The Turing universe in the context of enumeration reducibility

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The spectrum of relative definability

How can a set of natural numbers B be used to define a set of natural numbers A.

- There is an algorithm, which determines whether $x \in A$ using finite information about memberships in B: Turing reducibility.
- There is an algorithm, which enumerates instances of memberships in A from instances of memberships in B: enumeration reducibility.

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Turing reducibility

- $A \leq_T B$ iff χ_A is computable using oracle B.
- $A \leq_T B$ iff $A \oplus \overline{A}$ is c.e. in B.
- A is c.e. in B iff there is a c.e. set W such that $x \in A$ iff there are finite sets D_B and $D_{\overline{B}}$, such that $\langle x, D_B \oplus D_{\overline{B}} \rangle \in W$ and $D_B \oplus D_{\overline{B}} \subseteq B \oplus \overline{B}$.

Definition

$A \leq_e B$ if and only if there is a c.e. set W, such that $A = W(B) = \{x \mid \exists u(\langle x, u \rangle \in W \wedge D_u \subseteq B)\}$.

So A is c.e. in B if and only if $A \leq_e B \oplus \overline{B}$.

$A \leq_T B$ if and only if $A \oplus \overline{A} \leq_e B \oplus \overline{B}$.

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The structure of the enumeration degrees

- $A \equiv_e B$ if $A \leq_e B$ and $B \leq_e A$.
- $d_e(A) = \{B \mid A \equiv_e B\}$.
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- $\mathbf{0}_e = d_e(\emptyset) = \{W \mid W$ is c.e. $\}$.
- $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- $\mathcal{D}_e = \langle D_e, \leq, \vee, \mathbf{0}_e \rangle$ is an upper semi-lattice with least element.

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The total degrees

Proposition

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, *defined by* $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, *preserves the order and the least upper bound.*

The substructure of the total e-degrees is defined as $\mathcal{TOT} = \iota(D_T)$.

$$(\mathcal{D}_T, \leq_T, \vee, \mathbf{0}_T) \cong (\mathcal{TOT}, \leq_e, \vee, \mathbf{0}_e) \subseteq (\mathcal{D}_e, \leq_e, \vee, \mathbf{0}_e)$$

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More connections between $\mathcal{D}_T$ and $\mathcal{D}_e$

- B is c.e. in A if and only if $B \leq_e A \oplus \overline{A}$.
- Selman's Theorem: $A \leq_e B$ if and only if

$$\{X \mid B \text{ is c.e. in } X\} \subseteq \{X \mid A \text{ is c.e. in } X\}.$$

$$\left\{d_e(X \oplus \overline{X}) \mid B \leq_e X \oplus \overline{X}\right\} \subseteq \left\{d_e(X \oplus \overline{X}) \mid A \leq_e X \oplus \overline{X}\right\}.$$

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- Let $K_A = \{x \mid x \in W_x(A)\}$. Note that $K_A \equiv_e A$.
- The jump of A is $A' = K_A \oplus \overline{K_A}$. Then $d_e(A)' = d_e(A')$.
- The embedding ι preserves the jump operation.

$$(\mathcal{D}_T, \leq_T, \vee, \mathbf{0}_T, ') \cong (\mathcal{TOT}, \leq_e, \vee, \mathbf{0}_e, ') \subseteq (\mathcal{D}_e, \leq_e, \vee, \mathbf{0}_e, ')$$

Theorem (Soskov's Jump Inversion Theorem)

For every $\mathbf{x} \in \mathcal{D}_e$ there exists a total e-degree $\mathbf{a} \geq \mathbf{x}$, such that $\mathbf{a}' = \mathbf{x}'$.

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Computable model theory

Fix a countable relational structure $\mathcal{A} = (\mathbb{N}, R_1 \dots R_k)$.

Definition (Richter)

The degree spectrum of $\mathcal{A}$, denoted by $DS_T(\mathcal{A})$, is the set of Turing degrees of the diagrams of structures $\mathcal{B} \cong \mathcal{A}$.

If $DS_T(\mathcal{A})$ has a least member, it is the (Turing) degree of $\mathcal{A}$.

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Torsion-free abelian groups of rank 1

A torsion free abelian group of rank 1 G is (isomorphic to) a subgroup of $(\mathbb{Q}, +, =)$.

Definition

Let p be a prime number and $a \in G$.

$$h_p(a) = \begin{cases} \text{the largest } k, & \text{such that } p^k | a \text{ in } G; \\ \infty, & \text{if } \forall k (p^k | a \text{ in } G) . \end{cases}$$

Here $p^k | a$ in G if there exists $b \in G$ such that $p^k . b = a$.

Example: If $G = \mathbb{Q}$ then for all nonzero a and all p, $h_p(a) = \infty$, because for all k, $p^k . \frac{a}{p^k} = a$.
If $G = \mathbb{Z}$ then for all nonzero a and all but finitely many p, $h_p(a) = 0$.

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The type of G

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The characteristic of an element $a \in G$ is the sequence:

$$\chi(a) = (h_{p_0}(a), h_{p_1}(a), \ldots h_{p_n}(a), \ldots).$$

So if $a, b \neq 0$ then $\chi(a) =^* \chi(b)$.
The type of G, denoted $\chi(G)$ is the equivalence class of $\chi(a)$ for any $a \neq 0$ in G.

Baer noticed that there is a TFA1 group of every possible type.

Theorem (Baer)

Two torsion-free abelian groups of rank 1 are isomorphic if and only if they have the same type.

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Two torsion-free abelian groups of rank 1 are isomorphic if and only if they have the same type.

The type of G

Definition

The characteristic of an element $a \in G$ is the sequence:

$$\chi(a) = (h_{p_0}(a), h_{p_1}(a), \dots h_{p_n}(a), \dots).$$

So if $a, b \neq 0$ then $\chi(a) =^* \chi(b)$.
The type of G, denoted $\chi(G)$ is the equivalence class of $\chi(a)$ for any $a \neq 0$ in G.

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Let $S(G) = \{\langle i,j\rangle \mid j \leq$ the i-th element of $\chi(G)\}$.

Theorem (Downey, Jockusch)

The degree spectrum of G is precisely $\{deg_T(Y) \mid S(G)$ is c.e. in $Y\}$.

- Every set A can be coded (is m-equivalent) to a type of some group G.
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Least jump enumeration

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Theorem (Richter)

There is a non-c.e. set A such that A is c.e. in two sets B and C which form a minimal pair.

Hence there is a set A, such that $\mathcal{C}(A)$ does not have a member of least degree.

Theorem (Coles, Downey, Slaman)

For every sets A the set: $\mathcal{C}(A)' = \{X' \mid A \text{ is c.e. in } X\}$ *has a member of least degree.*

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Fix a countable relational structure $\mathcal{A} = (\mathbb{N}, R_1 \dots R_k)$.

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The enumeration degree spectrum of $\mathcal{A}$, denoted by $DS_e(\mathcal{A})$, is the set of e-degrees of the positive diagrams of structures $\mathcal{B} \cong \mathcal{A}$.

If $DS_e(\mathcal{A})$ has a least member, it is the (enumeration) degree of $\mathcal{A}$.

- Consider the structure $\mathcal{A}^+ = (\mathbb{N}, R_1, \overline{R_1} \dots R_k, \overline{R_k})$.
- Then $DS_e(\mathcal{A}^+) \subseteq \mathcal{TOT}$.
- In fact $DS_e(\mathcal{A}^+) = \{\iota(\mathbf{a}) \mid \mathbf{a} \in DS_T(\mathcal{A})\}$.
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TFA1 groups in the e-degrees

- Let G be a torsion-free abelian group of rank 1.
- Recall that the Turing degree spectrum of G is precisely $\{deg_T(Y) \mid S(G) \text{ is c.e. in } Y\}$.
- Note that the diagram of a group is enumeration equivalent to its positive diagram, as addition is a total function.
- In particular $DS_e(G) = DS_e(G^+) = \iota(DS_T(G))$.
- Denote $d_e(S(G))$ by $\mathbf{s}_G$-the type degree of G. The enumeration degree spectrum of G is:

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Defining the Turing jump operator

Theorem (Shore, Slaman)

The Turing jump operator is first order definable in $\mathcal{D}_T$.

1. Slaman and Woodin: The double jump is first order definable in $\mathcal{D}_T$.
2. For every $\mathbf{a} \not\leq_T \mathbf{0}'_T$ there is $\mathbf{g}$ such that $\mathbf{a} \vee \mathbf{g} = \mathbf{g}''$

Hence $\mathbf{0}'$ is the greatest degree which does not join any $\mathbf{g}$ to $\mathbf{g}''$.

Ingredient 1: Slaman and Woodin's analysis of the automorphisms of the Turing degrees and *"involves explicit translation of automorphism facts in definability facts via a coding of second order arithmetic".*

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Semi-recursive sets in the enumeration degrees

Definition (Jockusch)

A set of natural numbers A is semi-recursive if there is a total computable selector function s_A, such that $s_A(x, y) \in \{x, y\}$ and if $\{x, y\} \cap A \neq \emptyset$ then $s_A(x, y) \in A$.

- For every set A the set $L_A = \{\sigma \in 2^{<\omega} \mid \sigma \leq_L \chi_A\}$ is semi-recursive.

Theorem (Jockusch)

For every noncomputable set B there is a semi-recursive set $A \equiv_T B$ such that both A and $\overline{A}$ are not c.e.

Theorem (Arslanov, Cooper, Kalimullin)

If A is a semi-recursive set, which is not c.e. and not co-c.e then $d_e(A)$ and $d_e(\overline{A})$ form a minimal pair.

$$(\forall \mathbf{x} \in \mathcal{D}_e)((d_e(A) \vee \mathbf{x}) \wedge (d_e(\overline{A}) \vee \mathbf{x}) = \mathbf{x}).$$

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If A is a semi-recursive set, which is not c.e. and not co-c.e then $d_e(A)$ and $d_e(\overline{A})$ form a minimal pair.

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Semi-recursive sets in the enumeration degrees

Definition (Jockusch)

A set of natural numbers A is semi-recursive if there is a total computable selector function s_A, such that $s_A(x, y) \in \{x, y\}$ and if $\{x, y\} \cap A \neq \emptyset$ then $s_A(x, y) \in A$.

- For every set A the set $L_A = \{\sigma \in 2^{<\omega} \mid \sigma \leq_L \chi_A\}$ is semi-recursive.

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$\mathcal{K}$-pairs in the enumeration degrees

Definition (Kalimullin)

A pair of sets A, B are called a $\mathcal{K}$-pair if there is a c.e. set W, such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

- A trivial example is $\{A, U\}$ and $\{U, A\}$, where U is c.e.
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$\mathcal{K}$-pairs are invisible in the Turing universe

- $\mathcal{K}$-pairs are always quasi-minimal: the only total degree below either of them is $\mathbf{0}_e$.
- A consequence of the existence of nontrivial $\mathcal{K}$-pairs in $\mathcal{D}_e$ is that the Slaman-Shore property fails, there is a degree $\mathbf{a} \not\leq_e \mathbf{0}'_e$, such that for every $\mathbf{g}$, $\mathbf{a} \vee \mathbf{g} <_e \mathbf{g}''$.
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$\mathcal{K}$-pairs and the definability of the enumeration jump

Theorem (Kalimullin)

$\mathbf{0}'_e$ *is the largest degree which can be represented as the least upper bound of a triple* $\mathbf{a}, \mathbf{b}, \mathbf{c}$*, such that* $\mathcal{K}(\mathbf{a}, \mathbf{b})$*,* $\mathcal{K}(\mathbf{b}, \mathbf{c})$ *and* $\mathcal{K}(\mathbf{c}, \mathbf{a})$*.*

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The enumeration jump is first order definable in $\mathcal{D}_e$*.*

Theorem (S, Ganchev)

For every nonzero enumeration degree $\mathbf{u} \in \mathcal{D}_e$*,* $\mathbf{u}'$ *is the largest among all least upper bounds* $\mathbf{a} \vee \mathbf{b}$ *of nontrivial* $\mathcal{K}$*-pairs* $\{\mathbf{a}, \mathbf{b}\}$*, such that* $\mathbf{a} \leq_e \mathbf{u}$*.*

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The local structures

- The local structure of the Turing degrees $\mathcal{D}_T(\leq \mathbf{0}'_T)$ consists of all Δ^0_2 Turing degrees.
- The structure $\mathcal{R}$ consists of the of the computably enumerable degrees.
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- Recall that $\iota : \mathcal{D}_T \to \mathcal{D}_e$ preserves the jump, hence $\mathcal{D}_T(\leq \mathbf{0}')$ embeds in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

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- A pair of degrees $\mathbf{a}, \mathbf{b}$ are a splitting of $\mathbf{c}$ if $\mathbf{a} < \mathbf{c}$, $\mathbf{b} < \mathbf{c}$ and $\mathbf{a} \vee \mathbf{b} = \mathbf{c}$.

Theorem (Harrington)

There exists a c.e. Turing degree $\mathbf{a} <_T \mathbf{0}'_T$, such that no pair of c.e. degrees above $\mathbf{a}$ are a splitting of $\mathbf{0}'_T$.

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Definability in the local structures

Definition

1. For every $n \geq 1$ the class of low$_n$ degrees is $L_n = \{\mathbf{a} \leq \mathbf{0}' \mid \mathbf{a}^n = \mathbf{0}^n\}$.
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For every $n \geq 1$ the classes L_{n+1} and H_n are first order definable in $\mathcal{D}_T(\leq \mathbf{0}')$.

Note that the definability of L_1 in $\mathcal{D}_T(\leq \mathbf{0}'_T)$ or in $\mathcal{R}$ remains open.

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1. The theory of first order arithmetic can be interpreted in $\mathcal{R}$ and $\mathcal{D}_T(\leq \mathbf{0}')$.
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Definability in the local structure of the enumeration degrees

Initial motivation: Prove that the theory of first order arithmetic is interpretable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

Theorem (Slaman and Woodin)

A uniformly low antichain can be coded by parameters in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

1. Non-trivial Σ^0_2 $\mathcal{K}$-pairs are low.
2. A $\mathcal{K}$-system, a sequence of $\{\mathbf{a}_i\}_{i \in I}$ of e-degrees such that if $i \neq j$ then $\mathcal{K}(\mathbf{a}_i, \mathbf{a}_j)$, is an antichain.
3. Every nonzero Δ^0_2 e-degree bounds a countable $\mathcal{K}$-system.

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1. Non-trivial Σ^0_2 $\mathcal{K}$-pairs are low.
2. A $\mathcal{K}$-system, a sequence of $\{\mathbf{a}_i\}_{i\in I}$ of e-degrees such that if $i \neq j$ then $\mathcal{K}(\mathbf{a}_i, \mathbf{a}_j)$, is an antichain.
3. Every nonzero Δ^0_2 e-degree bounds a countable $\mathcal{K}$-system.

Definability in the local structure of the enumeration degrees

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$\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x})((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x})$

Is it enough to require that this formula is satisfied by all Σ^0_2 e-degrees?

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There is a first order formula $\mathcal{LK}(x, y)$, which defines the $\mathcal{K}$-pairs in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

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We know that:

- $\mathcal{TOT} \cap \mathcal{D}_e(\geq \mathbf{0}'_e)$ is first order definable.
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Is $\mathcal{TOT}$ first order definable in $\mathcal{D}_e$?

Recall that the total degrees are an automorphism base for $\mathcal{D}_e$.

A positive answer would connect the problems of the existence of a non-trivial automorphism in both structures.

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The end

Thank you!