Definability of the jump classes in the local structure of the enumeration degrees

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Reducibilities and Degree structures

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$$(\mathcal{D}_T, \leq_T, \vee, ', \mathbf{0}_T) \qquad (\mathcal{D}_e, \leq_e, \vee, ', \mathbf{0}_e)$$

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Proposition

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Theorem (Slaman, Woodin: Biinterpretability with parameters)

There is a way within $\mathcal{D}_T$ to represent the standard model of arithmetic $\langle \mathbb{N}, +, *, <, 0, 1 \rangle$ and each set of natural numbers X so that the relation

$$\vec{\mathbf{p}} \textit{ represents the set } X \textit{ and } \mathbf{x} \textit{ is the Turing degree of } X.$$

can be defined using a parameter $\mathbf{g}$ in $\mathcal{D}_T$ as a property of $\vec{\mathbf{p}}$ and $\mathbf{x}$.

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- The degrees below $\mathbf{0}^{(5)}$ form an automorphism base.
- Rigidity is equivalent to full biinterpretability.

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Method: "Involves explicit translation of automorphism facts in definability facts via a coding of second order arithmetic."

Semi-computable sets

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A is semi-computable if there is a total computable function s_A, such that $s_A(x, y) \in \{x, y\}$ and if $\{x, y\} \cap A \neq \emptyset$ then $s_A(x, y) \in A$.

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Theorem (Arslanov, Cooper, Kalimullin)

If A is a semi-computable set then for every X:

$$(d_e(X) \vee d_e(A)) \wedge (d_e(X) \vee d_e(\overline{A})) = d_e(X).$$

Kalimullin pairs

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Theorem (Kalimullin)

A pair of sets A, B is a $\mathcal{K}$-pair if and only if their enumeration degrees $\mathbf{a}$ and $\mathbf{b}$ satisfy:

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x} \in \mathcal{D}_e)((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x}).$$

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Theorem (Kalimullin)

$\mathbf{0}'_e$ is the largest degree which can be represented as the least upper bound of a triple $\mathbf{a}, \mathbf{b}, \mathbf{c}$, such that $\mathcal{K}(\mathbf{a}, \mathbf{b})$, $\mathcal{K}(\mathbf{b}, \mathbf{c})$ and $\mathcal{K}(\mathbf{c}, \mathbf{a})$.

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Corollary (Kalimullin)

The enumeration jump is first order definable in $\mathcal{D}_e$.

Maximal $\mathcal{K}$-pairs

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Theorem (Ganchev, S)

A nonzero degree $\mathbf{a} \leq \mathbf{0}_e'$ is total if and only if it is the least upper bound of a maximal $\mathcal{K}$-pair.

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Question (Ganchev, S)

Is the the join of every maximal $\mathcal{K}$-pair total?

Defining totallity in $\mathcal{D}_e$

Theorem (Cai, Ganchev, Lempp, Miller, S)

If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair in $\mathcal{D}_e$ then there is a semi-computable set C, such that $A \leq_e C$ and $B \leq_e \overline{C}$.

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1. Ganchev, S had observed that if $\mathcal{TOT}$ is definable by maximal $\mathcal{K}$-pairs then the image of the relation 'c.e. in' is definable for non-c.e. degrees.
2. A result by Cai and Shore allowed us to complete this definition.

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- If $\mathcal{D}_T$ is rigid then $\mathcal{D}_e$ is rigid.
- The automorphism analysis for the enumeration degrees follows.

The total degrees as an automorphism base

Theorem (Selman)

A is enumeration reducible to B if and only if
$\{\mathbf{x} \in \mathcal{TOT} \mid d_e(A) \leq \mathbf{x}\} \supseteq \{\mathbf{x} \in \mathcal{TOT} \mid d_e(B) \leq \mathbf{x}\}$.

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The total enumeration degrees form a definable automorphism base of the enumeration degrees.

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Question

Can we improve this bound further?

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$$\mathcal{R} \subseteq \mathcal{D}_T(\leq \mathbf{0}') \hookrightarrow \mathcal{D}_e(\leq \mathbf{0}'_e)$$

The local coding theorem

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Definition

A set of degrees $\mathcal{Z}$ contained in $\mathcal{D}_T(\leq \mathbf{0}')$ is *uniformly low* if it is bounded by a low degree and there is a sequence $\{Z_i\}_{i<\omega}$, representing the degrees in $\mathcal{Z}$, and a computable function f such that $\{f(i)\}^{\emptyset'}$ is the Turing jump of $\bigoplus_{j<i} Z_j$.

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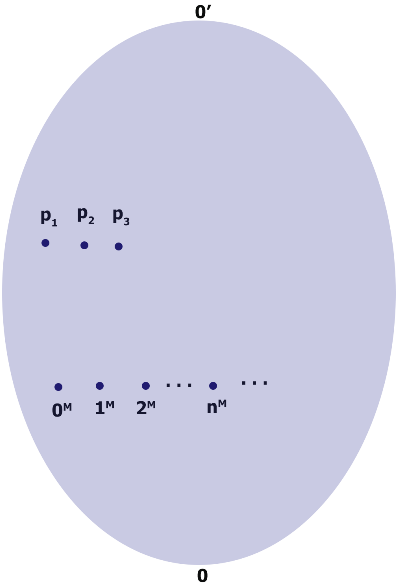
Theorem (Slaman and Woodin)

If $\mathcal{Z}$ is a uniformly low subset of $\mathcal{D}_T(\leq \mathbf{0}')$ then $\mathcal{Z}$ is definable from parameters in $\mathcal{D}_T(\leq \mathbf{0}')$.

The local coding theorem of Slaman and Woodin



Using parameters we can code a model of arithmetic $\mathcal{M} = (\mathbb{N}^{\mathcal{M}}, 0^{\mathcal{M}}, s^{\mathcal{M}}, +^{\mathcal{M}}, \times^{\mathcal{M}}, \leq^{\mathcal{M}})$.

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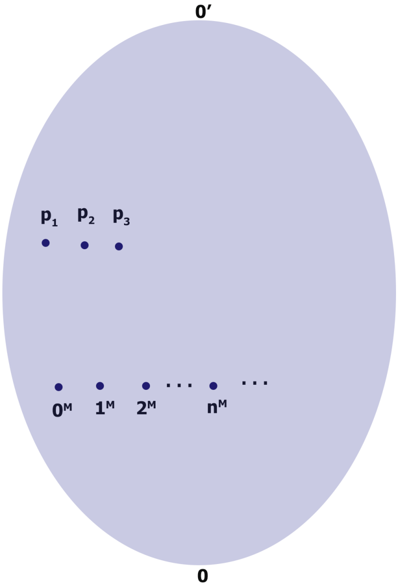

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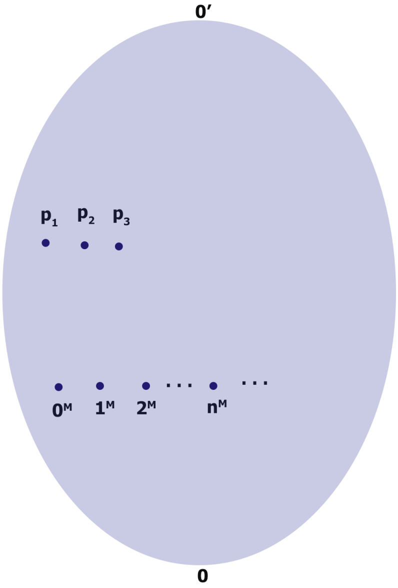

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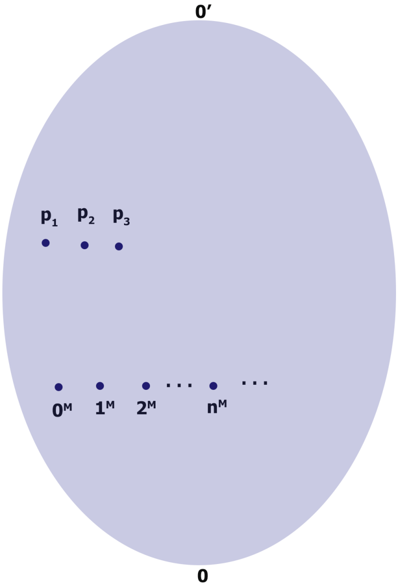

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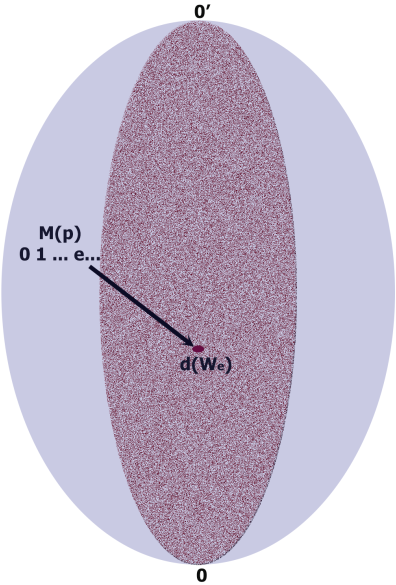
1. The set $\mathbb{N}^{\mathcal{M}}$ is definable with parameters $\vec{\mathbf{p}}$.
2. The graphs of s, $+$, $\times$ and the relation $\leq$ are definable with parameters $\vec{\mathbf{p}}$.
3. $\mathbb{N} \models \varphi$ iff $\mathcal{D}_T(\leq \mathbf{0}') \models \varphi_T(\vec{\mathbf{p}})$

An indexing of the c.e. degrees

Theorem (Slaman, Woodin)

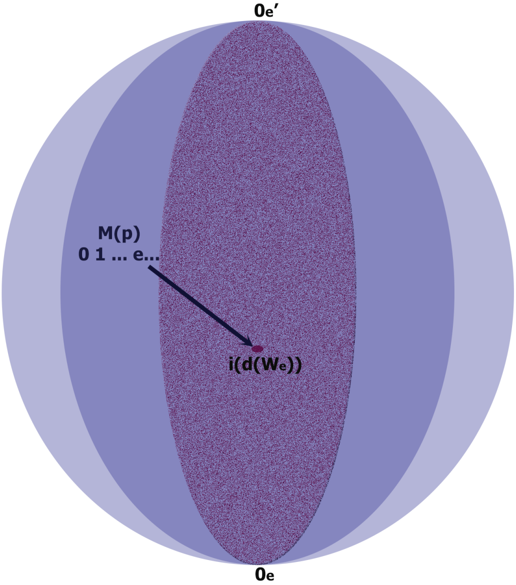
There are finitely many Δ^0_2 parameters which code a model of arithmetic $\mathcal{M}$ and an indexing of the c.e. degrees: a function $\psi : \mathbb{N}^{\mathcal{M}} \to \mathcal{D}_T(\leq \mathbf{0}')$ such that $\psi(e^{\mathcal{M}}) = d_T(W_e)$.



Towards a better automorphism base of $\mathcal{D}_e$

Theorem (Slaman, Woodin)

There are total Δ^0_2 parameters that code a model of arithmetic $\mathcal{M}$ and an indexing of the image of the c.e. Turing degrees.



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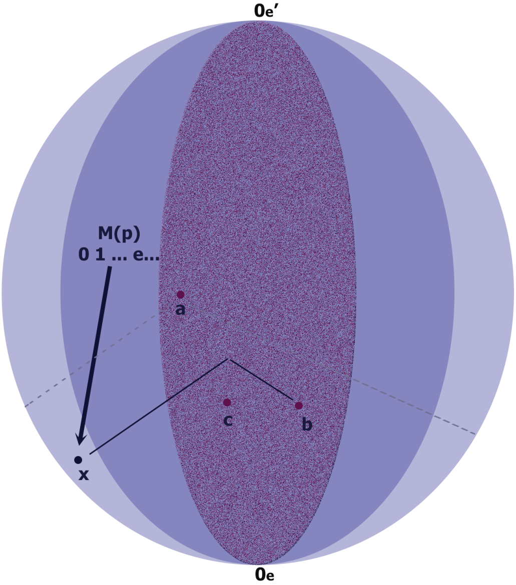
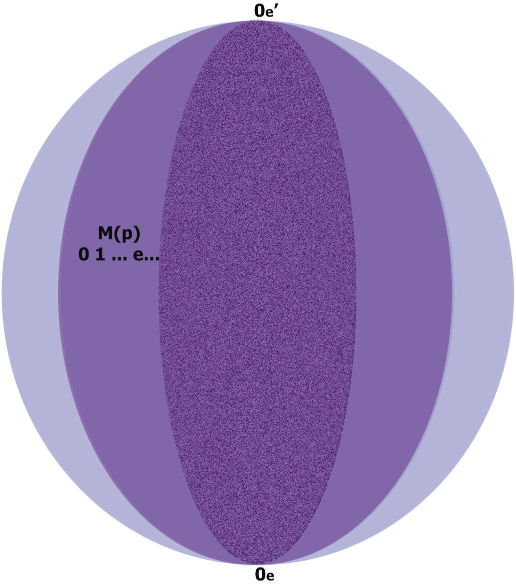
Idea: Can we extend this indexing to capture more elements in $\mathcal{D}_e$?



Towards a better automorphism base of $\mathcal{D}_e$

Theorem (Slaman, S)

If $\vec{p}$ defines a model of arithmetic $\mathcal{M}$ and an indexing of the image of the c.e. Turing degrees then $\vec{p}$ defines an indexing of the total Δ^0_2 enumeration degrees.



Proof flavour:

The image of the c.e. degrees

$\rightarrow$ The low co-d.c.e. e-degrees

$\rightarrow$ The low Δ^0_2 e-degrees

$\rightarrow$ The total Δ^0_2 e-degrees

The priority constructions

Theorem (Slaman, S)

Suppose $\mathbf{x}$ is a co-d.c.e. enumeration degree, $\mathbf{x}' = \mathbf{0}'_e$ and $\mathbf{y}$ is a Δ^0_2 enumeration degree, such that $\mathbf{y} \not\leq \mathbf{x}$. There are Δ^0_2 enumeration degrees $\mathbf{g}_i$ and Π^0_1 enumeration degrees $\mathbf{a}_i, \mathbf{c}_i, \mathbf{b}_i$ for $i = 1, 2$, such that:

1. $\mathbf{g}_i$ is the least element below $\mathbf{a}_i$ which joins $\mathbf{b}_i$ above $\mathbf{c}_i$;
2. $\mathbf{x} \leq \mathbf{g}_1 \vee \mathbf{g}_2$;
3. $\mathbf{y} \not\leq \mathbf{g}_1 \vee \mathbf{g}_2$.

The priority constructions

Theorem (Slaman, S)

Let $\mathbf{x}, \mathbf{y}$ be Δ^0_2 enumeration degrees, such that $\mathbf{x}' = \mathbf{0}'_e$ and $\mathbf{y} \not\leq \mathbf{x}$. There are Δ^0_2 enumeration degrees $\mathbf{g}_i$, Π^0_1 enumeration degrees $\mathbf{a}_i$ and low co-d.c.e. enumeration degrees $\mathbf{c}_i, \mathbf{b}_i$ for $i = 1, 2$, such that:

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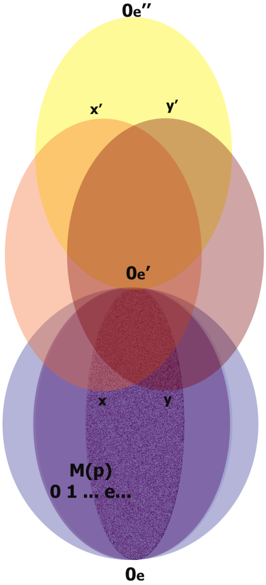
3. $\mathbf{y} \not\leq \mathbf{g}_1 \vee \mathbf{g}_2$.

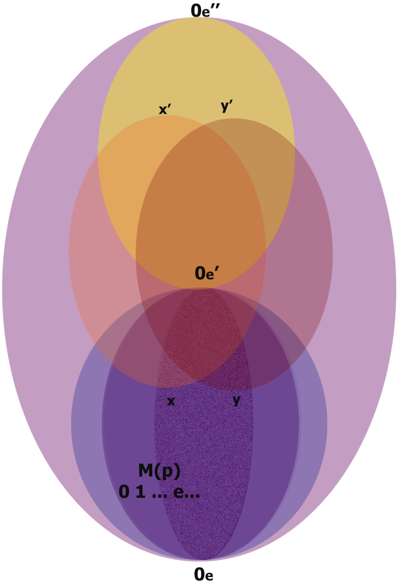
Moving outside the local structure

1. Extend to an indexing of all total degrees that are "c.e. in" and above some total Δ^0_2 enumeration degree.
   - The jump is definable.
   - The image of the relation "c.e. in" is definable.
2. Relativizing the previous theorem extend to an indexing of $\bigcup_{\mathbf{x}\leq\mathbf{0}'}\iota([\mathbf{x},\mathbf{x}'])$.

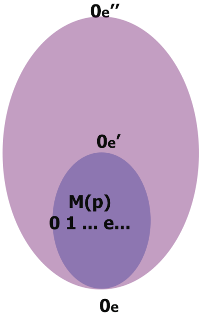


Moving outside the local structure

3. Extend to an indexing of all total degrees below $\mathbf{0}_e''$.



And now we iterate



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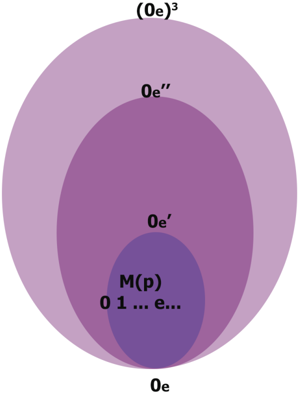

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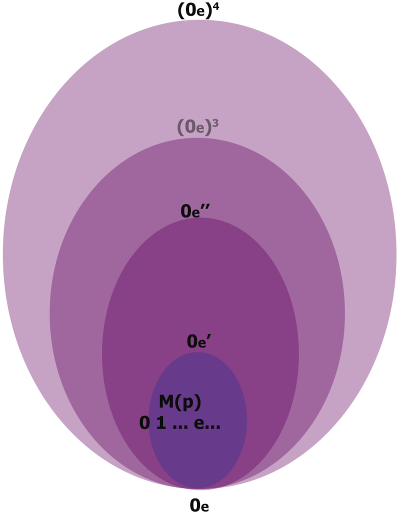

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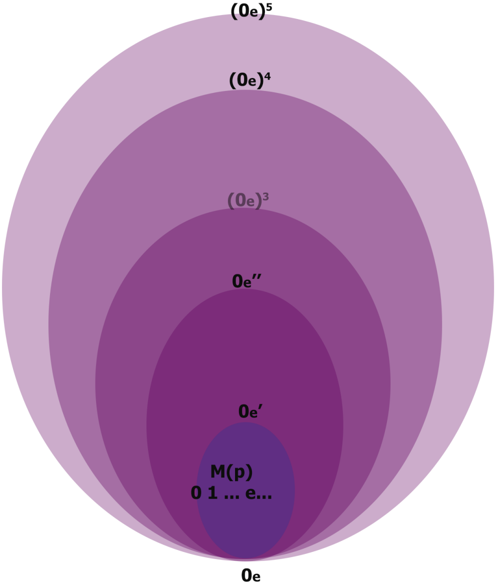

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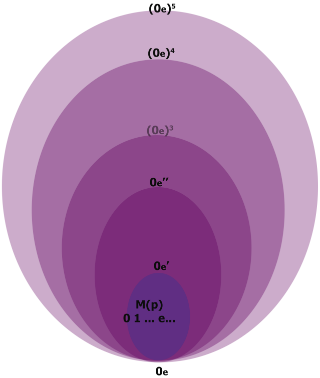

Theorem (Slaman, S)

Let n be a natural number and $\vec{p}$ be parameters that index the image of the c.e. Turing degrees. There is a definable from $\vec{p}$ indexing of the total Δ^0_{n+1} degrees.

Consequences

Theorem (Slaman, S)

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Consequences

Theorem (Slaman, S)

1. The enumeration degrees below $\mathbf{0}'_e$ are an automorphism base for $\mathcal{D}_e$.
2. The image of the c.e. Turing degrees is an automorphism base for $\mathcal{D}_e$.
3. If the structure of the c.e. Turing degrees is rigid then so is the structure of the enumeration degrees.

Definability in the local structure of the enumeration degrees

Theorem (Ganchev, S)

The class of $\mathcal{K}$-pairs below $\mathbf{0}_e'$ is first order definable in $\mathcal{D}_e(\leq \mathbf{0}_e')\ldots$

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Theorem (Ganchev, S)

The following classes of degrees are definable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$

1. The downwards properly Σ^0_2 degrees.
2. The upwards properly Σ^0_2 degrees.
3. The total enumeration degrees.

The jump hierarchy

Definition

- A degree $\mathbf{a}$ is Low_n if $\mathbf{a}^{(n)} = \mathbf{0}_T^{(n)}$.
- A degree $\mathbf{a}$ is High_n if $\mathbf{a}^{(n)} = \mathbf{0}_T^{(n+1)}$.

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The Low_1 enumeration degrees are first order definable in $\mathcal{D}_e(\leq \mathbf{0}_e')$: $\mathbf{a}$ is low if and only if every $\mathbf{b} \leq \mathbf{a}$ bounds a half of a $\mathcal{K}$-pair.

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Corollary (Nies, Shore, Slaman)

The total Low_{n+1} and the total High_n degrees are first order definable in $\mathcal{D}_e(\leq \mathbf{0}_e')$.

Total degrees in the local structure

Question

To what extent do the total enumeration degrees determine $\mathcal{D}_e(\leq \mathbf{0}_e')$?

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Note that Selman's theorem is not true locally. There are upwards properly Σ^0_2 enumeration degrees, degrees $\mathbf{a} < \mathbf{0}'_e$ such that the only total enumeration degree in $\mathcal{D}_e(\leq \mathbf{0}'_e)$ above them is $\mathbf{0}'_e$.

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Theorem (Soskov)

For every $\mathbf{a} \in \mathcal{D}_e$ there is a total $\mathbf{f} \geq \mathbf{a}$ such that $\mathbf{a}' = \mathbf{f}'$.

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Question

Is Soskov's jump inversion theorem true locally?

If it is true, then the jump classes would be fully definable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$:

$\mathbf{a} \in \text{Low}_{n+1}$ if and only if there is a total $\mathbf{f} \geq \mathbf{a}$, such that $\mathbf{f} \in \text{Low}_{n+1}$

$\mathbf{a} \in \text{High}_n$ if and only if every total $\mathbf{f} \geq \mathbf{a}$ are in High_n

The enumeration jump operator

Theorem (Ganchev, Sorbi)

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Theorem (Ganchev, S)

Let A be a Σ^0_2 set that is not c.e. There is a non c.e. set $B \leq_e A$ and a total set X such that

1. $B \leq_e X$.
2. $B' \equiv_e X'$.

Proof: Sacks Jump inversion + Good approximations of Σ^0_2 sets + The ability to "dump" elements in a constructed set $B = \Gamma(A)$ when they are not needed anymore.

Defining the jump classes in $\mathcal{D}_e(\leq \mathbf{0}'_e)$

$\mathbf{a} \in \mathrm{Low}_1$ if and only if

for every nonzero $\mathbf{b} \leq \mathbf{a}$

there is a nonzero $\mathbf{x} \leq \mathbf{b}$, such that $\mathbf{x}$ is half of a $\mathcal{K}$-pair.

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$\mathbf{a} \in \mathrm{Low}_{n+1}$ if and only if

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there is a nonzero $\mathbf{x} \leq \mathbf{b}$ and a total $\mathbf{f} \geq \mathbf{x}$, such that $\mathbf{f} \in \mathrm{Low}_{n+1}$.

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$\mathbf{a} \in \mathrm{High}_n$ if and only if

there is a nonzero $\mathbf{b} \leq \mathbf{a}$ such that

for every nonzero $\mathbf{x} \leq \mathbf{b}$ all total $\mathbf{f} \geq \mathbf{x}$ are in High_n.

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Theorem (Ganchev, S)

All jump classes are first order definable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

The end

Thank you!