Cupping Classes of Σ_2^0 Enumeration Degrees

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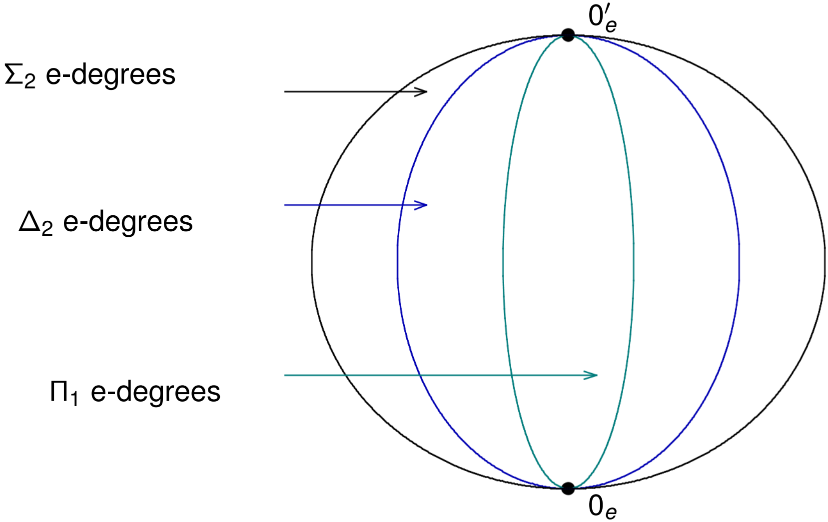
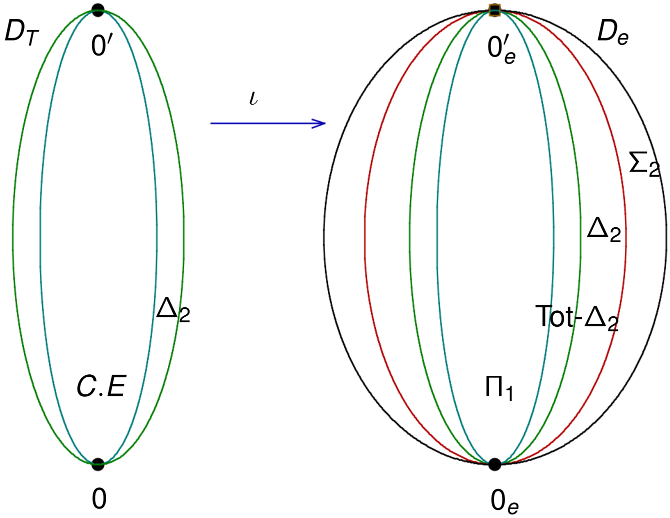
The local structure of the enumeration degrees



Transferring results from the Turing degrees

There is a natural embedding of the Turing degrees in the enumeration degrees. The images of Turing degrees under this embedding are the total e-degrees.



Cupping

We say that a degree **a** is cuppable if there exists a degree $\mathbf{b} < \mathbf{0}'_{\mathbf{e}}$ such that $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'_{\mathbf{e}}$.

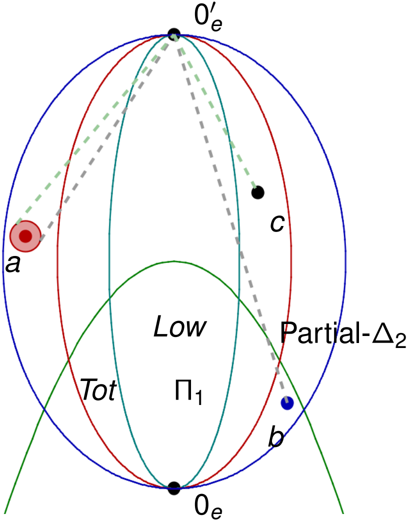
- Negative Results:
  (Cooper, Sorbi, Yi): There exists a nonzero Σ_2 enumeration degree that is not cuppable.
- Positive Results:
  (Cooper, Sorbi and Yi): Every nonzero Δ_2 e-degree is cuppable by a total incomplete Δ_2 e-degree.

  (S, Wu): Every nonzero Δ_2 e-degree is cuppable by a partial and low Δ_2 e-degree.

Cupping partners

Question

How much further can we limit the the search for cupping partners.



Reaching the first limit

Theorem

For every uniform sequence of incomplete Δ_2 enumeration degrees $\{\mathbf{a_n}\}_{n<\omega}$ there is a non-zero Δ_2 enumeration degree $\mathbf{b}$ such that $\mathbf{a_n} \cup \mathbf{b} \lneqq \mathbf{0_e'}$ for every n.

Proof: The Construction of a non-cuppable Σ_2 enumeration degree carried out against a uniform sequence of incomplete Δ_2 enumeration degrees.

Proof sketch

- Let $\{A_n\}_{n<\omega}$ be a list of representatives of the given enumeration degrees.
- Let $\{A_{n,s}\}_{s<\omega}$ be a good Δ_2 approximation to A_n.
  - $(\overset{\infty}{\exists} s)(A_s \subseteq A)$.
  - $Lim_s\, A_s(x) \downarrow$.

Proof sketch

We shall construct a Δ_2 set B satisfying the following requirements:

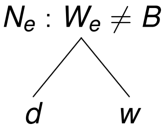
- For every natural number e we have a requirement:

$$N_e : W_e \neq B.$$

- For every j and every n we will have a requirement :

$$P_{j,n} : \Theta_j^{A_n,B} \neq \overline{K}.$$

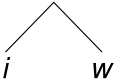
The N-strategy



- Select a witness x as a fresh number.
- If $x \notin W_e$ - do nothing (outcome w)
- If $x \in W_e$ then extract x from B (outcome d)

The P-strategy

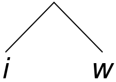
$$P_{j,n} : \Theta_j^{A_n,B} \neq \overline{K}$$



- Construct an e-operator Γ threatening to prove that $\Gamma^{A_n} = \overline{K}$.
- Perform cycles k of increasing length, monitoring each number $n < k$.

The P-strategy

$$P_{j,n} : \Theta_j^{A_n,B} \neq \overline{K}$$



$\underline{n \in \overline{K}}$: Search for an axiom in Θ_j that is valid on almost all stages. $Ax(n) = \langle n, D_A, D_B \rangle$.

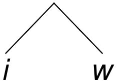
Valid $Ax(n)$ Enumerate $\langle n, D_A \rangle$ in Γ, go on to $n+1$.

Invalid $Ax(n)$ Then outcome i. Redefine $Ax(n)$, move on to $n+1$.

- Infinitely many times outcome $i \Rightarrow n \notin \Theta_j^{A_n,B}$.

The P-strategy

$$P_{j,n} : \Theta_j^{A_n,B} \neq \overline{K}$$



$\underline{n \notin \overline{K}}$ Rectify $\Gamma^A(n)$.

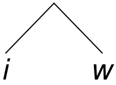
Incorrect For each axiom $\langle n, D_A \rangle \in \Gamma$, enumerate D_B back in B, outcome is w. Do not move on to next element.

- On all but finitely many stages: outcome $w \Rightarrow n \in \Theta_j^{A_n,B}$.

Correct Looks like $n \notin \Gamma^A$, restore B and go on to $n+1$.

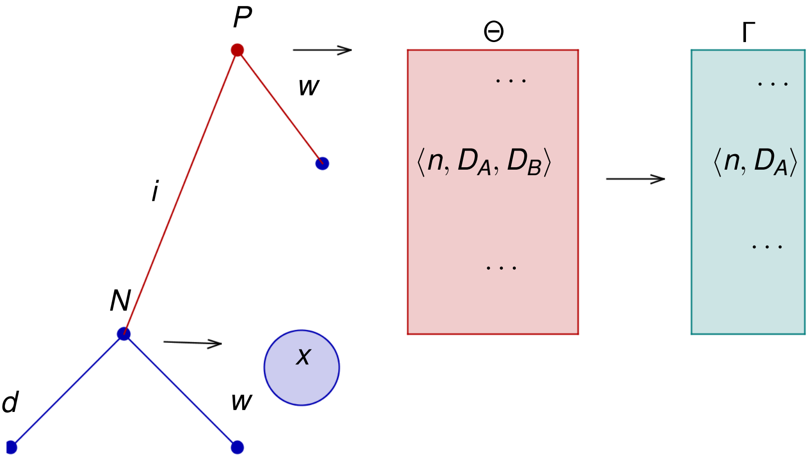
The P-strategy

$$P_{j,n} : \Theta_j^{A_n,B} \neq \overline{K}$$



- A_n is incomplete. Hence $\Gamma^{A_n} \neq \overline{K}$. Let n be the least difference.
- If $n \in \overline{K} \setminus \Gamma^{A_n}$ then Θ_j has failed to provide us with a valid axiom. Infinitely often - outcome i.
- If $n \in \Gamma^{A_n} \setminus \overline{K}$ then we have restored an axiom in Θ_j and it is valid forever. Cofinitely often outcome w.

The set B is Δ_2

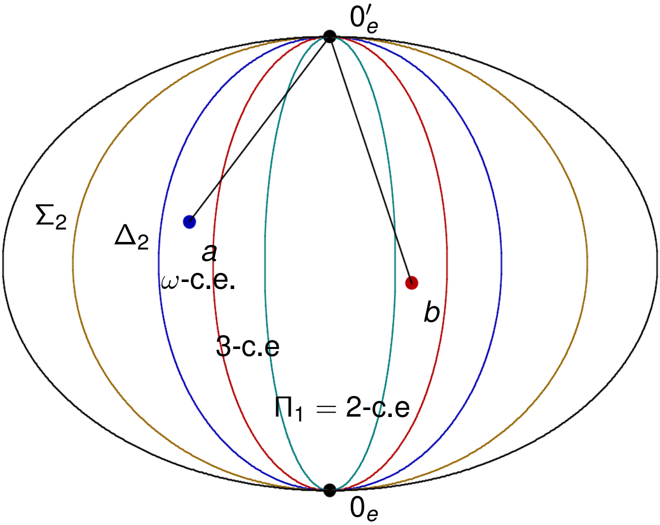


Looking at the local structure more closely

Definition

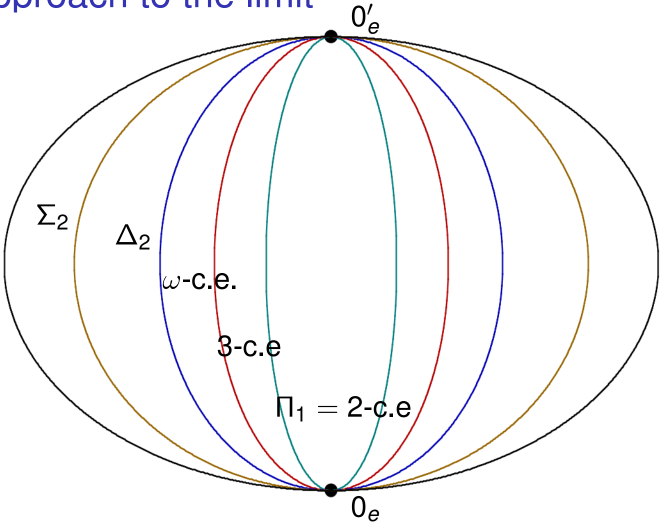
1. A set A is n-c.e. if there is a computable function f such that for each x, $f(x, 0) = 0$, $|\{s + 1 \mid f(x, s) \neq f(x, s + 1)\}| \leq n$ and $A(x) = \lim_s f(x, s)$.
2. A is ω-c.e. if there are two computable functions $f(x, s), g(x)$ such that for all x, $f(x, 0) = 0$, $|\{s + 1 \mid f(x, s) \neq f(x, s + 1)\}| \leq g(x)$ and $\lim_s f(x, s) \downarrow= A(x)$.
3. A degree **a** is n-c.e.(ω-c.e.) if it contains a n-c.e.(ω-c.e.) set.

Looking at the local structure more closely



Wu, S: For every non-zero ω-c.e. enumeration degree **a** there exists an incomplete 3-c.e. enumeration degree **b** that cups **a.**

Another approach to the limit



(Cooper, Seetapun and Li): In the Turing degrees there exists a single incomplete Δ_2 Turing degree d that cups every non-zero c.e. Turing degree.

Can we find a similar result for bigger classes?

The second limit

Theorem

For every incomplete Σ_2 enumeration degree $\mathbf{a}$ there exists a non-zero 3-c.e. enumeration degree $\mathbf{b}$ such that $\mathbf{a}$ does not cup $\mathbf{b}$.

Proof: Let A be a representative of the given Σ_2 e-degree with good approximation $\{A_s\}$. We shall construct two 3-c.e. sets X and Y so that one of them will have the required properties.

Requirements

- For every natural number e we have a requirement:

$$\mathcal{N}_e : W_e \neq X \wedge W_e \neq Y.$$

- For every i we will have a pair of requirements:

$$\mathcal{P}_i^0 : \Theta_i^{A,X} \neq \overline{K}.$$

$$\mathcal{P}_i^1 : \Psi_i^{A,Y} \neq \overline{K}.$$

We will ensure that: $(\forall i)(\mathcal{P}_i^0) \vee (\forall i)(\mathcal{P}_i^1)$.

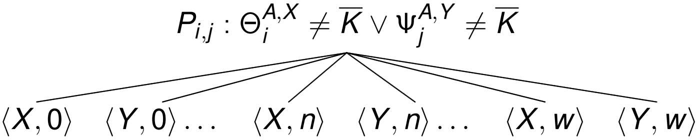
The P-strategy

$$P_{i,j} : \Theta_i^{A,X} \neq \overline{K} \vee \Psi_j^{A,Y} \neq \overline{K}$$

$$\langle X, 0 \rangle \quad \langle Y, 0 \rangle \ldots \quad \langle X, n \rangle \quad \langle Y, n \rangle \ldots \quad \langle X, w \rangle \quad \langle Y, w \rangle$$

- Construct an e-operator Γ threatening to prove that A is complete.
- Run cycles k scanning each element $n < k$. For every element n act as in the previous construction.

The P-strategy



$\underline{n \in \overline{K}}$: Search for a valid $Ax_\theta(n) = \langle n, D_{A,\theta}, D_X \rangle$ and $Ax_\psi = \langle n, D_{A,\psi}, D_Y \rangle$.

Invalid $Ax_\theta(n)$	Then outcome $\langle X, n \rangle$. Redefine $Ax_\theta(n)$, move on to $n+1$.
Invalid $Ax_\psi(n)$	Then outcome $\langle Y, n \rangle$. Redefine $Ax_\psi(n)$, move on to $n+1$.
Valid	Enumerate $\langle n, D_{A,\theta} \cup D_{A,\psi} \rangle$ in Γ, go on to $n+1$.

The P-strategy

$$P_{i,j} : \Theta_i^{A,X} \neq \overline{K} \vee \Psi_j^{A,Y} \neq \overline{K}$$

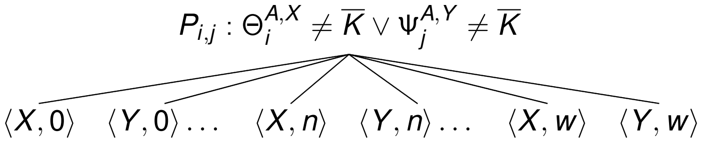
$$\langle X, 0 \rangle \quad \langle Y, 0 \rangle \ldots \quad \langle X, n \rangle \quad \langle Y, n \rangle \ldots \quad \langle X, w \rangle \quad \langle Y, w \rangle$$

$\underline{n \notin \overline{K}}$ Rectify $\Gamma^A(n)$.

Incorrect For each axiom $\langle n, D_{A,\theta} \cup D_{A,\psi} \rangle \in \Gamma$, enumerate D_X back in X or D_Y back in Y.
Choose the axiom for n valid the longest in Γ.
If Θ was restored: outcome $\langle X, w \rangle$.
If Ψ was restored: outcome $\langle Y, w \rangle$.

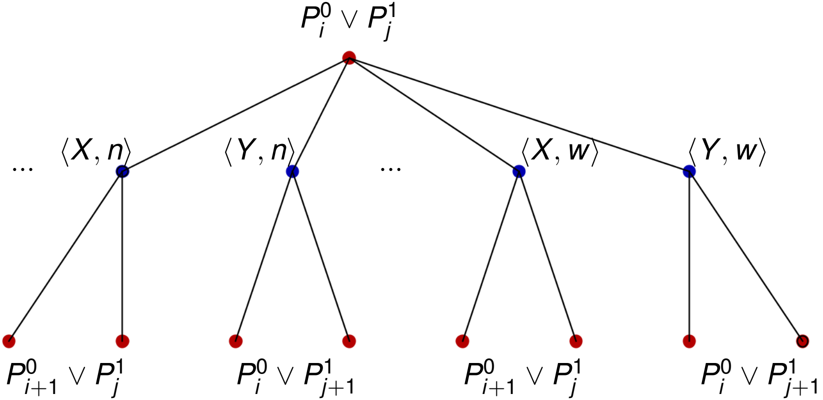
Correct Looks like $n \notin \Gamma^A$ then go on to $n + 1$.

The P-strategy



- A is incomplete. Hence $\Gamma^A \neq \overline{K}$. Let n be the least difference.
- After a certain stage s outcomes $\langle X, m \rangle$ and $\langle Y, m \rangle$ are not accessible.
- If $n \in \overline{K} \backslash \Gamma^A$ then
  - Θ_i has failed to provide us with a valid axiom.
  - Ψ_j has failed to provide us with a valid axiom.
- If $n \in \Gamma^A \backslash \overline{K}$ then
  - We have restored an axiom in Θ_i and it is valid forever.
  - We have restored an axiom in Ψ_j and it is valid forever.

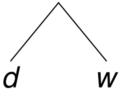
The tree



$P_i^0 \vee P_j^1$
...
$\langle X, n \rangle$
$\langle Y, n \rangle$
...
$\langle X, w \rangle$
$\langle Y, w \rangle$
$P_{i+1}^0 \vee P_j^1$
$P_i^0 \vee P_{j+1}^1$
$P_{i+1}^0 \vee P_j^1$
$P_i^0 \vee P_{j+1}^1$

The N-strategy

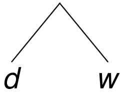
$$N_e : W_e \neq X \wedge W_e \neq Y$$



- Select a witness x as a fresh number.
- If $x \notin W_e$ - do nothing (outcome w)
- If $x \in W_e$ then extract x from both X and Y (outcome d)

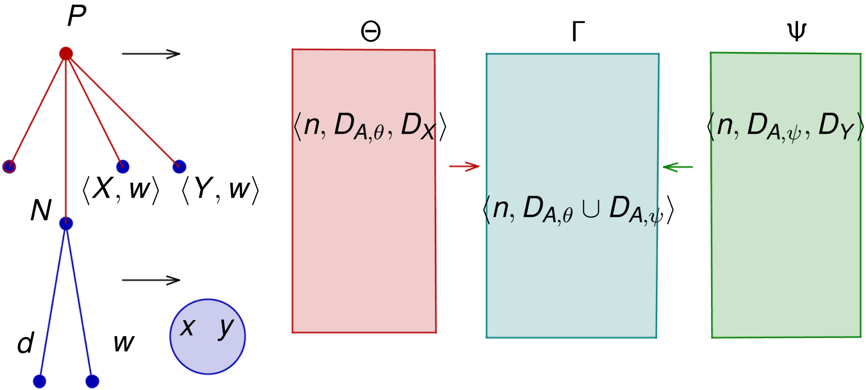
The N-strategy

$$N_e : W_e \neq X \wedge W_e \neq Y$$



- Permanently restrain x out of X but allow it to be enumerated back in Y.
- Select a second witness y - one that does not appear in any axiom seen sofar in the construction.
- If $y \notin W_e$ then - do nothing (outcome w)
- If $y \in W_e$ then extract and permanently restrain y from Y (outcome d)

Conflicts resolved



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