The automorphism group of the enumeration degrees

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Enumeration reducibility

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$A \leq_e B$ if there is a c.e. set W, such that

$$A = W(B) = \{x \mid \exists D(\langle x, D\rangle \in W \ \& \ D \subseteq B)\}.$$

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- $d_e(A) = \{B \mid A \leq_e B \,\&\, B \leq_e A\}$.
- $d_e(A) \leq d_e(B)$ if $A \leq_e B$.
- $\mathbf{0}_e = d_e(\emptyset)$ consists of all c.e. sets.
- $d_e(A \oplus B) = d_e(A) \vee d_e(B)$.
- $d_e(A)' = d_e(L_A \oplus \overline{L_A})$, where $L_A = \{e \mid e \in W_e(A)\}$.

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$\mathcal{D} = \langle D, \leq, \vee, ', \mathbf{0}\rangle$ is an upper semi-lattice with least element and jump operation.

What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

Proposition

$$A \leq_T B \Leftrightarrow A \oplus \overline{A} \text{ is c.e. in } B \Leftrightarrow A \oplus \overline{A} \leq_e B \oplus \overline{B}.$$

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$$(\mathcal{D}_T, \leq_T, \vee, ' \mathbf{0}_T) \cong (\mathcal{TOT}, \leq_e, \vee, ', \mathbf{0}_e) \subseteq (\mathcal{D}_e, \leq_e, \vee, ' \mathbf{0}_e)$$

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Theorem (Selman)

$A \leq_e B$ if and only if the set of total enumeration degrees above B is a subset of the set of total enumeration degrees above A.

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$A \leq_e B$ if and only if the set of total enumeration degrees above B is a subset of the set of total enumeration degrees above A. $\mathcal{TOT}$ is an automorphism base for $\mathcal{D}_e$.

Defining the Turing jump operator

Theorem (Shore, Slaman)

The Turing jump operator is first order definable in $\mathcal{D}_T$.

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1. The double jump is first order definable in $\mathcal{D}_T$: Slaman and Woodin's analysis of the automorphisms of the Turing degrees and *"involves explicit translation of automorphism facts in definability facts via a coding of second order arithmetic".*
2. An additional structural fact: for every $\mathbf{a} \not\leq_T \mathbf{0}'_T$ there is $\mathbf{g}$ such that $\mathbf{a} \vee \mathbf{g} = \mathbf{g}''$.

$\mathcal{K}$-pairs in the enumeration degrees

Definition (Kalimullin)

A pair of sets A, B are called a $\mathcal{K}$-pair if there is a c.e. set W, such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

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A pair of sets A, B are a $\mathcal{K}$-pair if and only if their enumeration degrees **a** *and* **b** *satisfy:*

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \leftrightharpoons (\forall \mathbf{x} \in \mathcal{D}_e)((\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x}).$$

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- There are no $\mathcal{K}$-pairs in the structure of the Turing degrees.

$\mathcal{K}$-pairs and the definability of the enumeration jump

Theorem (Kalimullin)

$\mathbf{0}'_e$ is the largest degree which can be represented as the least upper bound of a triple $\mathbf{a}, \mathbf{b}, \mathbf{c}$, such that $\mathcal{K}(\mathbf{a}, \mathbf{b})$, $\mathcal{K}(\mathbf{b}, \mathbf{c})$ and $\mathcal{K}(\mathbf{c}, \mathbf{a})$.

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$\mathcal{K}$-pairs and the definability of the enumeration jump

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Theorem (Ganchev, S)

For every nonzero enumeration degree $\mathbf{u} \in \mathcal{D}_e$, $\mathbf{u}'$ is the largest among all least upper bounds $\mathbf{a} \vee \mathbf{b}$ of nontrivial $\mathcal{K}$-pairs $\{\mathbf{a}, \mathbf{b}\}$, such that $\mathbf{a} \leq_e \mathbf{u}$.

Definability in the local structure of the enumeration degrees

Theorem (Ganchev, S)

The class of $\mathcal{K}$-pairs below $\mathbf{0}'_e$ is first order definable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

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The class of total degrees is first order definable in $\mathcal{D}_e(\leq \mathbf{0}'_e)$.

Open question

We know that:

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A positive answer would connect the problems of the existence of a non-trivial automorphism in both structures.

One step further in the dream world

Theorem (Ganchev,S)

For every nonzero enumeration degree $\mathbf{u} \in \mathcal{D}_e$,

$$\mathbf{u}' = max\left\{\mathbf{a} \vee \mathbf{b} \mid \mathcal{K}(\mathbf{a}, \mathbf{b})\ \&\ \mathbf{a} \leq_e \mathbf{u}\right\}.$$

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- Suppose that a degree is total if and only if it is the least upper bound of a maximal $\mathcal{K}$-pair.
- The relation $\mathbf{x}$ is c.e. in $\mathbf{u}$ would also be definable for total degrees by :

$$\exists\mathbf{a}\exists\mathbf{b}(\mathbf{x} = \mathbf{a} \vee \mathbf{b} \ \& \ \mathcal{K}(\mathbf{a},\mathbf{b}) \ \& \ \mathbf{a} \leq_e \mathbf{u}).$$

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- Then for total $\mathbf{u}$, our definition of the jump would read $\mathbf{u}'$ is the largest total degree, which is c.e. in $\mathbf{u}$.

Definability via automorphism analysis in $\mathcal{D}_e$

Slaman and Woodin: *Definability in Degree Structures*, 1995.

1. Coding theorem.
2. A characterization of an automorphism in terms of a countable object.
3. A finite automorphism base.

Effectively coding and decoding

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Definition

A countable relation $\mathcal{R} \subseteq \mathcal{D}_e^n$ is e-presented beneath a set A if there is a set $W \leq_e A$ such that
$\mathcal{R} = \{(\mathbf{d}_e(W_{i_1}(A)), \dots, \mathbf{d}_e(W_{i_n}(A))) \mid (i_1, \dots, i_n) \in W\}$.

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Theorem (Coding Theorem)

For every n there is a formula φ_n, such that for every countable relation on enumeration degrees $\mathcal{R} \subseteq \mathcal{D}_e^n$ which is e-presented beneath R there are parameters $\bar{\mathbf{p}} \leq_e \mathbf{d}_e(R)''$ such that
$\mathcal{R} = \{(\mathbf{x}_1, \dots, \mathbf{x}_n) \mid \mathcal{D}_e \models \varphi_n(\mathbf{x}_1, \dots, \mathbf{x}_n, \bar{\mathbf{p}})\}$.

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Theorem (Decoding Theorem)

Let $\mathcal{R} \subseteq \mathcal{D}_e^n$ be countable and coded by parameters $\bar{\mathbf{p}}$. Let $\mathbf{d}_e(P)$ be an upper bound on these parameters. Then there is a presentation W of $\mathcal{R}$, such that $W \leq_e P^5$.

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Denote by $\varphi(\mathbf{u}, \mathbf{u}') : \mathbf{u}' = \max\{\mathbf{a} \vee \mathbf{b} \mid \mathcal{K}(\mathbf{a}, \mathbf{b}) \ \& \ \mathbf{a} \leq_e \mathbf{u}\}$.

Theorem

Let $\mathcal{I} \subseteq \mathcal{D}_e$ be a jump ideal. For every element $\mathbf{u} \in \mathcal{I}$ we have the following equivalence: $\mathcal{I} \models \varphi_{\mathcal{J}}(\mathbf{u}, \mathbf{u}') \leftrightarrow \mathcal{D}_e \models \varphi_{\mathcal{J}}(\mathbf{u}, \mathbf{u}')$.

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Corollary

If ρ is an automorphism of a jump ideal $\mathcal{I}$ then $\rho(\mathbf{x}') = \rho(\mathbf{x})'$.

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Corollary

Let $\mathcal{I} \subseteq \mathcal{J}$ be jump ideals in $\mathcal{D}_e$. Let $\rho : \mathcal{J} \to \mathcal{J}$ be an automorphism of $\mathcal{J}$. Then $\rho \restriction \mathcal{I}$ is an automorphism of $\mathcal{I}$.

Example 2: Automorphisms are locally presented

Let $\mathcal{C} \subseteq \mathcal{D}_e$ be countable and e-presented beneath C. Let $\langle \mathbb{N}, 0, s, +, *, \mathcal{C}, \psi \rangle$ be the standard model of arithmetic together with a counting $\psi : \mathbb{N} \to \mathcal{C}$.

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Corollary

Let $\mathcal{I} \subseteq \mathcal{J}$ be jump ideals in $\mathcal{D}_e$. Let $\rho : \mathcal{J} \to \mathcal{J}$ be an automorphism of $\mathcal{J}$. If $\mathcal{I}$ is countable and e-presented beneath I and $I \in \mathcal{J}$ then $\rho \restriction \mathcal{I}$ is arithmetically presented in I.

Persistent automorphisms

Definition

Let $\mathcal{I} \subseteq \mathcal{D}_e$ be countable jump ideal. An automorphism $\rho : \mathcal{I} \to \mathcal{I}$ is called persistent if for every $\mathbf{x} \in \mathcal{D}_e$ there is a countable jump ideal $\mathcal{J}$ and an automorphism $\rho_1 : \mathcal{J} \to \mathcal{J}$ such that $\{\mathbf{x}\} \cup \mathcal{I} \subseteq \mathcal{J}$ and $\rho_1 \restriction \mathcal{I} = \rho$.

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Theorem

Let $\mathcal{I} \subseteq \mathcal{J}$ be countable jump ideals in $\mathcal{D}_e$. Every persistent automorphism of $\mathcal{I}$ can be extended to a persistent automorphism of $\mathcal{J}$.

Generic persistence

Definition

Let $\mathcal{I} \subseteq \mathcal{D}_e$ be a jump ideal. An automorphism $\rho : \mathcal{I} \to \mathcal{I}$ is generically persistent if for in some generic extension $V[G]$ in which $\mathcal{I}$ is countable, ρ is persistent.

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3. *Every persistent automorphism of a countable ideal $\mathcal{I} \subseteq \mathcal{D}_e$ can be extended to an automorphism π of $\mathcal{D}_e$.*

Arithmetically representing automorphisms of $\mathcal{D}_e$.

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Corollary

Let π be an automorphism of $\mathcal{D}_e$. There exists an arithmetic formula φ such that $\varphi(X, Y)$ is true if and only if $\pi(\mathbf{d}_e(X)) = \mathbf{d}_e(Y)$. There are therefore at most countably many automorphisms of $\mathcal{D}_e$.

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Corollary

The structure of the enumeration degrees $\mathcal{D}_e$ has an automorphism base consisting of:

1. *A single total degree $\mathbf{g}$.*
2. *A single quasiminimal degree $\mathbf{a}$.*
3. *The enumeration degrees below $\mathbf{0}_e^8$.*

Assigning reals

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1. A countable ω-model $\mathcal{M}$ of T.
2. a jump ideal $\mathcal{I}$ in $\mathcal{D}_e$.
3. A bijection $f : \mathcal{D}_e^{\mathcal{M}} \to \mathcal{I}$, such that for all $\mathbf{x}, \mathbf{y} \in \mathcal{D}_e^{\mathcal{M}}$, if $\mathcal{M} \models \mathbf{x} \geq \mathbf{y}$ then $f(\mathbf{x}) \geq f(\mathbf{y})$.

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Theorem

If $(\mathcal{M}, f, \mathcal{I})$ is an e-assignment of reals then $\mathcal{D}_e^{\mathcal{M}} = \mathcal{I}$ and f is an automorphism of $\mathcal{I}$.

Extendably assigning reals

Definition

An e-assignment of reals $(\mathcal{M}, f, \mathcal{I})$ is extendable if for every $\mathbf{z} \in \mathcal{D}_e$ there exists an e-assignment of reals $(\mathcal{M}_1, f_1, \mathcal{I}_1)$ such that $\mathcal{D}_e^{\mathcal{M}} \subseteq \mathcal{D}_e^{\mathcal{M}_1}$, $\mathcal{I} \cup \{\mathbf{z}\} \subseteq \mathcal{I}_1$ and $f \subseteq f_1$.

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Theorem

If $(\mathcal{M}, f, \mathcal{I})$ is an extendible e-assignment then there is an automorphism $\pi : \mathcal{D}_e \to \mathcal{D}_e$, such that for all $\mathbf{x} \in \mathcal{D}_e^{\mathcal{M}}$, $\pi(\mathbf{x}) = f(\mathbf{x})$.

Example 3: Interpreting automorphisms

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Theorem

Let $\mathbf{g}$ be the enumeration degree of an 8-generic $g \leq_e \emptyset^8$. Then the relation $Bi(\bar{\mathbf{c}}, \mathbf{d})$, stating that "$\bar{\mathbf{c}}$ codes a model of arithmetic with a unary predicate for X and $\mathbf{d}_e(X) = \mathbf{d}$" is definable in $\mathcal{D}_e$ using parameter $\mathbf{g}$. $\mathcal{D}_e$ is biinterpretable with second order arithmetic using parameters.

Definability in $\mathcal{D}_e$

Corollary

Let $R \subseteq (2^\omega)^n$ be relation definable in second order arithmetic and invariant under enumeration reducibility.

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2. *If $\mathcal{R}$ is invariant under automorphisms then $\mathcal{R}$ is definable without parameters in $\mathcal{D}_e$.*
   In particular the hyperarithmetic jump operation is first order definable in $\mathcal{D}_e$.

The end

Thank you!