Definability in the local structure of the enumeration degrees

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Preliminaries: The enumeration degrees

Definition

- $A \leq_e B$ iff there is a c.e. set W, such that $A = W(B) = \{x \mid \exists u(\langle x, u\rangle \in W \wedge D_u \subseteq B)\}$.
- $d_e(A) = \{B \mid A \leq_e B \;\&\; B \leq_e A\}$
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- $\mathbf{0}_e = d_e(\emptyset) = \{W \mid W \text{ is c.e. }\}$.
- $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- $d_e(A)' = d_e(A')$, where $A' = L_A \oplus \overline{L_A}$ and $L_A = \{x \mid x \in W_x(A)\}$.
- $\mathcal{D}_e = \langle D_e, \leq, \vee, ', \mathbf{0}_e\rangle$ is an upper semi-lattice with jump operation and least element.

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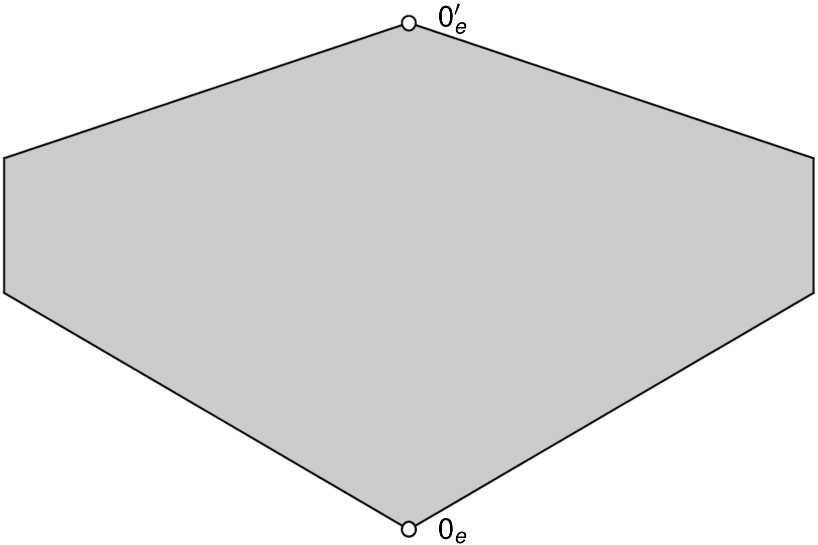
The total degrees

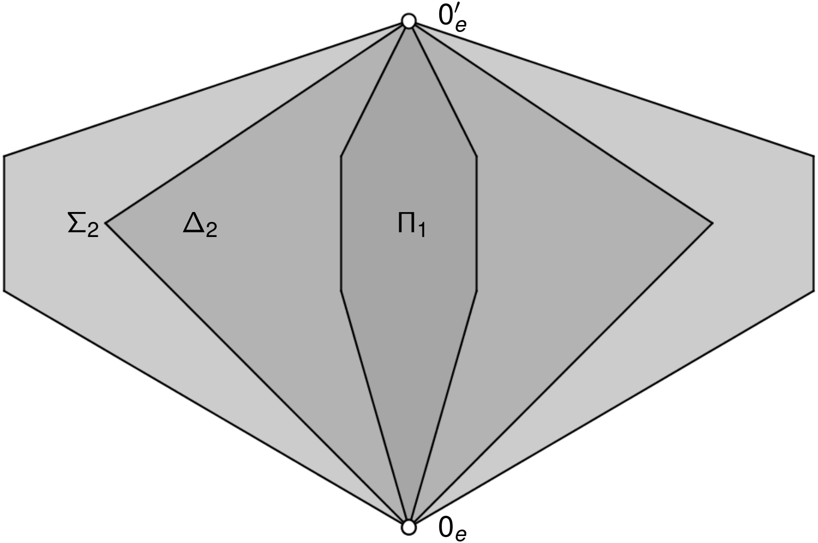
Proposition

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, preserves the order, the least upper bound and the jump operation:

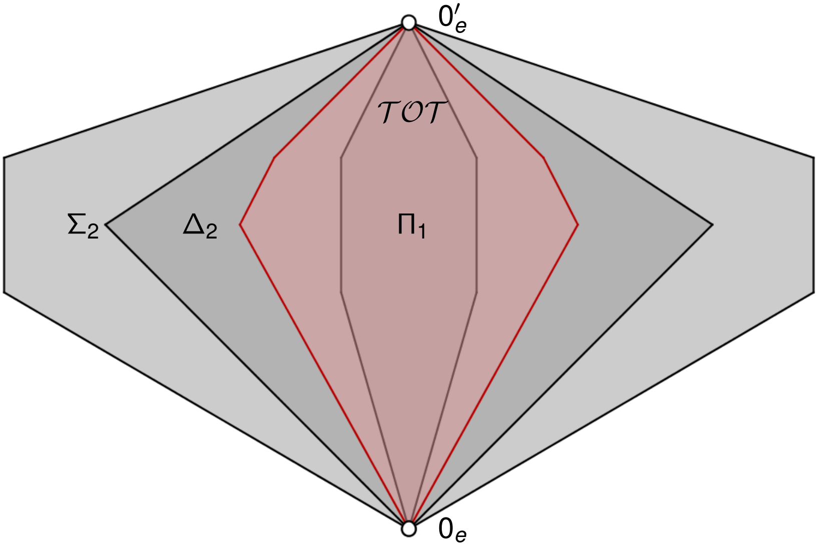
The sub structure of the total e-degrees is defined as $\mathcal{TOT} = \iota(D_T)$.



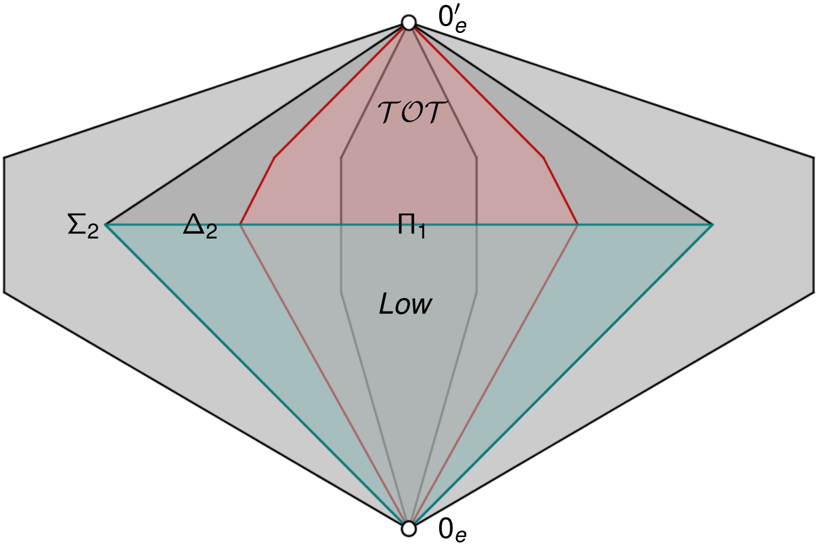
The local structure of the enumeration degrees $\mathcal{G}_e = \mathcal{D}_e(\leq 0'_e)$ consists of all degrees below $\mathbf{0}'_e$.



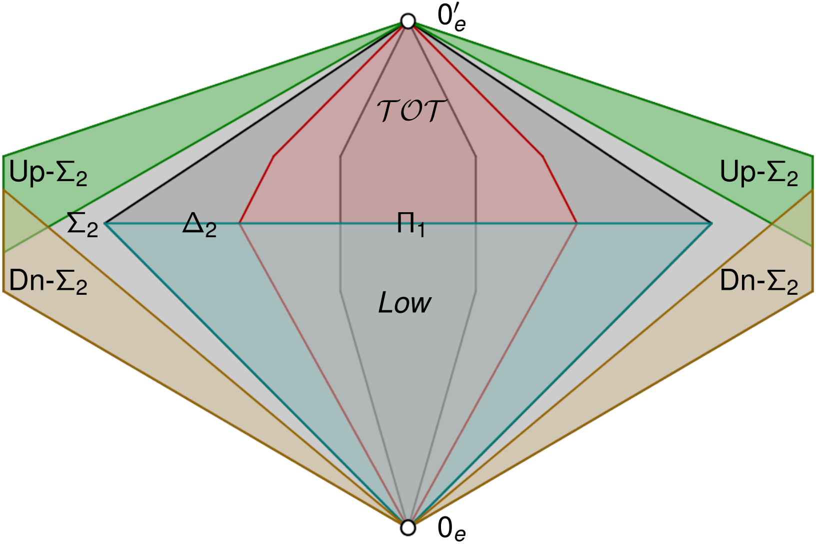
With respect to the arithmetic hierarchy the degrees can be partitioned into three classes.



The total degrees below $\mathbf{0}'_e$ are images of the Turing degrees below $\mathbf{0}'$.
Every total degree is Δ^0_2, but not all Δ^0_2 are total.



A degree is low if its jump is as low as possible: $\mathbf{0}'_e$. Every low degree is Δ^0_2.



The upwards properly Σ^0_2 have no incomplete Δ^0_2 above them. The downwards properly Σ^0_2 have no nonzero Δ^0_2 below them.

$\mathcal{K}$-pairs

Iskander Kalimullin: Definability of the jump operator in the enumeration degrees
Journal of Mathematical Logic (2003)

Definition

Let A, B be sets of natural numbers. The pair (A, B) is a $\mathcal{K}$-pair if there exists a c.e. set W, such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

$\mathcal{K}$-pairs: A trivial example

Example

Let V be a c.e set. Then (V, A) is a $\mathcal{K}$-pair (over $\emptyset$) for any set of natural numbers A.

Let $W = V \times \mathbb{N}$. Then $V \times A \subseteq W$ and $\overline{V} \times \overline{A} \subseteq \overline{W}$.

We will only be interested in non-trivial $\mathcal{K}$-pairs.

$\mathcal{K}$-pairs: A more interesting example

Definition (Jockusch)

A set of natural numbers A is semi-recursive if there is a computable function s_A such that for every pair of natural numbers (x, y):

1. $s_A(x, y) \in \{x, y\}$.
2. If $x \in A$ or $y \in A$ then $s_A(x, y) \in A$.

Theorem (Jockusch)

For every noncomputable set B there is a semi recursive set $A \equiv_T B$ such that both A and $\overline{A}$ are not c.e.

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$\mathcal{K}$-pairs: A more interesting example

Example

Let A be a semi-recursive set. Then $(A, \overline{A})$ is a $\mathcal{K}$-pair.

Proof:

$$\bar{s}_A(x,y) = \begin{cases} x \text{ , if } s_A(x,y) = y \\ y \text{ , if } s_A(x,y) = x. \end{cases} .$$

Let $W = \{\langle s_A(x,y), \bar{s}_A(x,y)\rangle \mid x, y \in \mathbb{N}\}$.

Then $A \times \overline{A} \subseteq W$ and $\overline{A} \times \overline{\overline{A}} = \overline{A} \times A \subseteq \overline{W}$.

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An order theoretic characterization of $\mathcal{K}$-pairs

Kalimullin has proved that the property of being a $\mathcal{K}$-pair is degree theoretic and first order definable in $\mathcal{D}_e$.

Theorem (Kalimullin)

(A, B) is a $\mathcal{K}$-pair if and only if the degrees $\mathbf{a} = d_e(A)$, $\mathbf{b} = d_e(B)$ have the following property:

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x} \in \mathcal{D}_e)((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x}).$$

Properties of $\mathcal{K}$-pairs in the local structure

Let A and B be Σ^0_2 degree which form a nontrivial $\mathcal{K}$-pair.

1. $A \leq_e \overline{B}$ and $\overline{A} \leq_e \overline{K}$.

   Fix x and let $W^{[x]} = \{y \mid \langle x, y\rangle\} \in W$.

   If $x \in A$ then $B \subset W^{[x]}$.

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   $x \in A$ iff $\exists y(y \in W^{[x]} \;\&\; y \in \overline{B})$.

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   Let $\mathbf{a} = d_e(A), \mathbf{b} = d_e(B)$ and $\mathbf{c} = d_e(C)$.

   Then for all $\mathbf{x}$:

$$\mathbf{x} \leq (\mathbf{c} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) \leq (\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x}.$$

3. The degrees of A and B form a minimal pair.

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3. The degrees of A and B form a minimal pair.
4. The enumeration degrees of A and B are quasi minimal, i.e. the only total degree bounded by either of them is $\mathbf{0}_e$.

   Assume towards a contradiction that $C \oplus \overline{C} \leq_e A$.

   Then $C \oplus \overline{C}$ and B form a $\mathcal{K}$-pair.

   So $B \leq_e \overline{C \oplus \overline{C}} = \overline{C} \oplus C \leq_e A$.

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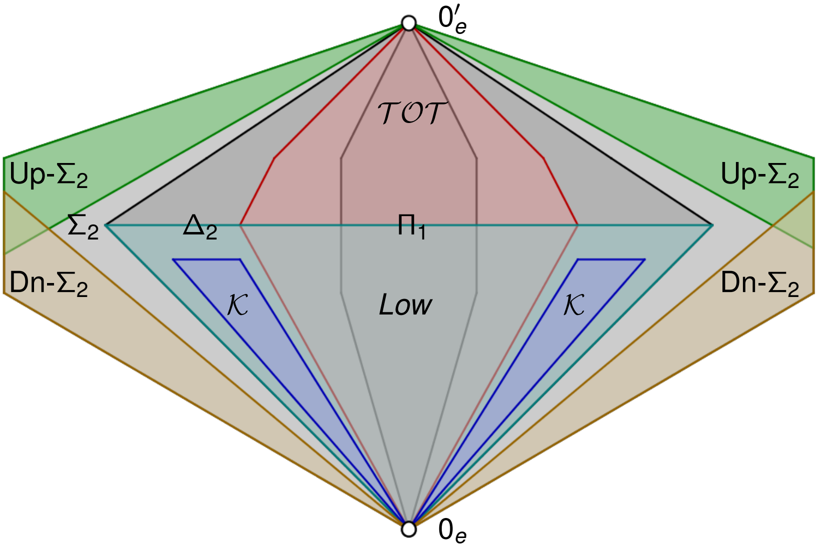

The $\mathcal{K}$-pairs in the local structure $\mathcal{G}_e$.

Local definability of $\mathcal{K}$-pairs

$$\mathcal{K}(\mathbf{a},\mathbf{b}) \leftrightharpoons (\forall \mathbf{x})((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x})$$

Is it enough to require that this formula is satisfied by all Σ^0_2 e-degrees?

Theorem (Kalimullin)

If (A, B) is not a $\mathcal{K}$-pair then there is a witness C computable from $A \oplus B \oplus K$ such that:

$$(d_e(A) \vee d_e(C)) \wedge (d_e(B) \vee d_e(C)) \neq d_e(C)$$

- If $\mathbf{a}$ and $\mathbf{b}$ are Δ^0_2 then C is also Δ^0_2 and $\mathcal{K}(\mathbf{a},\mathbf{b})$ ensures "$\mathbf{a}$ and $\mathbf{b}$ are a true $\mathcal{K}$-pair".
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Definition

A Σ^0_2 enumeration degree $\mathbf{a}$ is called *cuppable* if there is an incomplete Σ^0_2 e-degree $\mathbf{b}$, such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$.
If furthermore $\mathbf{b}$ is low, then $\mathbf{a}$ will be called *low-cuppable*.

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Step 1: Inspired by non-splitting

Theorem

If $\mathbf{u}$ *and* $\mathbf{v}$ *are* Σ^0_2 *enumeration degrees such that* $\mathbf{u} \vee \mathbf{v} = \mathbf{0}'_e$ *then* $\mathbf{u}$ *is low-cuppable or* $\mathbf{v}$ *is low-cuppable.*

Proof:

Uses a construction very similar to the construction of a non-splitting enumeration degree.

Theorem (S)

There is a degree $\mathbf{a} < \mathbf{0}'_e$ *such that for every pair of* Σ^0_2 *degrees* $\mathbf{u}$ *and* $\mathbf{v}$ *with* $\mathbf{u} \vee \mathbf{v} = \mathbf{0}'_e$ *then* $\mathbf{a}$ *cups* $\mathbf{u}$ *or* $\mathbf{v}$ *to* $\mathbf{0}'_e$.

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Theorem (Cooper, Sorbi, Yi)

Every nonzero Δ^0_2 degree is cuppable.

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Theorem (G, S)

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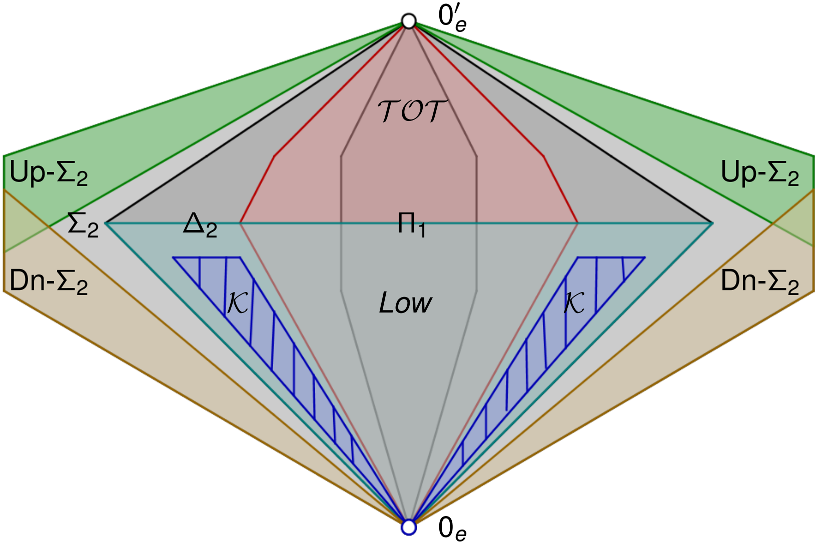

The first example of a definable class of degrees in the local structure: $\mathcal{K}$-pairs.

An easy consequence

If **a** bounds a nonzero Δ^0_2 degree then it bounds a nontrivial $\mathcal{K}$-pair.

If **a** is a downwards properly Σ^0_2 degree, then it bounds no $\mathcal{K}$-pair.

Corollary

The class of downwards properly Σ^0_2 is first order definable in $\mathcal{G}_e$ by the formula:

$$\mathcal{DP}\Sigma^0_2(\mathbf{x}) \rightleftharpoons \forall \mathbf{b}, \mathbf{c}[(\mathbf{b} \leq \mathbf{x} \mathbin{\&} \mathbf{c} \leq \mathbf{x}) \Rightarrow \neg \mathcal{LK}(\mathbf{b}, \mathbf{c})].$$

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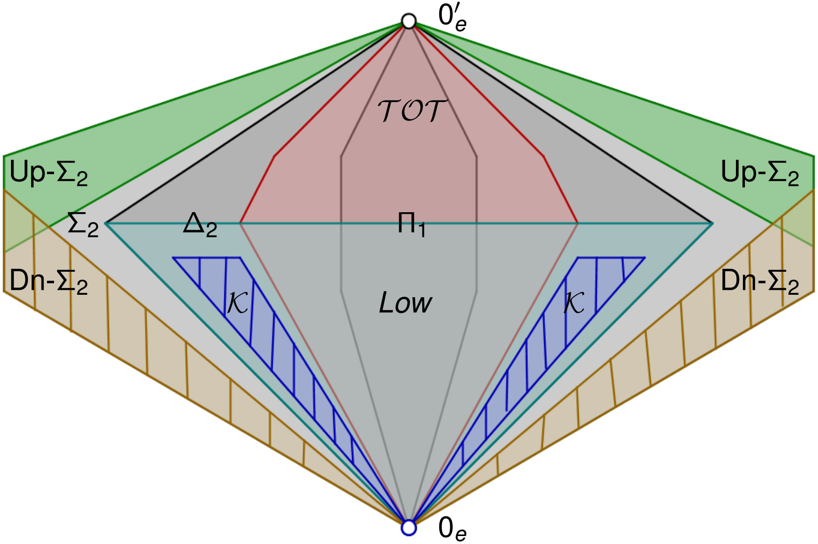

The second example of a definable class of degrees in the local structure: Downwards properly Σ^0_2 degrees.

The upwards properly Σ^0_2 degrees

Definition

$\mathbf{x}$ is upwards properly Σ^0_2 every $\mathbf{y} \in [\mathbf{x}, \mathbf{0}'_e)$ is properly Σ^0_2.

Theorem (Jockusch)

For every noncomputable set B there is a semi recursive set $A \equiv_T B$ such that both A and $\overline{A}$ are not c.e.

Corollary

Every nonzero total enumeration degree can be represented as the least upper bound of a nontrivial $\mathcal{K}$-pair.

The upwards properly Σ_2^0 degrees

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The upwards properly Σ^0_2 degrees

Theorem (Arslanov, Cooper, Kalimullin)

For every Δ^0_2 enumeration degree $\mathbf{a} < \mathbf{0}'_e$ there is a total enumeration degree $\mathbf{b}$ such that $\mathbf{a} \leq \mathbf{b} < \mathbf{0}'_e$.

So a degree $\mathbf{a}$ is upwards properly Σ^0_2 if and only if no element above it other than $\mathbf{0}'_e$ can be represented as the least upper bound of a nontrivial $\mathcal{K}$-pair.

Corollary

The class of upwards properly Σ^0_2 is first order definable in $\mathcal{G}_e$ by the formula :

$$\mathcal{U}P\Sigma^0_2(\mathbf{x}) \rightleftharpoons \forall \mathbf{c}, \mathbf{d}(\mathcal{LK}(\mathbf{c}, \mathbf{d}) \;\&\; \mathbf{x} \leq \mathbf{c} \vee \mathbf{d} \Rightarrow \mathbf{c} \vee \mathbf{d} = \mathbf{0}'_e).$$

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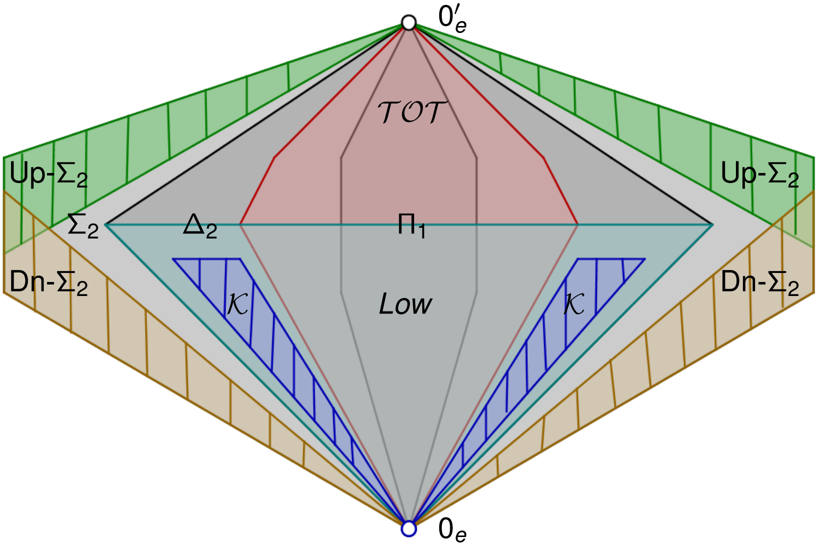

The third example of a definable class of degrees in the local structure: Upwards properly Σ^0_2 degrees.

Semi-recursive sets revisited

1. $A \leq_e \overline{B}$ and $\overline{A} \leq_e \overline{K}$.
2. If $C \leq_e A$ the C and B form a $\mathcal{K}$-pair.

Consider a nontrivial $\mathcal{K}$-pair of a semi recursive set and its complement: $\{A, \overline{A}\}$.

Assume that there is a $\mathcal{K}$-pair $\{C, D\}$ such that $A <_e C$ and $\overline{A} <_e D$.

By property (2) A forms a $\mathcal{K}$-pair with D.

By property (1) $D \leq_e \overline{A}$.

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Maximal $\mathcal{K}$-pairs

Definition

We say that $\{A, B\}$ is a maximal $\mathcal{K}$-pair if there is no $\mathcal{K}$-pair $\{C, D\}$, such that $A <_e C$ and $B <_e D$.

Corollary

Every nonzero total set is enumeration equivalent to the join of a maximal $\mathcal{K}$-pair.

Goal

Prove that the join of every maximal $\mathcal{K}$-pair is e-equivalent to a total set.

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Extending $\mathcal{K}$-pairs to maximal

Theorem

For every nontrivial Δ^0_2 $\mathcal{K}$-pair $\{A, B\}$ there is a $\mathcal{K}$-pair $\{C, \overline{C}\}$, such that $A \leq_e C$ and $B \leq_e \overline{C}$.

Basic tool: A dynamic characterization of $\mathcal{K}$-pairs

Lemma (Kalimullin)

A pair of non-c.e. Δ^0_2 sets A, B is a $\mathcal{K}$-pair if and only if there are Δ^0_2 approximations $\{A_s\}_{s<\omega}$ to A and $\{B_s\}_{s<\omega}$ to B, such that:

$$\forall s(A_s \subseteq A \ \vee \ B_s \subseteq B).$$

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Local definability of the total degrees

Denote by $\mathcal{MK}(\mathbf{x}, \mathbf{y})$ the first order formula that defines in $\mathcal{G}_e$ the set of degrees of maximal $\mathcal{K}$-pairs.

Corollary

The class of total degrees is first order definable in $\mathcal{G}_e$ by the formula:

$$\mathcal{TOT}(\mathbf{x}) \rightleftharpoons \mathbf{x} = \mathbf{0}_e \ \vee \ \exists \mathbf{c} \exists \mathbf{d} [\mathcal{MK}(\mathbf{c}, \mathbf{d}) \ \& \ \mathbf{x} = \mathbf{c} \vee \mathbf{d}.]$$

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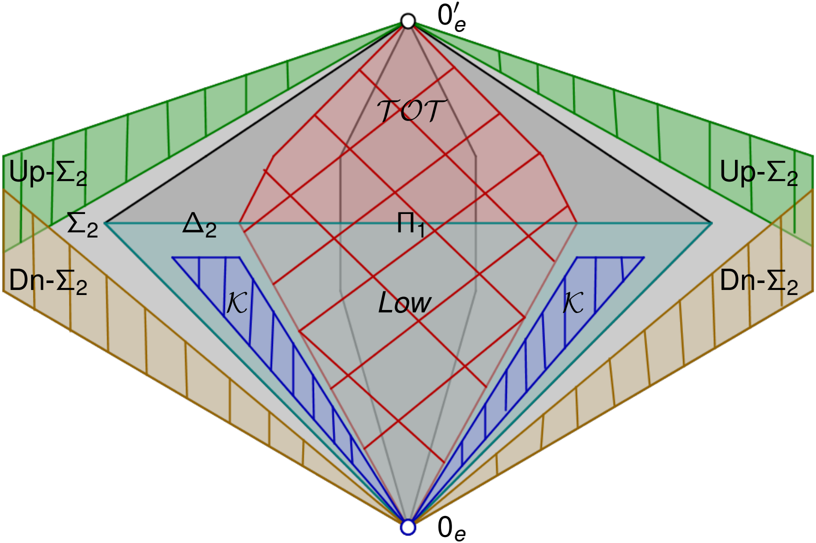

The fourth example of a definable class of degrees in the local structure: The total degrees.

One final consequence

Theorem (Giorgi, Sorbi, Yang)

Every non-low total degree bounds a downwards properly Σ^0_2 enumeration degree.

Corollary

The class of low total e-degrees is first order definable in $\mathcal{G}_e$ by the formula:

$$\mathcal{TL}(\mathbf{x}) \rightleftharpoons \mathcal{TOT}(\mathbf{x})\ \&\ \forall \mathbf{c} \leq \mathbf{x}[\neg \mathcal{D}P\Sigma^0_2(\mathbf{c})]$$

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One final consequence

Theorem (Soskov)

For every enumeration degree $\mathbf{x}$ *there is a total enumeration degree* $\mathbf{y}$*, such that* $\mathbf{x} < \mathbf{y}$ *and* $\mathbf{x}' = \mathbf{y}'$*.*

Thus a Σ^0_2 enumeration degree is low if and only if there is a low total Σ^0_2 enumeration degree above it.

Theorem (G, S)

The class of low e-degrees is first order definable in $\mathcal{G}_e$ *by the formula:*

$$\mathcal{LOW}(\mathbf{x}) \rightleftharpoons \exists \mathbf{y}[\mathbf{x} \leq \mathbf{y} \;\&\; \mathcal{TL}(\mathbf{y})]$$

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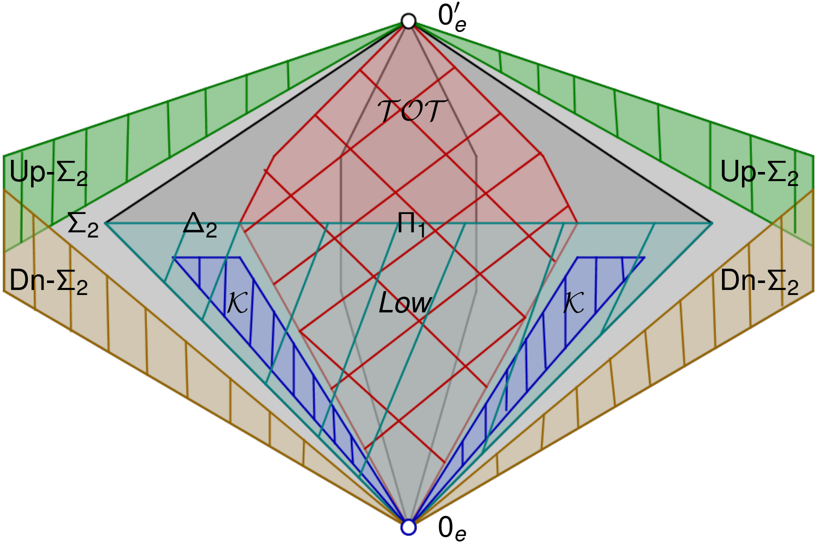

The fifth example of a definable class of degrees in the local structure: The low degrees.

The end

Thank you!