The Turing universe in the context of enumeration reducibility

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A fundamental goal of computability theory is to understand the way that objects relate to each other in terms of their information content. We wish to understand the relative information content between sets of natural numbers, how one subset of the natural numbers Y can be used to specify another one X. This specification can be computational, or arithmetic, or even by the application of a countable sequence of Borel operations. Each notion in the spectrum gives rise to a different model of relative computability. Which of these models best reflects the real world computation is under question.

The most widely used and studied model is the one based on computation: a set of natural numbers A is Turing reducible to set of natural numbers B if there is an effective procedure, by which given a natural number n and using the answers to finitely many membership questions to the oracle B we can correctly decide whether or not n is a member of A. By identifying sets that can be reduced to each other we obtain the partial order of the Turing degrees. Computable sets have least information content and form the least Turing degree $\mathbf{0}_T$. There is a natural way to combine the information content of two sets of natural numbers, A and B are combined into $A \oplus B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$, so that every pair of Turing degrees has a least upper bound. Finally, using a relativization of the halting problem, for every Turing degree $\mathbf{d}_T(A)$ we can find a degree which is strictly more complex, namely $\mathbf{d}_T(K_A)$, where K_A is the halting set, relativized to A, which we call the jump of the Turing degree $\mathbf{d}_T(A)$, and denote by $\mathbf{d}_T(A)'$. Thus the structure of the Turing degrees is an upper semi-lattice with least element and jump operation: $\mathcal{D}_T = (D_T, \leq_T, \mathbf{0}_T, \vee, ')$.

In this article we will examine the structure of the Turing degrees in the context, provided by a slightly weaker form of reducibility, based on the notion of enumeration rather than computation. This reducibility was introduced by Friedberg and Rogers [6] in 1959. A set of natural numbers A is enumeration reducible to a set of natural numbers B, if we can effectively transform any enumeration of the set B into an enumeration of the set A. There is a very close relationship between Turing reducibility and enumeration reducibility. To see this recall that an equivalent way of saying that $A \leq_T B$ is to say that both A and the complement of A can be enumerated using oracle B, or in other words $A \oplus \overline{A}$ is computably enumerable in B. Going deeper into the definition of *c.e. in*, we can say that there is a c.e. set W, whose members are pairs $\langle n, u \rangle$

⋆ Supported by a BNSF grant No. DMU 03/07/12.12.2011, by Sofia University Science Fund and by a Marie Curie international outgoing fellowship STRIDE (298471) within the 7th European Community Framework Programme

of a natural number and a code for a finite set D_u, so that $A \oplus \overline{A}$ can be represented as the set $\{n \mid \exists u(\langle n, u\rangle \in W \,\&\, D_u \subseteq B \oplus \overline{B})\}$. In this definition the oracle set $B \oplus \overline{B}$ and the reduced set $A \oplus \overline{A}$ have a very specific structure: they both contain all of the positive information about a set in their even part and all the negative information about the same set in their odd part. If we drop this requirement on the structure of the reduced set, we obtain a definition of the relation *c.e. in*. If we consider a more general form of this definition, by dropping the structural requirements on both the oracle and reduced set, we obtain a definition of enumeration reducibility.

Definition 1. *Let A and B be sets of natural numbers. $A \leq_e B$ if and only if there is a c.e. set W, such that $A = W(B) = \{x \mid \exists u(\langle x, u\rangle \in W \wedge D_u \subseteq B)\}$. The set W will be called an enumeration operator.*

From this analysis we immediately obtain the following relationship:

Proposition 1. *Let A and B be sets of natural numbers.*

1. *A is c.e. in B if and only if $A \leq_e B \oplus \overline{B}$.*
2. *$A \leq_T B$ if and only if $A \oplus \overline{A} \leq_e B \oplus \overline{B}$.*

Enumeration reducibility carries its own structure. By identifying sets that are enumeration reducible to each other, i.e. enumeration equivalent, we obtain the set of enumeration degrees D_e. This is a partial order, where $\mathbf{d}_e(A) \leq_e \mathbf{d}_e(B)$ if and only if $A \leq_e B$. The least element $\mathbf{0}_e$ consists of all computably enumerable sets. The way in which we obtained a least upper bound for Turing degrees, gives a least upper bound in the enumeration degrees, $\mathbf{d}_e(A \oplus B) = \mathbf{d}_e(A) \vee \mathbf{d}_e(B)$. Thus $\mathcal{D}_e = (D_e, \leq_e, \mathbf{0}_e, \vee)$ is as well an upper semi-lattice.

Proposition 1 yields a standard embedding $\iota : D_T \to D_e$ of the Turing degrees into the enumeration degrees, which preserves the order and the least upper bound. This embedding is defined by $\iota(\mathbf{d}_T(A)) = \mathbf{d}_e(A \oplus \overline{A})$. The image of the Turing degrees under this embedding is the set $\mathcal{TOT}$ of total enumeration degrees. The substructure $\mathcal{TOT}$ of the total enumeration degrees plays a central role in the study of the enumeration degrees. Selman [21] showed that enumeration reducibility can be characterized by the following:

Theorem 1. *Let $A, B \subseteq \omega$. $A \leq_e B$ if and only if*

$$\{X \mid B \text{ is c.e. in } X\} \subseteq \{X \mid A \text{ is c.e. in } X\}.$$

Translated in terms of enumeration reducibility using the first part of Proposition 1, this means that every enumeration degree is entirely characterized by the set of total degrees above it. In particular the set of total enumeration degrees is an automorphism base for the structure of the enumeration degrees.

We are still missing one ingredient for a complete analogy between the two structures: a jump operation, that agrees with the Turing jump under the standard embedding. The candidate for an analog of the halting set, which first comes to mind is $K_A^e = \{\langle n, e\rangle \mid n \in W_e(A)\}$. A closer look at this set shows that this first choice is not satisfactory, as K_A^e is of the same enumeration degree as A. This is not too surprising, in the Turing case the reason that the halting set is not computable comes from the fact that the complement of the halting set diagonalizes against all possible computations. In contrast to the Turing degrees, the enumeration degrees are not closed under complement. This is why, to get a jump operation, which actually jumps up, we need to use the complement of K_A^e. Thus we define $A' = K_A^e \oplus \overline{K_A^e}$. We will call sets A, such that $A \equiv_e A \oplus \overline{A}$, *total sets*. Total sets are always members of total degrees. Examples of total sets are graphs of total functions, from where the term *total* originates, and as we just saw - jumps of sets, by definition. In the reverse direction we have the following jump inversion theorem by Soskov [28].

Theorem 2. *For every enumeration degree* $\mathbf{x}$ *there exists a total enumeration degree* $\mathbf{a}$*, such that* $\mathbf{x} \leq_e \mathbf{a}$ *and* $\mathbf{x}' = \mathbf{a}'$*.*

It is not hard to see that this definition of the enumeration jump meets our requirements - it agrees with the Turing jump under the standard embedding ι. Thus the structure of the enumeration degrees provides a richer context in which we can study the structure of the Turing degrees. In this article we will argue that there are cases in which this approach is useful, shedding light on phenomena that can be observed, but not well explained by viewing the structure of the Turing degrees alone.

1 Computable model theory

Our first example comes from a theorem by Coles, Downey and Slaman [3]. For every set of natural numbers A, consider the set $\mathcal{C}(A) = \{X \mid A \text{ is c.e. in } X\}$. It follows from a result by Richter [19] that sets of this form do not always have a member of least Turing degree. Coles, Downey and Slaman show that if you instead look at $\mathcal{C}(A)' = \{X' \mid A \text{ is c.e. in } X\}$, then this set always has a member of least Turing degree.

Theorem 3. *For every sets* A *the set:* $\mathcal{C}(A)' = \{X' \mid A \text{ is c.e. in } X\}$ *has a member of least degree.*

They call this degree *the least jump enumeration of* A. The proof constructs this least jump enumeration using forcing with finite conditions.

The motivation for this result comes from computable model theory, and the notion of degree spectrum, used to characterize different structures. Fix a countable relational structure $\mathcal{A} = (\mathbb{N}, R_1 \dots R_k)$. The *degree spectrum* of $\mathcal{A}$, denoted by $DS_T(\mathcal{A})$, is the set of Turing degrees of the diagrams of structures $\mathcal{B} \cong \mathcal{A}$. If $DS_T(\mathcal{A})$ has a least member, it is the (Turing) degree of $\mathcal{A}$. Sometimes the spectrum of a structure does not behave nicely, and it is useful and more informative to consider the jump spectrum of a structure. The *jump spectrum* of $\mathcal{A}$ is $DS'_T(\mathcal{A}) = \{\mathbf{d}' \mid \mathbf{d} \in DS_T(\mathcal{A})\}$. If $DS'_T(\mathcal{A})$ has a least member, it is the (Turing) jump degree of $\mathcal{A}$.

Now we will consider one particular instance of this characterization problem. A torsion free abelian group G of rank 1 is (up to isomorphism) a subgroup of the additive group of the rational numbers $(\mathbb{Q}, +, =)$. Fix such a group G. For every prime number p and element $a \in G$ we introduce the notion the p-height of a in G as follows:

$$h_p(a) = \begin{cases} \text{the largest } k, & \text{such that } p^k|a \text{ in } G; \\ \infty, & \text{if } \forall k(p^k|a \text{ in } G)\ . \end{cases}$$

Here $p^k|a$ in G if there exists $b \in G$ such that $p^k.b = a$.

Example: If $G = \mathbb{Q}$ then for all nonzero a and all p, $h_p(a) = \infty$, because for all k, $p^k.\frac{a}{p^k} = a$. If $G = \mathbb{Z}$ then for all nonzero a and all but finitely many p, $h_p(a) = 0$.

In fact it is not difficult to see that in any torsion free abelian group G of rank 1 we have that for nonzero elements a, b and for all but finitely many prime numbers p, $h_p(a) = h_p(b)$. We can therefore assign to every element $a \in G$ an infinite sequence of numbers: $\chi(a) = (h_{p_0}(a), h_{p_1}(a), \dots h_{p_n}(a), \dots)$, which we will call the *characteristic* of a. So different elements of G have characteristics, which differ at only finitely many places, i.e. they are equivalent with respect to the equivalence relation in ω^ω which places sequences that differ in finitely many positions in the same equivalence class. The type of G, denoted $\chi(G)$ is the equivalence class of $\chi(a)$ for any $a \neq 0$ in G. Baer showed that first of all there are torsion free abelian groups of rank 1 of every possible type. But more importantly he proved that two torsion-free abelian groups of rank 1 are isomorphic if and only if they have the same type. Thus a torsion free abelian group of rank one is completely characterized by its type. And it turns out that this type can also be used to characterize the degree spectrum of every such group.

Let $S(G) = \{\langle i, j\rangle \mid j \leq \text{ the i-th element of } \chi(G)\}$. Here we are actually taking any nonzero element of G and using its characteristic to represent G. Note that $S(G)$ does not depend on the choice of this element, up to many-one equivalence. Downey and Jockusch (see [3]) had shown that the degree spectrum of G is precisely $\{\mathbf{d}_T(Y) \mid S(G) \text{ is c.e. in } Y\}$. Thus the result by Coles , Downey and Slaman can be restated as follows: Every torsion-free abelian group G of rank 1 has a jump degree. So we know this fact, but just from the world of Turing degrees it is not easy to explain the reason for this fact to be true. We now turn to the wider context of the enumeration degrees and view the same problem there.

Let us fix a countable relational structure $\mathcal{A} = (\mathbb{N}, R_1 \dots R_k)$. Soskov [29] introduced the notion of *enumeration degree spectrum.* The enumeration degree spectrum of $\mathcal{A}$, denoted by $DS_e(\mathcal{A})$, is the set of e-degrees of the positive diagrams of structures $\mathcal{B} \cong \mathcal{A}$. If $DS_e(\mathcal{A})$ has a least member, it is the (enumeration) degree of $\mathcal{A}$. The enumeration jump degree spectrum of $\mathcal{A}$, denoted by $DS'_e(\mathcal{A})$, is the set of enumeration jumps of elements in $DS_e(\mathcal{A})$. If $DS'_e(\mathcal{A})$ has a least member, it is the (enumeration) jump degree of $\mathcal{A}$.

We immediately observe that just like Turing degrees embed into the enumeration degree via the standard embedding ι, we have a connection between the Turing degree spectrum and the enumeration degree spectrum of structures. Let $\mathcal{A}$ be any structure. Consider the structure $\mathcal{A}^+ = (\mathbb{N}, R_1, \overline{R_1} \dots R_k, \overline{R_k})$. Firstly note that $DS_e(\mathcal{A}^+)$ consists entirely of total enumeration degrees. Secondly note that the positive diagram of A^+ is enumeration equivalent to the diagram of $\mathcal{A}$. Thus:

$$DS_e(\mathcal{A}^+) = \{\iota(\mathbf{a}) \mid \mathbf{a} \in DS_T(\mathcal{A})\} = \iota(DS_T(\mathcal{A})).$$

$\mathcal{A}$ has Turing degree $\mathbf{a}$ if and only if $\mathcal{A}^+$ has enumeration degree $\iota(\mathbf{a})$. Similarly $DS'_e(\mathcal{A}^+) = \{\iota(\mathbf{a})' \mid \mathbf{a} \in DS_T(\mathcal{A})\} = \{\iota(\mathbf{a}') \mid \mathbf{a} \in DS_T(\mathcal{A})\} = \iota(DS'_T(\mathcal{A}))$ and $\mathcal{A}$ has Turing jump degree $\mathbf{a}$ if and only if $\mathcal{A}^+$ has enumeration jump degree $\iota(\mathbf{a})$.

Soskov [29] considers the co-spectrum of a structure, the of enumeration degrees which are lower bounds to the enumeration degree spectrum of a structure. He shows that every countable ideal of enumeration degrees can be realized as the co-spectrum of a structure. The easier case of this theorem is when the ideal is a principal ideal. Soskov shows that every principal ideal $(\mathbf{a})$ is the co-spectrum of a torsion free abelian group of rank one, G. The top element of this ideal $\mathbf{a}$ is precisely the enumeration degree of the type of the group $S(G)$. He further makes the following observation.

Let G be a torsion-free abelian group of rank 1. The only relation in the language (apart from equality) is the graph of the total function $+$ which can also enumerate its negative instances thus $DS_e(G) = \{\iota(\mathbf{a}) \mid \mathbf{a} \in DS_T(G)\} = \left\{\mathbf{d}_e(Y \oplus \overline{Y}) \mid S(G) \text{ is c.e. in } Y\right\}$. We can apply the first part of Proposition 1 to simplify the description of the enumeration degree spectrum of G. Denote $\mathbf{d}_e(S(G))$ by $\mathbf{s}_G$ - the type degree of G. The enumeration degree spectrum of G is:

$$DS_e(G) = \{\mathbf{a} \mid \mathbf{a} \in \mathcal{TOT} \ \&\ \mathbf{s}_G \leq_e \mathbf{a}\}.$$

Now it is clear when a group has a degree and when not. G has an enumeration degree (and hence G has a Turing degree) if and only if the enumeration degree $\mathbf{s}_G$ is total. This follows from Selman's Theorem 1, as every enumeration degree is completely characterized by the set of total degrees above it. Furthermore, if G has an enumeration degree then it is precisely $\mathbf{s}_G$.

Consider the enumeration jump spectrum of G.

$$DS'_e(G) = \{\mathbf{a}' \mid \mathbf{a} \in \mathcal{TOT} \ \&\ \mathbf{s}_G \leq_e \mathbf{a}\}.$$

By the monotonicity of the enumeration jump all members of DS'_e are greater than or equal to $\mathbf{s}'_G$. By Soskov's Jump Inversion Theorem 2 there is a total degree $\mathbf{a} \geq \mathbf{s}_G$ such that $\mathbf{a}' = \mathbf{s}'_G$. Thus the enumeration jump spectrum of G always has a least element and it is $\mathbf{s}'_G$. This gives an alternative, more informative proof of the result by Coles, Downey and Slaman.

In recent work Soskov has shown a different application of properties of enumeration degrees to computable model theory. In this case the theme is jump inversion of spectra of structures. It had been previously shown [12] that for every successor ordinal α if the Turing degree spectrum of a structure $\mathcal{A}$ lies in the cone above $\mathbf{0}^\alpha$ then there is a structure $\mathcal{B}$, whose α-th jump spectrum is the spectrum of $\mathcal{A}$. The case for limit ordinals was not known. In [27] Soskov shows that ω-jump inversion is not always possible. The reason for this negative result is seen when one considers the co-set, the set of lower bounds, of the image of the ω-th jump spectrum of a structure: it is shown that every member of this co-set is bounded by a total enumeration degree in the same co-set. Thus an example of a structure for which ω-jump inversion fails is given by a torsion free abelian group G of rank one, such that $\mathbf{d}_e(S(G))$ is not total and above $\mathbf{0}_e^\omega$.

2 Definability of the jump operator

A major theme in degree theory has been the definability theme. Among the most notable definability results in the Turing degrees is Shore and Slaman's [23] proof of the first order definability of the Turing jump operator. The proof of this theorem relies on two main ingredients - the definability of the double jump and a proof of the following structural property: for every Turing degree $\mathbf{a}$ which is not Δ_2^0 there is a Turing degree $\mathbf{g}$ such that $\mathbf{a} \vee \mathbf{g} = \mathbf{g}''$. Since any Δ_2^0 degree $\mathbf{a} \leq \mathbf{0}_T'$ obviously does not satisfy this property ($\mathbf{a} \vee \mathbf{g} \leq \mathbf{0}_T' \vee \mathbf{g} \leq \mathbf{g}' < \mathbf{g}''$), this gives a first order definition of $\mathbf{0}_T'$. Relativizing, one gets the definition of the jump operation for every Turing degree degree modulo the definability of the double jump. The first ingredient is proved with a forcing construction, known as Kumabe-Slaman forcing. The definability of the double jump relies on Slaman and Woodin's [26] analysis of the automorphism group of the Turing degrees. Slaman and Woodin show that every countable relation on the Turing degrees can be coded by finitely many parameters, the Coding Theorem. This shows in particular that one can interpret the theory of second order arithmetic in the Turing degrees, an earlier result due to Simpson [24]. Then using methods from set theory, Slaman and Woodin show that every automorphism of the Turing degrees has an arithmetic presentation, that it is completely determined by its action on one element and that it fixes the cone above $\mathbf{0}_T''$. Further they show that the biinterpretability conjecture is true modulo finitely many parameters and from this obtain that every relation in the Turing degrees which is induced by a degree invariant relation on 2^ω, which is definable in second order arithmetic, is definable in $\mathcal{D}_T$ using parameters. If the relation is in addition invariant under automorphisms, then it is definable without parameters. This gives the definability of the double jump. As is stated in [23], this makes the definition of the double jump, and hence the jump operation, in the Turing degrees very far from natural. It cannot be stated in terms of a structural property, similar to the one used for the definition of $\mathbf{0}_T'$ from the double jump. In the enumeration degrees this is not the case. And the reason for this is the existence of pairs of enumeration degrees with very special properties.

Recall Jockusch's [15] definition of a semi-recursive set. A set of natural numbers A is *semi-recursive* if there is a total computable selector function s_A, such that $s_A(x, y) \in \{x, y\}$ and if $\{x, y\} \cap A \neq \emptyset$ then $s_A(x, y) \in A$. For example for every set A the set of finite binary strings, which are to the left of the characteristic function of A in the usual lexicographical ordering, $L_A = \{\sigma \in 2^{<\omega} \mid \sigma \leq_L \chi_A\}$, is semi-recursive. Given two finite binary strings the selector function always picks the one which is smaller with respect to $\leq_L$. Jockusch [15] showed that in fact for every non-computable set B there is a semi-recursive set $A \equiv_T B$ such that both A and $\overline{A}$ are not c.e.

Arslanov, Cooper and Kalimullin [2] noticed that the enumeration degrees of semi-recursive sets have very interesting structural properties. If A is a semi-recursive set then $\mathbf{d}_e(A)$ and $\mathbf{d}_e(\overline{A})$ form a minimal pair in a very strong sense:

$$(\forall \mathbf{x} \in \mathcal{D}_e)((\mathbf{d}_e(A) \vee \mathbf{x}) \wedge (\mathbf{d}_e(\overline{A}) \vee \mathbf{x}) = \mathbf{x}).$$

It is then natural to wonder if this statement can be transformed into an if and only if statement. Kalimullin showed that it can, but first we need to introduce a generalization of semi-recursive sets.

Definition 2. *A pair of sets A, B is called a $\mathcal{K}$-pair if there is a c.e. set W, such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.*

A trivial example of a $\mathcal{K}$-pair is $\{A, U\}$, where A is arbitrary and U is c.e.. The c.e. set witnessing this is $\mathbb{N} \times U$. If A is a semi-recursive set, then $\{A, \overline{A}\}$ is a $\mathcal{K}$-pair, witnessed by the set $W = \{\langle x, y\rangle \mid s_A(x, y) = x\}$.

Kalimullin [16] showed that the notion of $\mathcal{K}$-pairs captures precisely the strong minimal pair property that semi-recursive sets have.

Theorem 4. *A pair of sets A, B are a $\mathcal{K}$-pair if and only if their enumeration degrees $\mathbf{a}$ and $\mathbf{b}$ satisfy: $\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x} \in \mathcal{D}_e)((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x})$.*

$\mathcal{K}$-pairs are unique to the structure of the enumeration degrees. They are always quasi-minimal, i.e. the only total degree below either of them is $\mathbf{0}_e$. A consequence of the existence of nontrivial $\mathcal{K}$-pairs in $\mathcal{D}_e$, which are not below $\mathbf{0}_e'$, is that the Slaman-Shore property used to define the Turing jump fails in the enumeration degrees: there is a degree $\mathbf{a} \not\leq_e \mathbf{0}_e'$, such that for every degree $\mathbf{g}$, $\mathbf{a} \vee \mathbf{g} <_e \mathbf{g}''$. This shows that there are no $\mathcal{K}$-pairs in the structure of the Turing degrees. And furthermore there is no hope that the enumeration jump can be defined using a similar technique to the one used for the definition of the Turing jump.

Nevertheless Kalimullin [16] showed that the enumeration jump is first order definable by a very natural structural property. He showed that $\mathbf{0}_e'$ is the largest degree which can be represented as the least upper bound of a triple $\mathbf{a}, \mathbf{b}, \mathbf{c}$, such that $\mathcal{K}(\mathbf{a}, \mathbf{b})$, $\mathcal{K}(\mathbf{b}, \mathbf{c})$ and $\mathcal{K}(\mathbf{c}, \mathbf{a})$. Using a relativized version of $\mathcal{K}$-pairs, he then obtained the definability of the jump operator in $\mathcal{D}_e$. An alternative definition, which does not even use relativization is given by Ganchev and Soskova [7].

Theorem 5. *For every nonzero enumeration degree $\mathbf{u} \in \mathcal{D}_e$, $\mathbf{u}'$ is the largest among all least upper bounds $\mathbf{a} \vee \mathbf{b}$ of nontrivial $\mathcal{K}$-pairs $\{\mathbf{a}, \mathbf{b}\}$, such that $\mathbf{a} \leq_e \mathbf{u}$.*

As a consequence of this result we obtain that the set of total degrees $\mathbf{a}$ above $\mathbf{0}_e'$ is also first order definable in $\mathcal{D}_e$: a degree above $\mathbf{0}_e'$ is total if and only if it is the jump of some enumeration degree.

3 The local structures

Two important substructures of the Turing degrees are the local structure of the Turing degrees $\mathcal{D}_T(\leq \mathbf{0}'_T)$, consisting of all Δ^0_2 Turing degrees and its substructure $\mathcal{R}$, consisting of the of the computably enumerable degrees, i.e. Turing degrees which contain c.e. sets. The local structure of the enumeration degrees $\mathcal{D}_e(\leq \mathbf{0}'_e)$, consists of all Σ^0_2 enumeration degrees. Recall that $\iota : \mathcal{D}_T \to \mathcal{D}_e$ preserves the jump, hence $\mathcal{D}_T(\leq \mathbf{0}')$ embeds in $\mathcal{D}_e(\leq \mathbf{0}'_e)$. It is not difficult to show that $\iota(\mathcal{R})$ is precisely the substructure of the Π^0_1 enumeration degrees. Thus the two local substructures of the Turing degrees live inside the local structure of the enumeration degrees. The three structures are different both in terms of the proper inclusions between their domains and in terms of their theories. Cooper [4] showed that $\mathcal{D}_e(\leq \mathbf{0}'_e)$ is dense, hence not elementary equivalent to $\mathcal{D}_T(\leq \mathbf{0}'_T)$. Ahmad [1] showed that the diamond can be embedded in $\mathcal{D}_e(\leq \mathbf{0}'_e)$, hence the theory of the local structure of the enumeration degrees differs also from the theory of the c.e. Turing degrees.

One possible advantage of the embeddability of the Turing structures into $\mathcal{D}_e(\leq \mathbf{0}'_e)$ is that algebraic results proved about the larger structure reveal more information about the smaller structures. We next describe an instance of this idea. A pair of degrees $\mathbf{a}, \mathbf{b}$ form a splitting of $\mathbf{c}$ if $\mathbf{a} < \mathbf{c}$, $\mathbf{b} < \mathbf{c}$ and $\mathbf{a} \vee \mathbf{b} = \mathbf{c}$. The existence of various splittings and non-splittings in a structure is important if one wishes to understand its two quantifier theory, i. e. the set two quantifier sentences true in the structure. The history of splitting results for the c.e. degree goes hand in hand with the evolution of the methods used to prove them, in particular the priority method. Harrington [13], generalizing a result by Lachlan [17], showed that there exists a c.e. Turing degree $\mathbf{a} < \mathbf{0}'_T$, i.e. an incomplete c.e. degree, such that no pair of c.e. degrees above $\mathbf{a}$ are a splitting of $\mathbf{0}'_T$. This came to be known as Harrington's non-splitting theorem and the method used in its proof as the monster priority method. Later on a similar method was used by Harrington and Shelah [14] to show the undecidability of the theory of the c.e. degree. Cooper and Soskova [5] pushed this result further to its limit, by considering this structural property within the local structure of the enumeration degrees. By Sack's splitting theorem, relativized to any Δ^0_2 Turing degree it follows that there is a Δ^0_2 splitting of $\mathbf{0}'_T$ above any incomplete Turing degree. So the only question, which remained unanswered was if one can find a splitting consisting of one c.e. member and one Δ^0_2 member above any incomplete c.e. degree. Cooper and Soskova [5] showed that this is not true:

Theorem 6. *There exists a Π^0_1 enumeration degree $\mathbf{a} <_e \mathbf{0}'_e$, such that no pair of a Π^0_1 and Σ^0_2 e-degrees above $\mathbf{a}$ are a splitting of $\mathbf{0}'_e$.*

So transferring back, via the inverse of the standard embedding ι, there is a c.e. Turing degree $\mathbf{a} <_T \mathbf{0}'_T$, such that no pair of a c.e. degree and a Δ^0_2 degree above $\mathbf{a}$ are a splitting of $\mathbf{0}'_T$. This method was then extended in [31] to show that the full analog of Harrington's non-splitting theorem holds in the local structure of the enumeration degrees.

Theorem 7. *There is an incomplete Σ_2^0 enumeration degree above which $\mathbf{0}'_e$ cannot be split.*

The definability theme for the local structures has also been widely explored. Here one cannot talk about definability of the jump operator, but one can look at a hierarchy of classes of degrees defined in terms of the strength of their jumps.

Definition 3. *Let $n \geq 1$*

1. *The class of low$_n$ degrees is $L_n = \{\mathbf{a} \leq \mathbf{0}' \mid \mathbf{a}^n = \mathbf{0}^n\}$.*
2. *The class of high$_n$ degrees is $H_n = \left\{\mathbf{a} \leq \mathbf{0}' \mid \mathbf{a}^n = \mathbf{0}^{n+1}\right\}$*

The definability of the classes L_{n+1} and H_n for all $n \geq 1$ in $\mathcal{R}$ was shown by Nies, Shore and Slaman [18] and in $\mathcal{D}_T(\leq \mathbf{0}')$ by Shore[22]. The proofs of these theorems have the same flavor as the proof of the definability of the jump operation. First it is shown that the theory of first order arithmetic can be interpreted in $\mathcal{R}$ and $\mathcal{D}_T(\leq \mathbf{0}')$. This is then used to show that there is a definable way of mapping a degree $\mathbf{a}$ to a set A in every coded model of arithmetic so that $A'' \in \mathbf{a}''$. Thus every relation which is invariant under double jump and definable in first order arithmetic is definable in the corresponding degree structure. An additional argument, building on top if the first one is then devised to show that H_1 is also first order definable. These methods are however not sufficiently powerful to capture the definability of the low degrees, L_1, in either structure. The definability of L_1 in $\mathcal{D}_T(\leq \mathbf{0}'_T)$ or in $\mathcal{R}$ remains open.

Now lets turn to the local structure of the enumeration degrees $\mathcal{D}_e(\leq \mathbf{0}'_e)$. The first quite important question to settle concerns the complexity of the theory of the local structure, more precisely the question of whether or not one can interpret the theory of true arithmetic in $\mathcal{D}_e(\leq \mathbf{0}'_e)$. Slaman and Woodin [25] prove a coding theorem for the global structure, showing that $Th(\mathcal{D}_e)$ is computably isomorphic to second order arithmetic and a limited effective version of the coding theorem, that is enough to show that the local theory is undecidable, but not sufficient to characterize its complexity fully.

Theorem 8. *Every uniformly low antichain can be coded by parameters in the local structure $\mathcal{D}_e(\leq \mathbf{0}'_e)$.*

Ganchev and Soskova [9] notice that $\mathcal{K}$-pairs can be used to obtain precisely this kind of antichains. In fact every half of a nontrivial $\mathcal{K}$-pair is a uniform low bound to an antichain $\{\mathbf{a}_i\}_{i\in\omega}$ of e-degrees such that if $i \neq j$ then $\{\mathbf{a}_i, \mathbf{a}_j\}$ is a $\mathcal{K}$-pair. Thus if the property "**a** and **b** form a $\mathcal{K}$-pair" is first order definable in the local structure, then one could use Theorem 8 to show that the theory of first order arithmetic can be interpreted in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

Unfortunately Kalimullin's global definition of $\mathcal{K}$-pairs given in Theorem 4 starts with a universal quantifier. It is not clear and still open if this formula, interpreted in the local structure, is still a first order definition of the true $\mathcal{K}$-pairs. However, using an additional structural property of the members in $\mathcal{D}_e(\leq \mathbf{0}'_e)$, which is reminiscent of Theorem 7 and proved by a similar technique as the one used there, Ganchev and Soskova [8] prove that $\mathcal{K}$-pairs in the local structure of the enumeration degrees form a definable class.

Theorem 9. *There is a first order formula $\mathcal{LK}(x, y)$, which defines the $\mathcal{K}$-pairs in $\mathcal{D}_e(\leq \mathbf{0}_e')$.*

This enables Ganchev and Soskova [10] to complete the original idea for interpreting arithmetic in the local structure.

Theorem 10. *The first order theory of $\mathcal{D}_e(\leq \mathbf{0}_e')$ is computably isomorphic to first order theory of true arithmetic.*

The definability of $\mathcal{K}$-pairs in the local structure, turned out to be a key unlocking the definability of many other classes, including the downwards properly Σ_2^0 enumeration degrees, the upwards properly Σ_2^0 enumeration degrees. Extending a result of Giorgi, Sorbi and Yang [11], Ganchev and Soskova [7] show that in addition the class L_1 is first order definable, by a very natural property.

Theorem 11. *An enumeration degree* $\mathbf{a}$ *is low*$_1$ *if and only if every degree* $\mathbf{b} \leq_e \mathbf{a}$ *bounds a* $\mathcal{K}$*-pair.*

So in contrast to the Turing structures $\mathcal{D}_T(\leq \mathbf{0}_T')$ and $\mathcal{R}$, where only L_1 cannot be shown to be definable, here in the local structure of the enumeration degrees we can show that this class is definable by a very natural property and it is not known whether or not the other jump classes are definable. The next result tips the scale in favor of $\mathcal{D}_e(\leq_e \mathbf{0}_e')$ at least in terms of richness of definable classes.

Recall that a semi-recursive set A and its complement $\overline{A}$ form a special example of $\mathcal{K}$-pair. In terms of structure this $\mathcal{K}$-pair has one additional property - it is maximal. A $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ is maximal if there does not exists a $\mathcal{K}$-pair $\{\mathbf{c}, \mathbf{d}\}$, with $\mathbf{a} < \mathbf{c}$ or $\mathbf{b} < \mathbf{d}$. Ganchev and Soskova [7] show that in the local structure of the enumeration degrees maximality is precisely the structural notion which captures $\mathcal{K}$-pairs of a semi-recursive set and its complement. By Jockusch's theorem there is a non c.e. and non co-c.e. semi-recursive set in every Turing degree. In e-degree terms this means that a nonzero enumeration degree is total if and only if it can be represented as the least upper bound of a maximal $\mathcal{K}$-pair.

Theorem 12. *The class of total enumeration degrees is first order definable in $\mathcal{D}_e(\leq \mathbf{0}_e')$.*

4 An open question

From what we have said so far, it follows that the set of total degrees that are comparable with $\mathbf{0}_e'$ is first order definable in $\mathcal{D}_e$. The open question of interest to us, first set by Rogers [20], concerns the definability of the class of all total enumeration degrees.

As noted above it follows from Selman's Theorem 1 that the class of total enumeration degrees is an automorphism base for the $\mathcal{D}_e$. Thus the definability of the total enumeration degrees would link the two major open problems: the existence of nontrivial automorphism for the Turing degrees and the existence of a nontrivial automorphism of the enumeration degrees. A positive answer to the second question would yield a positive answer to the first question.

One possible solution to the definability of the total degrees would be to extend the characterization that proves the definability of the total degrees in the local structure. Jockusch's result for the existence of semi-recursive sets is valid for every Turing degree, hence one direction is already known to be true: every nonzero total set is enumeration equivalent to the join of the components of a maximal $\mathcal{K}$-pair. The first order definability of the total enumeration degrees would then follow, if it were true that maximality is the additional structural property needed to capture $\mathcal{K}$-pairs of the form $\{A, \overline{A}\}$.

This definition would then relate in a nice way to the definition of the enumeration jump given in Theorem 5. Consider the relation *c.e. in* between Turing degrees defined by: $\mathbf{x}$ is c.e. in $\mathbf{u}$ if there are sets $X \in \mathbf{x}$ and $U \in \mathbf{u}$, such that X is c.e. in U. Ganchev and Soskova [7] show that if $\mathbf{x}$ and $\mathbf{u}$ are Turing degrees such that $\mathbf{u}$ is nonzero then $\mathbf{x}$ is c.e. in $\mathbf{u}$ if and only if there is a $\mathcal{K}$-pair $\{A, \overline{A}\}$ such that $\mathbf{d}_e(A) \leq_e \iota(\mathbf{u})$ and $\iota(\mathbf{x}) = \mathbf{d}_e(A) \vee \mathbf{d}_e(\overline{A})$. Thus if every maximal $\mathcal{K}$-pair is of the form $\{A, \overline{A}\}$ for some A then the total degrees would be definable and the relation *c.e. in* between nonzero total degrees would be definable. The definition of the enumeration jump given in Theorem 5 restricted to the total degrees can then be read as $\mathbf{u}'$ is the largest total enumeration degree which is c.e. in $\mathbf{u}$. The natural definition of the total enumeration degrees proposed above remains currently out of reach.

In recent work Soskova [30] has investigated how much of the techniques for the analysis of the automorphism group of the Turing degrees by Slaman and Woodin [26] can be applied to study the automorphism group of the enumeration degrees. The obtained results bring the definition of the class of total enumeration degrees one step closer - namely a parameter away. The obtained results mirror the originals, it is shown that the automorphism group of the enumeration degrees is at most countable, every member has an arithmetic presentation and that there is an automorphism basis consisting of a single element. It is further shown that every relation in the enumeration degrees which is induced by a degree invariant relation on 2^ω, definable in second order arithmetic, is definable in $\mathcal{D}_e$ using parameters. As a consequence we obtain that:

Theorem 13. *The class of total enumeration degrees is definable with parameters in the structure of the enumeration degrees.*

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