Cupping Classes of Σ_2^0 Enumeration Degrees

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Abstract. We prove that no subclass of the Σ_2^0 enumeration degrees containing the 3-c.e. enumeration degrees can be cupped to $\mathbf{0_e'}$ by a single Σ_2^0 enumeration degree.

1 Introduction

In an upper semi-lattice with greatest element $\langle \mathcal{A}, \leq, \vee, 1 \rangle$ we say that an element $\mathbf{a}$ is cuppable if there exists an element $\mathbf{b} \neq \mathbf{1}$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{1}$. Posner and Robinson showed that every degree in $\mathcal{D}_T(\leq 0')$ is cuppable. Cooper and Yates [5] showed the existence of a non-cuppable c.e. degree in the semi-lattice of the computably enumerable degrees. Meanwhile Cooper, Seetapun and (independently) Li proved that there exists a single Δ_2^0 Turing degree that cups every non-zero c.e. degree.

In this paper we consider cupping properties of the local degree structure of the enumeration degrees below $\mathbf{0_e'}$. Intuitively we say that a set A is *enumeration reducible* to a set B, denoted as $A \leq_e B$, if there is an effective procedure to enumerate A given any enumeration of B. By identifying sets that are reducible to each other we obtain a degree structure, the structure of the enumeration degrees $\langle \mathcal{D}_e, \leq \rangle$. It is an upper semi-lattice with jump operator and least element $\mathbf{0_e}$, the collection of all computably enumerable sets. The semi-lattice of the enumeration degrees can be considered as an extension of the semi-lattice of the Turing degrees, as the second semi-lattice can be embedded in the first, via an order theoretic embedding ι preserving the least upper bound and the jump operator.

An important substructure of $\mathcal{D}_e$ is given by the Σ_2^0 enumeration degrees. Cooper [2] proved that the Σ_2^0 enumeration degrees are the enumeration degrees below $\mathbf{0_e'}$. There is a natural hierarchy of classes of enumeration degrees within this substructure. The Π_1^0 enumeration degrees, which are exactly the images of the c.e. Turing degrees under ι, form the smallest class. Further classes can be obtained by considering the n-c.e. degrees for every $n \leq \omega$. Cooper [3] proved that the 2-c.e. enumeration degrees coincide with the Π_1^0 enumeration degrees. Thus the second class in our hierarchy consists of all 3-c.e. enumeration degrees. The

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last proper subclass of the Σ^0_2 enumeration degrees in this hierarchy comprises of all Δ^0_2 enumeration degrees.

In [6], Cooper, Sorbi and Yi proved that every nonzero Δ^0_2 enumeration degree can be cupped by a total Δ^0_2 enumeration degree, in contrast to the Σ^0_2 enumeration degrees where non-cuppable degrees exist. Soskova and Wu [10] examined the cupping properties of the Δ^0_2 enumeration degrees further and showed that every non-zero Δ^0_2 enumeration degree can be cupped by a 1-generic Δ^0_2, hence partial and low, enumeration degree. Furthermore they showed that every non-zero ω-c.e. enumeration degree can be cupped by a 3-c.e enumeration degree. These results exhibit the flexibility that we have when searching for cupping partners of enumeration degrees in each of the proper subclasses of the Σ^0_2 enumeration degrees and motivate the initial goal of this project: to find a single degree from a larger class that cups all non-zero enumeration degrees from a smaller class. One such example we obtain immediately by transferring Cooper, Seetapun and Li's result to the enumeration degrees via ι, namely that there exists a single Δ^0_2 enumeration degree which cups all non-zero Π^0_1 enumeration degrees. In this paper we prove that any other attempt at a result of this kind is doomed to failure as for every incomplete Σ^0_2 enumeration degree $\mathbf{a}$ there exists a non-zero member of the second class, a non-zero 3-c.e. enumeration degree $\mathbf{b}$, such that $\mathbf{b}$ is not cupped by $\mathbf{a}$ to $\mathbf{0_e'}$.

Theorem 1. *Let* $\mathbf{a}$ *be an incomplete* Σ^0_2 *enumeration degree. There exists a non-zero* 3*-c.e enumeration degree* $\mathbf{b}$ *such that* $\mathbf{a} \vee \mathbf{b} \neq \mathbf{0_e'}$.

Notation and terminology below are based on that of [4] and [9].

2 Requirements and Strategies

We shall start by giving a formal definition of enumeration reducibility and the n-c.e. degrees and then move on to establish the requirements and basic strategies for the proof of the main theorem.

Definition 1. *A set* A *is enumeration reducible* ($\leq_e$) *to a set* B *if there is a c.e. set* Φ *such that:*

$$n \in A \Leftrightarrow \exists u(\langle n, u\rangle \in \Phi \wedge D_u \subseteq B),$$

where D_u *denotes the finite set with code* u *under the standard coding of finite sets. We will refer to the c.e. set* Φ *as an enumeration operator and its elements will be called axioms.*

We say that an enumeration degree is n-c.e. ($n \leq \omega$) if it contains an n-c.e. set:

Definition 2. *1. For* $n < \omega$ *a set* A *is* n*-c.e. if there is a computable function* f *such that for each* x, $f(x, 0) = 0$, $|\{s \mid f(x, s) \neq f(x, s+1)\}| \leq n$ *and* $A(x) = \lim_s f(x, s)$.

2. *A is ω-c.e. if there are two computable functions $f(x,s), g(x)$ such that for all x, $f(x,0)=0$, $|\{s \mid f(x,s) \neq f(x,s{+}1)\}| \leq g(x)$ and $A(x) = \lim_s f(x,s)$.*

Let A be a representative of the given Σ^0_2 enumeration degree. Let $\{A_s\}_{s<\omega}$ be a good Σ^0_2 approximation to A as defined in [7]. A good Σ^0_2 approximation to A is a Σ^0_2 approximation with infinitely many good stages s at which $A_s \subseteq A$.

We shall construct two 3-c.e sets X and Y, so that ultimately the degree of one of them will have the requested properties. Consider the following group of requirements:

- Let $\{\Theta_i\}_{i<\omega}$ and $\{\Psi_i\}_{i<\omega}$ be enumerations of all enumeration operators. For every i we will have a pair of requirements:
$$\mathcal{P}^0_i : \Theta_i^{A,X} \neq \overline{K}.$$
$$\mathcal{P}^1_i : \Psi_i^{A,Y} \neq \overline{K}.$$
- Let $\{W_e\}_{e<\omega}$ be an enumeration of all c.e. sets. For every natural number e we have a requirement:
$$\mathcal{N}_e : W_e \neq X \wedge W_e \neq Y.$$

We shall construct the sets X and Y so that for all e the requirement $\mathcal{N}_e$ is satisfied, thus both X and Y have non-zero e-degree, and if $\mathcal{P}^j_i$ is not satisfied then for all i' the requirement $\mathcal{P}^{1-j}_{i'}$ is satisfied, thus the degree of at least one of the sets $A \oplus X$ or $A \oplus Y$ is incomplete. The construction shall be carried out on a tree of strategies. Each node of the tree shall be assigned either an $\mathcal{N}$-requirement or a $\mathcal{P}^0$- and a $\mathcal{P}^1$-requirement. On each stage we shall construct a finite path in the tree of strategies visiting some of the nodes and allowing them to act towards satisfying one of the assigned requirements. The intention is that there will be a leftmost path of nodes that is visited on infinitely many stages, providing a successful outcome to all strategies on it.

Each $\mathcal{P}$-node in the tree of strategies α is associated with a pair of requirements: $\mathcal{P}^0_\alpha$ and $\mathcal{P}^1_\alpha$. It will prove that at least one of them is satisfied. To do this the strategy constructs an e-operator Γ_α, threatening to prove that $A \geq_e \overline{K}$. The strategy α performs cycles of increasing length. On the k-th cycle it examines all elements $n = 0, 1, \dots k$ in turn. If the element n belongs to $\overline{K}$ and to both sets $\Theta_\alpha^{A,X}$ and $\Psi_\alpha^{A,Y}$ then it will enumerate an axiom in Γ_α which comprises of the A-parts of the two axioms for n in Θ_α and in Ψ_α that have been valid the longest. If later the element n leaves the approximation of $\overline{K}$ then the strategy shall restore one of the axioms in either Θ_α or Ψ_α by enumerating the corresponding X-part back in X or Y-part back in Y. The strategy shall then wait until it has observed a change in A that rectifies the operator Γ_α. As A is incomplete the strategy will eventually include in its cycles an element n such that $\Gamma^A_\alpha(n) \neq \overline{K}(n)$. If $n \in \Gamma^A_\alpha$ then $n \in \Gamma^A_\alpha[s]$ on all but finitely many stages s. Thus eventually $\Gamma^A_\alpha(n)$ will not be rectified by any change in A and α will have a finitary outcome proving the successful diagonalization. If $n \notin \Gamma^A_\alpha$ by the properties of a good approximation we have that on infinitely many stages s, in fact on all good stages, $n \notin \Gamma_\alpha^A[s]$. Thus infinitely often α will discover that at least one of the operators Θ_α or Ψ_α has failed to provide it with an axiom that is permanently valid, i.e. infinitely often α will have proof that $\Theta_\alpha^{A,X}(n) = 0$ or $\Psi_\alpha^{A,Y}(n) = 0$. This will give an infinitary outcome.

An $\mathcal{N}$-strategy β working on W_β would like to prove that $W_\beta \neq X$ and $W_\beta \neq Y$. The obvious strategy for β would be to select a witness x_β and wait until $x_\beta \in W_\beta$. We assume that the sets X and Y start off as ω, then during the construction the strategies extract or enumerate back elements in the sets. Thus if x_β never enters W_β the strategy will be successful. If the element does enter W_β then the strategy will extract x_β from both sets X and Y and again will have proved a difference. This strategy is in conflict with the need of higher priority $\mathcal{P}$-strategies to restore axioms by enumerating elements back in one of the sets X or Y. Therefore the strategy for β will have to be more elaborate.

It will start off as the original strategy: select a witness x_β as a fresh number and wait until $x_\beta \in W_\beta$. If this never happens then the requirement will be satisfied. Otherwise extract x_β from both sets X and Y. Suppose a higher priority strategy α requires that x_β be enumerated back in X or Y. In this case β shall choose a new witness y_β that has not been used in any axiom sofar, restrain X and let x_β be enumerated back in Y. From this point on any axiom that appears in the construction shall necessarily have $x_\beta \notin X$, thus x_β and y_β cannot appear in the same axiom. The strategy β will wait again until y_β enters W_β and then extract it from Y. Should a higher priority α require that y_β be enumerated back in one of the sets then β will only give permission to enumerate back in X.

3 Outcomes, Parameters and the Tree of Strategies

Consider first a $\mathcal{P}$-strategy α working on $\mathcal{P}_\alpha^0$ and $\mathcal{P}_\alpha^1$. It will have infinitely many outcomes: for every $n \in \omega \cup \{w\}$ two outcomes $\langle X, n\rangle$ and $\langle Y, n\rangle$ arranged as follows:

$$\langle X,0\rangle <_L \langle Y,0\rangle <_L \langle X,1\rangle <_L \langle Y,1\rangle \ldots <_L \langle X,w\rangle <_L \langle Y,w\rangle$$

For each outcome the first element of the pair indicates which requirement has been satisfied. Let $O_{\mathcal{P}}$ denote the set of all possible outcomes of α. The next $\mathcal{P}$-strategy below outcomes $\langle X, n\rangle$ shall be associated with a new $\mathcal{P}^0$-requirement and the same $\mathcal{P}^1$-requirement. Similarly the next $\mathcal{P}$-strategy below outcomes $\langle Y, n\rangle$ will be associated with the same $\mathcal{P}^0$-requirement and a different $\mathcal{P}^1$-requirement. Thus if $\mathcal{P}_i^j$ never gets satisfied then all $\mathcal{P}_i^{1-j}$ must be.

The strategy α will have a parameter Γ_α, the e-operator that it will construct when visited. At initialization Γ_α is set to the empty set. The strategy will also have parameters k_α denoting the current cycle of the strategy and $n_\alpha < k_\alpha$ denoting the current element of the cycle that α is working with. On initialization the values of the parameters are set to $k_\alpha = 1$ and $n_\alpha = 0$. Furthermore for each element $n < \omega$ the strategy α shall have one more parameter $D_\alpha(n)$, a list of all pairs of X- and Y-parts of axioms from Θ_α and Ψ_α respectively, for which the

A-parts are used in axioms for n in Γ_α. Initially the values of all such lists will be $\emptyset$. Finally it will have two parameters $Ax^\theta_\alpha(n)$ and $Ax^\psi_\alpha(n)$ denoting axioms in Θ_α and Ψ_α respectively which will be candidates for the construction of a new axiom in Γ_α, initially undefined.

An $\mathcal{N}$-strategy β shall have two outcomes $d <_L w$, $O_\mathcal{N} = \{d, w\}$. It has parameters x_β, y_β, which will be undefined when β is initialized. Furthermore on initialization β will give up any restraint it has imposed sofar.

Let $O = O_\mathcal{P} \cup O_\mathcal{N}$ be the collection of all possible outcomes and R the collection of all requirements. The tree of strategies is a computable function T with domain a downwards closed subset of $O^{<\omega}$ and range a subset of $R^2 \cup R$ with the following inductive definition:

1. $T(\emptyset) = \langle \mathcal{P}^0_0, \mathcal{P}^1_0 \rangle$.
2. Let α be in the domain of T and α be a $\langle \mathcal{P}^0_i, \mathcal{P}^1_j \rangle$-node. Then $\alpha\hat{\ }o$, where $o \in O_\mathcal{P}$, is also in the domain of T and $T(\alpha\hat{\ }o) = \mathcal{N}_{|\alpha|/2}$.
3. Let β be an $\mathcal{N}$-node in the domain of T. Then $\beta = \alpha\hat{\ }o$, where α is a $\langle \mathcal{P}^0_i, \mathcal{P}^1_j \rangle$-node for some i and j. Then $\beta\hat{\ }o'$, where $o' \in O_\mathcal{N}$, is in the domain of T. If $o = \langle X, n \rangle$ for some $n \in \omega \cup \{w\}$ then $T(\beta\hat{\ }o') = \langle \mathcal{P}^0_{i+1}, \mathcal{P}^1_j \rangle$. If $o = \langle Y, n \rangle$ for some $n \in \omega \cup \{w\}$ then $T(\beta\hat{\ }o') = \langle \mathcal{P}^0_i, \mathcal{P}^1_{j+1} \rangle$.

4 Construction

We shall perform the construction on stages. On each stage s we shall approximate the sets X and Y by constructing cofinite sets X_s and Y_s. We shall also construct a string δ_s of length s through the domain of T. We shall say that a node $\gamma \subset \delta_s$ is visited on stage s, also that s is a γ-true stage. On true stages strategies will be allowed to modify their parameters and choose an outcome. At the end of stage s we shall initialize all nodes to the right of δ_s.

On stage 0 all nodes are initialized and $X_0 = Y_0 = \omega$, $\delta_0 = \emptyset$.

Suppose we have constructed δ_t, X_t and Y_t for $t < s$. We construct $\delta_s(n)$ with an inductive definition. The sets X_s and Y_s shall be obtained by allowing the strategies visited on stage s to modify the approximations X_{s-1}, Y_{s-1} obtained on the previous stage. Define $\delta_s(0) = \emptyset$. Suppose that we have constructed $\delta_s \upharpoonright n$. If $n = s$, we end this stage and move on to $s+1$. Otherwise we visit the strategy $\delta_s \upharpoonright n$ and let it determine its outcome o. Then $\delta_s(n) = o$. We have two cases depending on the type of the node $\delta_s \upharpoonright n$.

I. If $\delta_s \upharpoonright n = \alpha$ is a $\mathcal{P}$-node, we perform the following actions:
Let s^- be the previous α-true stage if α has not been initialized since and $s^- = s$ otherwise. The strategy α will inherit the values of its parameters from stage s^- and during its actions it can change their values several times. Thus we will omit the subscript indicating the stage when we discuss α's parameters. If the current element n_α does not need further actions we shall move on to the next element. As we will do this in several cases we shall describe the actions that we take here and use the phrase **reset the parameters**. Denote the current values of n_α by n and of k_α by k. We *reset the*

parameters by changing the values of the parameters as follows: $n_\alpha := n+1$ if $n < k$, otherwise $n = k$ and we set $k_\alpha := k+1$, $n_\alpha := 0$. In both cases we initialize the strategies extending $\alpha\hat{}\langle X, w\rangle$ and $\alpha\hat{}\langle Y, w\rangle$.

1. Let $k = k_\alpha$ and $n = n_\alpha$. Let s_n^- be the previous stage when n was examined, if α has not been initialized since, $s_n^- = s$ otherwise.
2. If $n \in \overline{K}[s]$ and $n \in \Gamma_\alpha^A[t]$ for all stages t with $s_n^- < t \leq s$ then *reset the parameters* and go to step 1.
3. If $n \in \overline{K}[s]$, but $n \notin \Gamma_\alpha^A[t]$ on some stage t with $s_n^- < t \leq s$ then:
   - a.X If $Ax_\alpha^\theta(n)$ is not defined, then define it as the axiom that has been valid longest including on all stages $s_n^- < t \leq s$ and move on to step *a.Y*. If there is no such axiom then let the outcome be $\langle X, n\rangle$ and *reset the parameters*.
   - b.X If $Ax_\alpha^\theta(n)$ is defined but was not valid on some stage t with $s_n^- < t \leq s$, then cancel its value (make it undefined) and let the outcome be $\langle X, n\rangle$, *reset the parameters*. Otherwise go to step *a.Y*.
   - a.Y If $Ax_\alpha^\psi(n)$ is not defined, then define it as the axiom that has been valid longest including on all stages $s_n^- < t \leq s$ and move on to step *c*. If there is no such axiom then let the outcome be $\langle Y, n\rangle$ and *reset the parameters*.
   - b.Y If $Ax_\alpha^\theta(n)$ is defined but was not valid on some stage t with $s_n^- < t \leq s$, then cancel its value (make it undefined) and let the outcome be $\langle Y, n\rangle$, *reset the parameters*. Otherwise go to step *c*.
   - c. If both $Ax_\alpha^\theta(n) = \langle n, A_\theta, X_\theta\rangle$ and $Ax_\alpha^\psi(n) = \langle n, A_\psi, Y_\psi\rangle$ are defined and have been valid on all stages t with $s_n^- < t \leq s$ then enumerate in Γ_α the axiom $\langle n, A_\theta \cup A_\psi\rangle$. Enumerate $\langle X_\theta, Y_\psi\rangle$ in $D_\alpha(n)$. *Reset the parameters* and go back to step 1.
4. If $n \notin \overline{K}[s]$ and $n \notin \Gamma_\alpha^A[t]$ on some stage t: $s_n^- < t \leq s$ *reset the parameters* and go back to step 1.
5. Suppose $n \notin \overline{K}[s]$ but $n \in \Gamma_\alpha^A[t]$ on all t such that $s_n^- < t \leq s$. For every pair $\langle X_\theta, Y_\psi\rangle \in D_\alpha(n)$ find the highest priority $\mathcal{N}$-strategy $\beta \supset \alpha$ that has permanently restrained an element $x \in X_\theta$ out of X or $y \in Y_\psi$ out of Y. If there is such a strategy β and it has a permanent restraint on X, enumerate Y_ψ in $Y[s]$; if it has a permanent restraint on Y, enumerate X_θ back in $X[s]$. Otherwise if there is no such strategy enumerate Y_ψ back in $Y[s]$. Choose the axiom $\langle n, A_\theta \cup A_\psi\rangle$ in Γ_α^A that has been valid the longest. Let X_θ and Y_ψ be the corresponding X and Y parts of the axioms $\langle n, A_\theta, X_\theta\rangle \in \Theta$ and $\langle n, A_\psi, Y_\psi\rangle \in \Psi$
   - a. If $X_\theta \subseteq X[s]$ then this will ensure that $n \in \Theta_\alpha^{A,X}[s]$. Let the outcome be $\langle X, w\rangle$. Note that we will not reset the parameters at this point, thus the construction will keep going through this step while there is no change in A.
   - b. If $X_\theta \not\subseteq X[s]$ then $Y_\psi \subseteq Y[s]$ and this will ensure that $n \in \Psi_\alpha^{A,Y}[s]$. Let the outcome be $\langle Y, w\rangle$.

II. If $\delta_s \restriction n = \beta$ is an $\mathcal{N}$-node, we perform the following actions:
Let s^- be the previous β-true stage if β has not been initialized since, go to the step indicated on stage s^-. Otherwise $s^- = s$ and go to step 1.

1. Define x_β as a fresh number, one that has not appeared in the construction so far and is bigger than s. Go to the next step.
2. If $x_\beta \notin W_\beta[s]$ then let the outcome be $o = w$, return to this step on the next stage. Otherwise go to the next step.
3. Extract x_β from $X[s]$ and $Y[s]$. Restrain permanently x_β out of X. Let the outcome be $o = d$, go to the next step on the next stage.
4. If $x_\beta \in Y[s]$ then define y_β as a fresh number, initialize all strategies of lower priority than β and go to the next step. Otherwise $o = d$, return to this step on the next stage.
5. If $y_\beta \notin W_\beta$ then let the outcome be $o = w$. Return to this step on the next stage. Otherwise go to the next step.
6. If y_β is not yet restrained then restrain y_β permanently out of Y and extract y_β from $Y[s]$. Let the outcome be $o = d$, return to this step on the next stage.

This completes the construction.

5 Proof

The tree is infinitely branching and therefore there is a risk that there might not be a path in the tree that is visited infinitely often. However we shall start the proof by establishing some basic facts about the relationship between strategies.

For clarity we shall define one more notation. Let α be a $\mathcal{P}$-strategy. For every axiom $Ax = \langle n, A_\theta \cup A_\psi \rangle \in \Gamma_\alpha$ we shall connect a corresponding entry $\langle n, A_\theta, X_\theta, A_\psi, Y_\psi \rangle$ so that $\langle n, A_\theta, X_\theta \rangle \in \Theta_\alpha$ and $\langle n, A_\psi, Y_\psi \rangle \in \Psi_\alpha$ are the corresponding axioms used to construct Ax.

Lemma 1. *Let β be an $\mathcal{N}$-strategy, not initialized after stage s_i. If β has a witness x_β that is extracted by β on stage $s_x > s_i$ then $x_\beta \notin X[t]$ on all $t \geq s_x$. If β has a witness y_β that is extracted from Y on stage $s_y > s_x$ then $y_\beta \notin Y[t]$ on all $t \geq s_y$.*

Proof. Assume inductively that the lemma is true for higher priority $\mathcal{N}$-strategies.

Let s_i be the last stage on which β is initialized. Suppose β chooses the witness x_β on stage $s_1 > s_i$. By induction the lemma is true for any higher priority strategy $\beta' \geq \beta$, as if β' is initialized after stage s_i then β would be initialized as well and this does not happen by assumption. Furthermore we claim that:

Claim. Any witness which is permanently extracted by β' is extracted before stage s_1.

Indeed suppose that β' permanently extracts a new witness on stage $s_2 > s_1$. Then on stage s_2 the strategy β' has outcome d. Thus if $\beta >_L \beta'$ or $\beta \supseteq \beta'\hat{\ }w$ then β would be initialized on stage s_2 contrary to assumption. This leaves us with the only possibility that $\beta \supseteq \beta'\hat{\ }d$. Then on stage s_1, as β was visited, β' was visited and had outcome d. As β' is not initialized after stage s_1 and permanently extracts a new witness on stage s_2 it must be the case that β' permanently extracts a witness $y_{\beta'}$ from Y and $x_{\beta'}$ was already extracted before or on stage s_1. It follows that between stages s_1 and s_2, β' has selected this new witness $y_{\beta'}$ passing through $II.4$ of the construction and initializing all lower priority strategies including β. This leads again to a contradiction with the assumption that β is not initialized after stage s_1 and hence the claim is correct.

Thus on stage s_1 all witnesses of higher priority strategies that are ever permanently restrained out of either set X or Y are already permanently restrained out of X or Y. On stage s_1 the strategy β selects x_β as a fresh number, i.e. one that has not appeared in the construction sofar. And on stage s_x the witness x_β is permanently restrained out of X.

Now we will prove again inductively but this time on the stage t, that $x_\beta \notin X[t]$ on all stages $t \geq s_x$.

So suppose this is true for $t < s_3$ and that on stage $s_3 > s_x$ a $\mathcal{P}$-strategy α is visited and reaches point $I.5$ of the construction. Suppose α wants to enumerate X_θ or Y_ψ back in X or Y respectively for the axiom $\langle n, A_\theta \cup A_\psi \rangle$ in Γ_α with corresponding entry $\langle n, A_\theta, X_\theta, A_\psi, Y_\psi \rangle$. We have the following cases to consider:

1. Suppose $\alpha > \beta$. If $\alpha >_L \beta\hat{\ }d$ then α is initialized on stage s_x. If $\alpha \subset \beta\hat{\ }d$, then α was initialized on stage s_i and was not accessible before stage s_x. Thus the axiom $\langle n, A_\theta \cup A_\psi \rangle$ was enumerated in Γ_α on stage t with $s_x \leq t < s_3$, on which both $\langle n, A_\theta, X_\theta \rangle$ and $\langle n, A_\psi, Y_\psi \rangle$ were valid i.e. $X_\theta \subseteq X[t]$ and $Y_\psi \subseteq Y[t]$. By induction $x_\beta \notin X[t]$ hence $x_\beta \notin X_\theta$ and thus α does not enumerate x_β back in X.
2. Suppose $\alpha < \beta$. If $\alpha <_L \beta$ then β would be initialized on stage s_3, hence $\alpha \subset \beta$. Suppose the axiom $\langle n, A_\theta \cup A_\psi \rangle$ was enumerated in Γ_α on stage t. If $t \leq s_1$ then by the choice of x_β as a fresh number on stage s_1 we have that $x_\beta \notin X_\theta$. If $t > s_1$ then both $\langle n, A_\theta, X_\theta \rangle$ and $\langle n, A_\psi, Y_\psi \rangle$ were valid on stage t i.e. $X_\theta \subseteq X[t]$ and $Y_\psi \subseteq Y[t]$. By $I.5$ of the construction α will consider all $\mathcal{N}$-strategies that extend it and select the one with highest priority that has permanently restrained an element out of either set X or Y.
   Consider $\beta' < \beta$. By our *Claim* any witness $x_{\beta'}$ or $y_{\beta'}$ of β' that is ever permanently restrained out of X or Y is already restrained out on stage s_1 and by induction on all stages $s \geq s_1$ including on stage t. Thus X_θ and Y_ψ do not contain any witness of β'. As this is true for an arbitrary strategy β' of higher priority than β, if $x_\beta \in X_\theta$ then β will be the strategy selected by α and α will choose to enumerate Y_ψ back in Y.
   Thus again α does not enumerate x_β back in X.

To prove the second part of the lemma suppose y_β is selected on stage s_4 and extracted on stage s_y. Because $s_1 < s_4$ and all strategies of lower priority than β are initialized on stage s_4 the interactions between β and other strategies are dealt with in the same way as in the case when we were considering x_β. The only thing left for us to establish is that β does not come into conflict with itself. So suppose that on stage $s_5 > s_y$ a $\mathcal{P}$-strategy α is visited and reaches point $I.5$ of the construction. Suppose α wants to enumerate X_θ or Y_ψ back in X or Y respectively for the axiom $\langle n, A_\theta \cup A_\psi\rangle$ in Γ_α with corresponding entry $\langle n, A_\theta, X_\theta, A_\psi, Y_\psi\rangle$. We will prove that if $x_\beta \in X_\theta$ then $y_\beta \notin Y_\psi$. Let t be the stage on which the axiom $\langle n, A_\theta \cup A_\psi\rangle$ was enumerated in Γ_α. If $t < s_4$ then $y_\beta \notin Y_\psi$ by the choice of y_β on stage s_4 as a fresh number. If $t \geq s_4 > s_x$ then we have already proved that $x_\beta \notin X[t]$. The axiom $\langle n, A_\theta, X_\theta\rangle$ was valid on stage t, thus $X_\theta \subseteq X[t]$, and hence $x_\beta \notin X_\theta$.

This completes the induction step and the proof of the lemma. □

Lemma 2. *Let α be a $\mathcal{P}$-strategy, visited infinitely often and not initialized after stage s_i. If α performs finitely many cycles then:*

(1)There is a stage $s_n \geq s_i$ after which the value of n_α does not change.

(2) On all α-true stages $t > s_n$, α has either outcome $\langle X, w\rangle$ or outcome $\langle Y, w\rangle$.

(3) There is a stage $s_d \geq s_n$ such that on all α-true stages $t > s_d$, α has the same outcome o .

(4) If $o = \langle X, w\rangle$ then $\Theta_\alpha^{A,X} \neq \overline{K}$ and if $o = \langle Y, w\rangle$ then $\Psi_\alpha^{A,Y} \neq \overline{K}$.

Proof. It follows from the construction and the definition of the action *reset the parameters* that if the value of n_α changes infinitely often, then there will be infinitely many cycles. Thus part (1) of the lemma is true. Let s_n be the stage after which the value of n_α does not change. The only case when the value of the parameter $n_\alpha = n$ is not reset is when $n \notin \overline{K}$ and $n \in \Gamma_\alpha^A[t]$ on all stages t since the last time n was examined on stage s_n^-, thus α will have only outcomes $\langle X, w\rangle$ or $\langle Y, w\rangle$ on all stages after s_n and part (2) is true. It follows from part $I.5$ of the construction and the fact that n_α does not change any longer that on all stage $t > s_n$, $n \in \Gamma_\alpha^A[t]$. From the properties of a good approximation and under these circumstances $n \in \Gamma_\alpha^A$. Then there will be an axiom $\langle n, A_\theta \cup A_\psi\rangle \in \Gamma_\alpha$ which is valid on all but finitely many stages. Select the axiom which is valid longest. This axiom has corresponding entry $\langle n, A_\theta, X_\theta, A_\psi, Y_\psi\rangle$. The strategy α will eventually be able to spot this precise axiom, after possibly finitely many wrong guesses. So after a stage $s_d \geq s_n$ the strategy α will consider this axiom to select its outcome.

On stage s_n either $X_\theta \subset X[s_n]$ or $Y_\psi \subset Y[s_n]$. As we initialize all strategies below outcomes $\langle X, w\rangle$ and $\langle Y, w\rangle$ whenever we reset the parameters, we can be sure that $\mathcal{N}$-strategies visited on stages $t > s_n$ of lower priority than α will not extract any elements of $X_\theta \cup Y_\psi$ from X or Y. Higher priority $\mathcal{N}$-strategies will not extract any elements at all, otherwise α would be initialized. Thus if $X_\theta \subseteq X[s_n]$ then for all stages $t \geq s_n$ we have $X_\theta \subseteq X[t]$ and similarly if $Y_\psi \subseteq Y[s_n]$ then for all stages $t \geq s_n$ we have $Y_\psi \subseteq Y[t]$.

Suppose $X_\theta \subseteq X[s_n]$. Then on stages $t \geq s_d$ the strategy α will always have outcome $\langle X, w\rangle$. The axiom $\langle n, A_\theta, X_\theta\rangle \in \Theta_\alpha$ will be valid on all stages $t \geq s_d$, thus $n \in \Theta^{A,X}$, and $n \notin \overline{K}$.

If $X_\theta \nsubseteq X[s_n]$ then there is a strategy $\beta \supset \alpha$ which is permanently restraining some element x out of X on stage s_n. Then $\beta <_L \alpha\hat{}\langle X, w\rangle$ as strategies extending $\alpha\hat{}\langle X, w\rangle$ or to the right of it are in initial state on stage s_n and do not have any restraints. This strategy β will not be initialized on stages $t \geq s_n$ according part (2) of this lemma and the choice of $s_n > s_i$. By Lemma 1 $x \notin X_t$ on all $t \geq s_n$.

Hence case $I.5.b$ of the construction is valid on all $t \geq s_d$. Thus α will have outcome $\langle Y, w\rangle$ on all stages $t \geq s_d$ and $n \in \Psi^{A,Y}$. This proves parts (3) and (4) of the lemma. □

Lemma 3. *Let α be a $\mathcal{P}$-strategy, visited infinitely often and not initialized after stage s_i. If v is an element such that $\Gamma^A_\alpha(v) = \overline{K}(v)$ then there is a stage s_v after which the outcomes $\langle X, v\rangle$ and $\langle Y, v\rangle$ are not accessible any longer.*

Proof. If α has finitely many cycles then by Lemma 2 there will be a stage s_n after which $\langle X, v\rangle$ and $\langle Y, v\rangle$ are not accessible. Suppose there are infinitely many cycles.

If $v \notin \overline{K}$ then there is a stage s_v at which v exits $\overline{K}$. Then after stage s_v the outcomes $\langle X, v\rangle$ and $\langle Y, v\rangle$ are not accessible.

If $v \in \Gamma^A_\alpha$ then there is an axiom in Γ_α that is valid on all but finitely many stages, say on all stages $t \geq s'_v$. If α is on its k-th cycle during stage s'_v then let s_v be the beginning of the $(k+2)$-nd cycle. Then after stage s_v whenever α considers v part $I.2$ of the construction will be valid and hence α will never have outcome $\langle X, v\rangle$ or $\langle Y, v\rangle$. □

Lemma 4. *Let α be a $\mathcal{P}$-strategy, visited infinitely often and not initialized after stage s_i. If α performs infinitely many cycles, then there is leftmost outcome o that α has on infinitely many stages and*

(1) If $o = \langle X, u\rangle$ then $\Theta^{A,X}_\alpha(u) \neq \overline{K}(u)$.

(2) If $o = \langle Y, u\rangle$ then $\Psi^{A,Y}_\alpha(u) \neq \overline{K}(u)$.

Proof. The set A is not complete by assumption, hence $\Gamma^A_\alpha \neq \overline{K}$. Let u be the least difference between the sets. By Lemma 3 for every $v < u$ the outcomes $\langle X, v\rangle$ and $\langle Y, v\rangle$ are not visited on stages $t > s_v$. Let s_0 be a stage bigger than $\max\{s_v | v < u\}$. As α has infinitely many cycles there will be infinitely many stages $t > s_0$ on which $n_\alpha[t] = u$. If $u \notin \overline{K}$ and $u \in \Gamma^A_\alpha$ then there is a stage $s_1 > s$ such that on all stages $t > s_1$ we have $u \in \Gamma^A_\alpha[t]$ and $u \notin \overline{K}[t]$ and when α considers u on the first stage after s_1, it will never move on to the next element, and α would have finitely many cycles. Hence $u \in \overline{K}$ and $u \notin \Gamma^A_\alpha$.

(1) If $u \notin \Theta^{A,X}_\alpha$ then all axioms for u in Θ_α are invalid on infinitely many stages. Let t be any stage $t \geq s_0$. We will prove that there is a stage $t' \geq t$ on which α has outcome $\langle X, u\rangle$. As $u \notin \Gamma^A_\alpha$ and $\{A_s\}_{s<\omega}$ is a good approximation to A there are infinitely many stages s on which $u \notin \Gamma^A_\alpha[s]$ and hence part $I.3$ of the construction will be valid on infinitely many stages on which we consider u. Let $t_1 \geq t$ on which $n_\alpha[t_1] = u$ and part $I.3$ of the construction is true. If $Ax^\theta_\alpha(u)$ is not defined and we are not able to define it as there is no appropriate axiom in Θ_α valid for long enough then α will have outcome $\langle X, u\rangle$ on stage t_1, hence $t' = t_1$ proves the claim. Otherwise $Ax^\theta_\alpha(u)$ is defined on stage t_1 and by assumption there are infinitely many stages $t \geq t_1$ on which it is invalid. Let $t_2 > t_1$ be the next stage when $Ax^\theta_\alpha(u)$ is invalid and let $t' \geq t_2$ be the first stage

after t_2 on which again $n_\alpha[t'] = u$ and part $I.3$ of the construction is true. By $I.3.b.X$ of the construction α will have outcome $\langle X, u\rangle$ on stage t'.

(2) Now assume that $u \in \Theta_\alpha^{A,X}$. Then there is an axiom $\langle u, A_\theta, X_\theta\rangle \in \Theta_\alpha$ valid on all but finitely many stages. Select the axiom, say Ax, that is valid the longest. Then $Ax_\alpha^\theta(u)$ will have a permanent value Ax after a certain stage s_1. It follows that $u \notin \Psi_\alpha^{A,Y}$ as otherwise we would be able to find an axiom in $\Psi_\alpha^{A,Y}$ valid on all but finitely many stages, and construct an axiom in Γ_α valid on all but finitely many stages. Now a similar argument as the one used in part (1) of this lemma proves that α will have outcome $\langle Y, u\rangle$ on infinitely many stages. □

As an immediate corollary from Lemmas 2, 3 and 4 we obtain the existence of the true path:

Corollary 1. *There exists an infinite path through the tree of strategies with the following properties:*

(1) $\forall n \exists^\infty s[f \upharpoonright n \subseteq \delta_s]$

(2) $\forall n \exists s_l(n) \forall t > s_l(n)[\delta_t \not<_L f \upharpoonright n]$

(3) $\forall n \exists s_i(n) \forall t > s_i(n)[f \upharpoonright n$ *is not initialized on stage* $t]$.

Corollary 2. *X and Y are not c.e.*

Proof. For every requirement $\mathcal{N}_e$ there is an $\mathcal{N}_e$-strategy β along the true path, visited infinitely often and not initialized on any stage $t > s_i$. Let x_β and y_β be the final values of β's witnesses. If $\beta\hat{\ }w \subset f$ then there is an element $u \in \{x_\beta, y_\beta\}$ that never enters W_e. The way each $\mathcal{N}_e$-strategy chooses its witnesses ensures that only β can extract u form either of the sets X or Y. The construction and the definition of the true path ensure that β does not extract u from X and Y on any stage. Hence $u \in X \cap Y$ and $u \notin W_e$.

If $\beta\hat{\ }d \subset f$ then $x_\beta \in W_e$ and there is a β-true stage s_x on which β extracts x_β from X and Y. By Lemma 1 $x_\beta \notin X[t]$ on all stages $t \geq s_x$. If on any stage $t \geq s_x$ we have that $x_\beta \in Y[t]$ then β selects y_β on its next true stage. As the true outcome is d, $y_\beta \in W_e[t']$ on some stage $t' \geq t$. Then on the next β-true stage $s_y \geq t'$ the strategy β will permanently restrain y_β out of Y and by Lemma 1 we have that $y_\beta \notin Y$. □

Corollary 3. *$A \oplus X \neq \overline{K}$ or $A \oplus Y \neq \overline{K}$.*

Proof. Consider the $\mathcal{P}$-nodes on the true path. From the definition of the tree it follows that either for every $\mathcal{P}_e^0$-requirement there is a node on the tree α which is associated with $\mathcal{P}_e^0$ or there is a fixed requirement $\mathcal{P}_e^0$ associated with all but finitely many nodes. In the latter case there is a node on the true path for every $\mathcal{P}_e^1$-requirement.

Suppose there is a node on the tree for each $\mathcal{P}_e^0$-requirement then $A \oplus X \neq \overline{K}$. Assume for a contradiction $\Theta_e^{A,X} = \overline{K}$ and let $\alpha \subset f$ be the last node associated with $\mathcal{P}_e^0$. Then α has true outcome $\langle X, u\rangle$ for some $u \in \omega \cup \{w\}$. It follows from Lemma 2 and Lemma 4 that $\Theta_e^{A,X} \neq \overline{K}$.

The case when there is a node for every $\mathcal{P}_e^1$-requirement yields by a similar argument that $A \oplus Y \neq \overline{K}$. □

Lemma 5. *The sets X and Y are 3-c.e.*

Proof. We can easily obtain a 3-c.e. approximation of each of the sets X and Y from the one constructed. Define $\hat{X}_s = X_s \upharpoonright s$ and $\hat{Y}_s = Y_s \upharpoonright s$.

It follows from the construction that elements extracted from X and Y are necessarily witnesses of $\mathcal{N}$-strategies. An $\mathcal{N}$-strategy extracts a witness n only once on its entry in the approximation of a fixed c.e. set, hence necessarily after stage $n+1$. The $\mathcal{P}$-strategies only ever enumerate elements back in the sets X and Y, by their definition.

Suppose therefore that n is the witness x_β for an $\mathcal{N}$-strategy β. Then n appears in the defined approximations $\{\hat{X}_s\}_{s<\omega}$ and $\{\hat{Y}_s\}_{s<\omega}$ on stage $n+1$. If β never extracts x_β then we are done - as no other strategy can extract it. If β extracts x_β then it does so only once on stage s_x when it goes through $II.3$ and moves on to $II.4$ on the next stage. In order for β to return to step $II.3$ of the construction it will have to be initialized and will select new witnesses. Thus after its extraction on stage s_x from both $\hat{X}_{s_x}$ and $\hat{Y}_{s_x}$, the number x_β can only be enumerated back in either set and hence $|\{s\ |\hat{X}_{s-1}(x_\beta) \neq \hat{X}_s(x_\beta)\}| \leq 3$ and $|\{s\ |\hat{Y}_{s-1}(x_\beta) \neq \hat{Y}_s(x_\beta)\}| \leq 3$.

If n is the witness y_β then it will never be extracted from X thus $|\{s\ |\hat{X}_{s-1}(y_\beta) \neq \hat{X}_s(y_\beta)\}| = 1$. If it is ever extracted from Y it is extracted only once by β on the first stage it reaches step $II.6$. After that y_β is already restrained by β and whenever β executes step $II.6$ it will ignore the first sentence of the instruction and just have outcome $o = d$. Thus again $|\{s\ |\hat{Y}_{s-1}(y_\beta) \neq \hat{Y}_s(y_\beta)\}| \leq 3$. □

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References

1. S. B. Cooper, *Partial degrees and the density problem*, J. Symb. Log. **47** (1982), 854-859.
2. S. B. Cooper, *Partial Degrees and the density problem. part 2: the enumeration degrees of the Σ_2 sets are dense*, J. Symb. Log. **49** (1984), 503-513.
3. S. B. Cooper, *Enumeration reducibility, nondeterminitsic computations and relative computability of partial functions*, Recursion Theory Week, Oberwolfach 1989, Lecture Notes in Mathematics **1432** (1990), 57-110.
4. S. B. Cooper, *Computability Theory*, Chapman & Hall/CRC Mathematics, Boca Raton, FL, New York, London, 2004.
5. S.B. Cooper, *On a theorem of C.E.M. Yates*, handwritten notes, 1973.
6. S. B. Cooper, A. Sorbi, X. Yi, *Cupping and noncupping in the enumeration degrees of Σ_2^0 sets*, Ann. Pure Appl. Logic **82** (1996), 317-342.
7. A. H. Lachlan and R. A. Shore, *The n-rea enumeration degrees are dense*, Arch. Math. Logic **31** (1992), 277–285.
8. D. Posner, R. Robinson, *Degrees joining to* $0'$, J. Symb. Log. **46** (1981), 714-722.
9. R. I. Soare, *Recursively enumerable sets and degrees*, Springer-Verlag, Berlin, Heidelberg, London, New York, Paris, Tokyo, 1987.

10. M. Soskova, G. Wu, *Cupping Δ^0_2 enumeration degrees to* $0'$, Computability in Europe 2007, Lecture Notes in Computer Science **4497** (2007), 727-738.