A Generalization of Harrington's Nonsplitting Theorem

Mariya I. Soskova

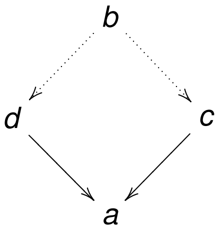
University of Leeds
Department of Pure Mathematics

24.08.2006

Lachlan's Nonsplitting theorem

Theorem

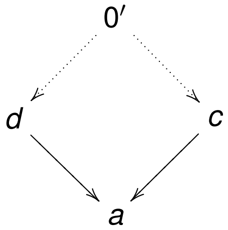
There exist c.e. degrees $a < b$ such that b can not be split over a.



Harrington's Nonsplitting theorem

Theorem

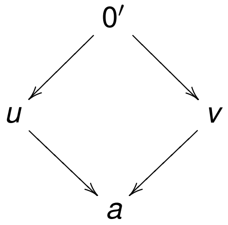
There exists a c.e. degree $a < 0'$ such that $0'$ can not be split over a.



Arslanov's splitting theorem

Theorem

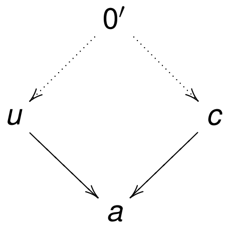
There is a d.c.e. splitting of $0'$ *above each c.e. degree* $a < 0'$.



The strongest nonsplitting theorem

Theorem

There exists a computably enumerable degree $a < 0'$ such that there exists no nontrivial cuppings of c.e. degrees in the Δ_2 degrees above a.



The semi-lattice of the enumeration degrees

Definition

1. A set A is enumeration reducible to a set B ($A \leq_e B$), if there is a c.e. set Φ such that

$$n \in A \Leftrightarrow \exists D(\langle n, [D] \rangle \in \Phi \wedge D \subset B)$$

2. A is enumeration equivalent to B ($A \equiv_e B$) if $A \leq_e B$ and $B \leq_e A$
3. Let $d_e(A) = \{B | A \equiv_e B\}$.
4. $(D_e, <, \cup, ', 0_e)$ is the semi-lattice of the enumeration degrees

Embedding the Turing degrees into the enumeration degrees

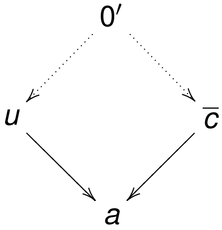
There exists an order theoretic embedding $\iota : D_T \to D_e$ with following properties.

1. ι preserves least element, joins and jump operators
2. The c.e. Turing degrees embed exactly onto the Π_1 enumeration degrees
3. There are partial Δ_2 degrees.

Main Result

Theorem

There exists a Π_1 e-degree $a < 0'_e$ such that there exist no nontrivial cuppings of Π_1 e-degrees in the Δ_2 e-degrees above a.



The Requirements

We will construct the Π_1 sets A and E

- For all enumeration operators Ψ:

$$N_\Psi : E \neq \Psi^A$$

- For each pair of a Δ_2 set U and a Π_1 set $\overline{W}$ and each enumeration operator Θ:

$$P_{\Theta,U,W} : E = \Theta^{U,\overline{W}} \Rightarrow (\exists \Gamma, \Lambda)[\overline{K} = \Gamma^{U,A} \vee \overline{K} = \Lambda^{\overline{W},A}]$$

The naive N strategy

- Select a witness $x \in E$ for N_Ψ
- Wait for $x \in \Psi^A$
- Extract x from E and restrain each $y \in A \upharpoonright \psi(x)$.

The naive P strategy

Good approximations

- If $E \neq \Theta^{U,\overline{W}}$, the requirement is trivially satisfied.
- We monitor the length of agreement $l(E, \Theta^{U,\overline{W}})$ and act only on expansionary stages.
- We define a good approximating sequence to the set $U \oplus \overline{W}$ with following properties
  - Infinitely many good stages:$\forall n \exists s (U \oplus \overline{W} \upharpoonright n \subseteq (U \oplus \overline{W})_s \subseteq U \oplus \overline{W})$.
  - Stability: $\forall n \exists s_0 \forall s > s_0 (U \oplus \overline{W} \upharpoonright n = (U \oplus \overline{W})_s \upharpoonright n)$
  - If $\Theta(U \oplus \overline{W}) = E$, then there are infinitely many expansionary stages

The naive P strategy

On expansionary stages construct an enumeration operator Γ, so that $\Gamma^{U,A} = \overline{K}$

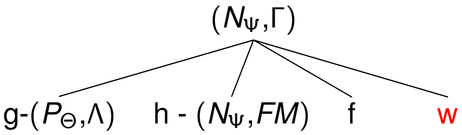
- For each $n < l$, $n \in \overline{K}$: axiom $\langle n, U \upharpoonright (u(n)+1), A \upharpoonright (\gamma(n)+1) \rangle \searrow \Gamma$.
- If the axiom becomes invalid - a change in $U \upharpoonright (u(n)+1)$, but store the old axiom in a list $Old(n)$.
- If n exits $\overline{K}$, extract $\gamma(n)$ from A of all valid axioms for n in Γ.

Combining the two strategies

- A-restraint by N_Ψ conflicts the need to rectify Γ
- Choose threshold d and try to achieve $\gamma(n) > \psi(x)$ for all $n \geqslant d$
- Extract x from E. Return of $l(E, \Theta^{U,\overline{W}})$ forces U or $\overline{W}$ to change.
- U-change: lift the gamma markers and preserve the restraint
- $\overline{W}$-change - start over with new witness, implement the backup strategy which insures $\Lambda^{\overline{W},A} = \overline{K}$

The detailed N_Ψ strategy

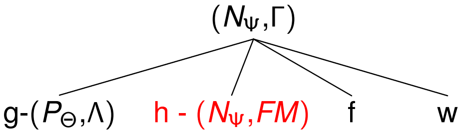
Initialization



1. Choose a new threshold d and a new witness $x > d$, $x \in E$.
2. Wait for $x < l$. $(o = w)$
3. Extract all markers $\gamma(d)$ - old and new and empty the list $Old(n)$ for $n \geq d$. Define $u(d)$ new, bigger than $\theta(x)$.
4. For every element $y \leq x$, $y \in E$ enumerate in the list *Axioms* the current valid axiom from Θ.

The detailed N_Ψ strategy

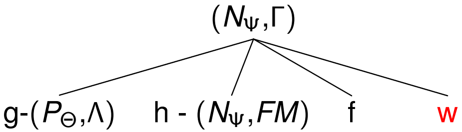
Honestification



- Scan the list *Axioms*. If for any element $y \leq x$, $y \in E$ the listed axiom is not valid anymore, then:
  1. Update the list *Axioms*
  2. Extract all markers $\gamma(d)$ - old and new and empty the list $Old(n)$ for $n \geq d$. Define $u(d)$ new, bigger than $\theta(x)$.(o = h)
- Wait for $x \searrow \Psi^A$ (o = w).

The detailed N_Ψ strategy

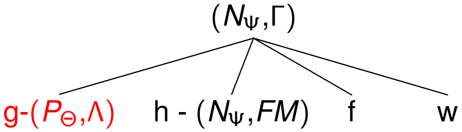
Honestification



- Scan the list *Axioms*. If for any element $y \leq x$, $y \in E$ the listed axiom is not valid anymore, then:
  1. Update the list *Axioms*
  2. Extract all markers $\gamma(d)$ - old and new and empty the list $Old(n)$ for $n \geq d$. Define $u(d)$ new, bigger than $\theta(x)$. (o = h)
- Wait for $x \searrow \Psi^A$ (o = w).

The detailed N_Ψ strategy

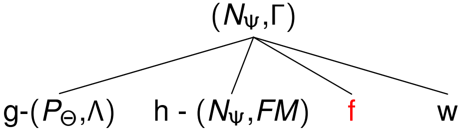
Attack



- Extract x from E. The outcome (o = g)lets the backup strategy synchronize its attack with this one.
- Choose x' to be the least element extracted from E during the attack, with corresponding axiom $\langle x', U_x, W_x\rangle \in Axioms$
- Unsuccessful attack - $W_x \not\subseteq \overline{W}$: cancel x and start over from initialization. The attack is successful for the backup strategy (o = g).

The detailed N_Ψ strategy

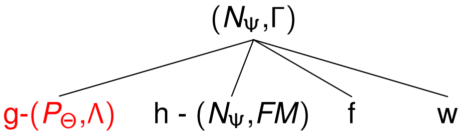
Attack



- Successful attack - $W_x \subseteq \overline{W}$, hence there is a useful change in U (o = f).
- Keep an eye on U - it may later on change back due to its tricky Δ_2 nature.
- If so - $W_x \not\subseteq \overline{W}$ - delayed successful attack for backup strategy.

The detailed N_Ψ strategy

Attack



- Successful attack - $W_x \subseteq \overline{W}$, hence there is a useful change in U (o = f).
- Keep an eye on U - it may later on change back due to its tricky Δ_2 nature.
- If so - $W_x \not\subseteq \overline{W}$ - delayed successful attack for backup strategy.

Bibliography

- S. B. Cooper, *Computability Theory*, Chapman & Hall/CRC Mathematics, Boca Raton, FL, 2004.
- P. G. Odifreddi, *Classical Recursion Theory, Volume II*, North-Holland/Elsevier, Amsterdam, Lausanne, New York, Oxford, Shannon, Singapore, Tokyo 1999.
- R. I. Soare, *Recursively enumerable sets and degrees*, Springer-Verlag, Heidelberg, 1987.
- L. Harrington, *Understanding Lachlan's Monster Paper*, Notes
- A.H. Lachlan, R.A. Shore, *The n-rea Enumeration Degrees are Dense*, Arch. Math. Logic (1992)31 : 277-285.
- S.D. Leonhardi, *Generalized Nonsplitting in the Recursively Enumerable Degrees*
- R. Soare, *Notes on Lachlan's Monster Theorem*