\}$ be a maximal $\mathcal{K}$-pair, such that $\mathbf{g} = \mathbf{a} \vee \mathbf{b}$. Finally note that $\mathbf{b}$ is definable from $\mathbf{a}$, it is the largest enumeration degree, which forms a $\mathcal{K}$-pair with $\mathbf{a}$. Thus if $\pi(\mathbf{a}) = \mathbf{a}$ then $\pi(\mathbf{b}) = \mathbf{b}$ and hence $\pi(\mathbf{g}) = \mathbf{g}$ and by Part (1) $\pi = id$.

(3) Follows directly from Part (1).

□

7. Interpreting automorphisms in $\mathcal{D}_e$

Slaman and Woodin [17] define the notion of *extendable assignment of reals* and use it show $\mathcal{D}_T$ is biinterpretable with second order arithmetic using parameters. We follow their ideas to obtain similar results about $\mathcal{D}_e$.

Definition 10. *An e-assignment of reals consists of*

(1) *A countable ω-model $\mathcal{M}$ of T, the finitely axiomatizable theory consisting of the fragment of ZFC with Σ_1 replacement and Σ_1 comprehension;*
(2) *A function f and a countable jump ideal $\mathcal{I}$ in $\mathcal{D}_e$ such that $f : \mathcal{D}_e^{\mathcal{M}} \to \mathcal{I}$ is a bijection and for all $\mathbf{x}, \mathbf{y} \in \mathcal{D}_e^{\mathcal{M}}$, if $\mathcal{M} \models \mathbf{x} \geq \mathbf{y}$ then $f(\mathbf{x}) \geq f(\mathbf{y})$.*

Theorem 14. *If $(\mathcal{M}, f, \mathcal{I})$ is an e-assignment of reals then $\mathcal{D}_e^{\mathcal{M}} = \mathcal{I}$.*

Proof. Both $\mathcal{D}_e^{\mathcal{M}}$ and $\mathcal{I}$ are countable jump ideals in $\mathcal{D}_e$ and f is an isomorphism between those ideals. By Theorem 5 the enumeration jump is first order definable in the jump ideals $\mathcal{D}_e^{\mathcal{M}}$ and $\mathcal{I}$ by the same first order formula $\varphi_{\mathcal{J}}$. Thus f preserves the jump: for every $\mathbf{x} \in \mathcal{D}_e^{\mathcal{M}}$ we have $f(\mathbf{x}') = f(\mathbf{x})'$. By a proof similar to that of Theorem 6 we show that for every $\mathbf{x} \in \mathcal{D}_e^{\mathcal{M}}$, there is a representative $X \in \mathbf{x}$ which is arithmetic relative to a representative $F(X) \in f(\mathbf{x})$. Indeed, by *Example 1* of applications of the Coding Theorem there are parameters $\bar{\mathbf{p}} \leq \mathbf{x}'$, hence in $\mathcal{D}_e^{\mathcal{M}}$, which code a model of the structure $\langle \mathbb{N}, 0, s, +, *, X\rangle$. Then $f(\bar{\mathbf{p}}) \leq f(\mathbf{x}') = f(\mathbf{x})'$ will code the same structure. By the Decoding Theorem X is e-reducible to a member of $f(\mathbf{x})^6$ and hence $\mathbf{x} \in \mathcal{I}$. Thus $\mathcal{D}_e^{\mathcal{M}} \subseteq \mathcal{I}$. Now fix $\mathbf{y} \in \mathcal{I}$, $Y \in \mathbf{y}$, and parameters $\bar{\mathbf{q}} \leq \mathbf{y}'$, hence in $\mathcal{I}$, which code a model of the structure $\langle \mathbb{N}, 0, s, +, *, Y\rangle$.

Then $f^{-1}(\bar{\mathbf{q}}) \leq f^{-1}(\mathbf{y}') = f^{-1}(\mathbf{y})'$ will code the same structure. By the Decoding Theorem Y is e-reducible to a member of $f^{-1}(\mathbf{y})^6$ and hence $\mathbf{y} \in \mathcal{D}_e^{\mathcal{M}}$. So $\mathcal{I} = \mathcal{D}_e^{\mathcal{M}}$. □

Definition 11. *An e-assignment of reals $(\mathcal{M}, f, \mathcal{I})$ is extendable if for every $\mathbf{z} \in \mathcal{D}_e$ there exists an e-assignment of reals $(\mathcal{M}_1, f_1, \mathcal{I}_1)$ such that $\mathcal{D}_e^{\mathcal{M}} \subseteq \mathcal{D}_e^{\mathcal{M}_1}$, $\mathcal{I} \cup \{\mathbf{z}\} \subseteq \mathcal{I}_1$ and $f \subseteq f_1$.*

Theorem 15. *If $(\mathcal{M}, f, \mathcal{I})$ is an extendible e-assignment then there is an automorphism $\pi : \mathcal{D}_e \to \mathcal{D}_e$, such that for all $\mathbf{x} \in \mathcal{D}_e^{\mathcal{M}}$, $\pi(\mathbf{x}) = f(\mathbf{x})$.*

Proof. By Theorem 14 $\mathcal{D}_e^M = \mathcal{I}$. Thus f is an automorphism of the countable ideal $\mathcal{I}$. By extendability we have that for every $\mathbf{z}$, f can be extended to an automorphism of a jump ideal, containing $\mathcal{I}$ and $\mathbf{z}$, which is precisely the definition of a persistent automorphism. By Theorem 11 f can be extended to a global automorphism π. □

Example 3. We give a third example in the series of examples of applications of the Coding Theorem. If $(\mathcal{M}, f, \mathcal{I})$ is an e-assignment of reals, then we can interpret this in the enumeration degrees: we can fix a countable set of enumeration degrees, that will represent the elements of the domain of $\mathcal{M}$, i.e. fix some injection $\psi : M \to \mathcal{D}_e$, then $\psi(M)$ is the required countable set of enumeration degrees. We can define a countable binary relation $R_\in$ on these degrees exactly mimicking the relation $\in^{\mathcal{M}}$, i.e. $R_\in(\psi(x), \psi(y))$ if and only if $x \in^{\mathcal{M}} y$. Then we can represent f as a countable binary relation on the enumeration degrees that represent the elements of $\mathcal{D}_e^{\mathcal{M}}$ and the elements of the ideal $\mathcal{I}$. All of these are countable relations on degrees an so by the Coding Theorem there are parameters $\bar{\mathbf{m}}$, $\bar{\mathbf{f}}$ and $\bar{\mathbf{i}}$ coding these sets.

On the other hand the property "$\bar{\mathbf{m}}$, $\bar{\mathbf{f}}$ and $\bar{\mathbf{i}}$ code an e-assignment of reals" is a definable property of $\bar{\mathbf{m}}$, $\bar{\mathbf{f}}$ and $\bar{\mathbf{i}}$. We can require that $R_\in$ coded by $\bar{\mathbf{m}}$ satisfies the finitely many axioms of T. To ensure the coded model is an ω-model, we can require that the definable element representing $\omega^{\mathcal{M}}$ is isomorphic to the domain of some standard model of arithmetic, say the one from Example 1. By the definability of the enumeration jump, we can require that the set coded by $\bar{\mathbf{i}}$ is a jump ideal. Finally we can definably identify the element $\mathbf{d}_e$ representing $\mathcal{D}_e^{\mathcal{M}}$ in the model coded by $\bar{\mathbf{m}}$ and require that the relation coded by $\bar{\mathbf{f}}$ is a bijection between the degrees $\mathbf{x}$ such that $R_\in(\mathbf{x}, \mathbf{d}_e)$ and the ideal coded by $\bar{\mathbf{i}}$, which preserves $\leq$.

We can furthermore deduce that the property "$\bar{\mathbf{m}}$, $\bar{\mathbf{f}}$ and $\bar{\mathbf{i}}$ code an extendible e-assignment of reals" is a definable property of $\bar{\mathbf{m}}$, $\bar{\mathbf{f}}$ and $\bar{\mathbf{i}}$. In addition to the requirements above we need to add that for every $\mathbf{z}$ there exist parameters $\bar{\mathbf{m}}_1$, $\bar{\mathbf{f}}_1$ and $\bar{\mathbf{i}}_1$, coding an e-assignment of reals, such that the set of enumeration degrees in the model coded by $\bar{\mathbf{m}}$ is extended by the set of enumeration degrees in the model coded by $\bar{\mathbf{m}}_1$, the ideal coded by $\bar{\mathbf{i}}_1$ contains $\mathbf{z}$ and the elements of the ideal coded by $\bar{\mathbf{i}}$ and that the relation coded by $\bar{\mathbf{f}}_1$ extends the relation coded by $\bar{\mathbf{f}}$.

Theorem 16. *Let $\mathbf{g}$ be the enumeration degree of an 8-generic $g \leq_e \emptyset^8$. Then the relation $Bi(\bar{\mathbf{c}}, \mathbf{d})$, stating that "$\bar{\mathbf{c}}$ codes a model of arithmetic with a unary predicate for D and $\mathbf{d}_e(D) = \mathbf{d}$" is definable in $\mathcal{D}_e$ using parameter $\mathbf{g}$. Thus $\mathcal{D}_e$ is biinterpretable with second order arithmetic using parameters.*

Proof. Consider the following definable property of $\bar{\mathbf{c}}$ and $\mathbf{d}$: there exist parameters $\bar{\mathbf{m}}$, $\bar{\mathbf{f}}$ and $\bar{\mathbf{i}}$, such that:

(1) $\bar{\mathbf{m}}$, $\bar{\mathbf{f}}$ and $\bar{\mathbf{i}}$ code an extendible e-assignment of reals $(\mathcal{M}, f, \mathcal{I})$.
(2) $\bar{\mathbf{c}}$ codes a model of arithmetic with a predicate for a set D.
(3) The set D is an element of $\mathcal{M}$ and $f(\mathbf{d}_e(D)^{\mathcal{M}}) = \mathbf{d}$.
(4) For the set g, given by its arithmetic definition, interpreted in $\mathcal{M}$, $f(\mathbf{d}_e(g)^{\mathcal{M}}) = \mathbf{g}$.

If $\bar{\mathbf{c}}$ and $\mathbf{d}$ satisfy this property then by Theorem 15 f is a persistent automorphism of $\mathcal{I}$ which can be extended to a global automorphism π. As $\mathcal{M}$ is an ω-model, arithmetic definitions are interpreted correctly in $\mathcal{M}$, hence $\mathbf{d}_e(g)^{\mathcal{M}} = \mathbf{g}$ and $\mathbf{d}_e(D)^{\mathcal{M}} = \mathbf{d}_e(D)$. Now $\pi(\mathbf{g}) = f(\mathbf{g}) = \mathbf{g}$. As $\mathbf{g}$ is the degree of an 8-generic, by Corollary 4 π is the identity. Thus $\mathbf{d} = f(\mathbf{d}_e(D)) = \pi(\mathbf{d}_e(D)) = \mathbf{d}_e(D)$.

If $\bar{\mathbf{c}}$ and $\mathbf{d}$ are such that $Bi(\bar{\mathbf{c}}, \mathbf{d})$ then we let $\mathcal{M}$ be any countable ω-model of T which contains D as an element, $\mathcal{I} = \mathcal{D}_e^{\mathcal{M}}$ and $f = id$. By the discussion in Example 3 there are parameters which code this model and satisfy the clauses of the definable property. □

Corollary 5. *Let $R \subseteq (2^\omega)^n$ be relation definable in second order arithmetic and invariant under enumeration reducibility.*

(1) *The relation $\mathcal{R} \subseteq \mathcal{D}_e^n$ defined by $\mathcal{R}(\mathbf{d}_e(X_1), \dots, \mathbf{d}_e(X_n)) \leftrightarrow R(X_1, \dots, X_n)$ is definable in $\mathcal{D}_e$ with one parameter $\mathbf{g}$. In particular $\mathcal{TOT}$ is definable with one parameter.*
(2) *If $\mathcal{R}$ is invariant under automorphisms then $\mathcal{R}$ is definable without parameters in $\mathcal{D}_e$. In particular the hyperarithmetic jump operation is first order definable in $\mathcal{D}_e$.*

Proof. Suppose that φ is the defining formula in second order arithmetic for R. Then $(\mathbf{x}_1, dots, \mathbf{x}_n) \in \mathcal{R}$ if and only if there exist parameters $\bar{\mathbf{c}}_1, \dots, \bar{\mathbf{c}}_n$, such that:

(1) For every $i \in \{1, \dots, n\}$, $\bar{\mathbf{c}}_i$ codes model of arithmetic with a predicate for a set X_i;
(2) The interpretation of $\varphi(X_1, \dots, X_n)$ (a formula in second order arithmetic) is true in $\mathcal{D}_e$.
(3) For every $i \in \{1, \dots, n\}$, $Bi(\bar{\mathbf{c}}_i, \mathbf{x}_i)$, the relation from Theorem 16.

As $\mathcal{TOT}$ is a relation which is induced by the enumeration degree invariant relation $Tot(X) \leftrightarrow \exists Y(X \equiv_e Y \oplus \overline{Y})$, it follows that $\mathcal{TOT}$ is definable in $\mathcal{D}_e$ with parameter the enumeration degree $\mathbf{g}$ of an 8-generic, used in the definition of the relation Bi.

If $\mathcal{R}$ is invariant under automorphism then we can replace the third clause by merely requiring that the set coded by $\mathbf{c}_i$ is automorphic to $\mathbf{x}_i$. This achieved without the use of parameters by a omitting the last (the fourth) clause in the proof of Theorem 16, which gives the first order definition of the relation Bi.

Let π be an automorphism of $\mathcal{D}_e$. Let X be a set and $\Pi(X) \in \pi(\mathbf{d}_e(X))$. By Theorem 12 for every $\mathbf{y} \geq {\mathbf{0}_e}^4$, we have that $\pi(\mathbf{y}) = \mathbf{y}$. Hence $\Pi(X) \leq_e X^4$ and $X \leq_e \Pi(X)^4$. Thus X and $\Pi(X)$ have the same arithmetic degree. From this it follows that the relation *arithmetic equivalence* is invariant under automorphism and hence by the argument above first order definable in $\mathcal{D}_e$. There is however a simpler proof of this fact, which uses only the Coding Theorem and the definability of the enumeration jump. If X and Y have the same arithmetic degree then X and Y have the same hyperarithmetic jump. Thus the relation $HypJ(X, Y) \leftrightarrow Y \equiv_e \mathcal{O}^X$ is invariant under automorphisms. Indeed $\pi(\mathbf{d}_e(\mathcal{O}^X)) = \mathbf{d}_e(\mathcal{O}^X)$ as $\mathbf{d}(\mathcal{O}^X) \geq {\mathbf{0}_e}^4$

and $\mathcal{O}^{\Pi(X)} = \mathcal{O}^X$, as X and $\Pi(X)$ have the same arithmetic degree. By the argument above the hyperarithmetic jump operation is first order definable in $\mathcal{D}_e$. □

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