CUPPING AND DEFINABILITY IN THE LOCAL STRUCTURE OF THE ENUMERATION DEGREES

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ABSTRACT. We show that every splitting of $\mathbf{0}'_e$ in the local structure of the enumeration degrees, $\mathcal{G}_e$, contains at least one low-cuppable member. We apply this new structural property to show that the classes of all $\mathcal{K}$-pairs in $\mathcal{G}_e$, all downwards properly Σ^0_2 enumeration degrees and all upwards properly Σ^0_2 enumeration degrees are first order definable in $\mathcal{G}_e$.

1. INTRODUCTION

Enumeration reducibility introduced by Friedberg and Rogers [11] arises as a way to compare the computational strength of the positive information contained in sets of natural numbers. A set A is enumeration reducible to a set B if given any enumeration of the set B, one can effectively compute an enumeration of the set A. The induced structure of the enumeration degrees $\mathcal{D}_e$ is an upper semilattice with least element and jump operation. This structure can be viewed as an extension of the structure of the Turing degrees, as there is an embedding $\iota : \mathcal{D}_T \Rightarrow \mathcal{D}_e$ which preserves the order, the least upper bound and the jump operation.

The jump operation gives rise to a local substructure, $\mathcal{G}_e$, consisting of all degrees in the interval enclosed by the least degree and its first jump. The elements of the local structure of the enumeration degrees can be characterized in terms of their relationship to the arithmetical hierarchy. Cooper [4] shows that the elements of $\mathcal{G}_e$ are precisely the enumeration degrees which contain Σ^0_2 sets, or equivalently are made up entirely of Σ^0_2 sets, which we call Σ^0_2 degrees. Naturally the arithmetical hierarchy gives rise to a substructure of $\mathcal{G}_e$, the substructure of the Δ^0_2 enumeration degrees, the enumeration degrees which contain Δ^0_2 sets. This is a proper substructure of $\mathcal{G}_e$, as there are *properly Σ^0_2 enumeration degrees*, degrees which do not contain any Δ^0_2 set. Another way to partition the elements of $\mathcal{G}_e$ is in terms of the jump hierarchy. We distinguish between low and non-low enumeration degrees, where a degree is low if its enumeration jump is as low as possible, namely $\mathbf{0}'_e$. In terms of their relationship with the Turing degrees the elements of $\mathcal{G}_e$ can be divided into total enumeration degrees, ones that are images of Turing degrees under the embedding ι, or non-total degrees, ones that are not.

Each of these subclasses of $\mathcal{G}_e$ is defined by singling out a property of the sets that comprise an element in the class. For example McEvoy [18] proves that an enumeration degree is low if and only if all of its members are Δ^0_2. An enumeration degree is total if and only if it contains a set of the form $A \oplus \overline{A}$. One of the goals in the study of the local structure of the enumeration degrees is to find a

relationship between the natural information content of the sets in a Σ_2^0 degree and its definability in $\mathcal{G}_e$. In this article we give the first example of such a relationship, we prove the local definability of the enumeration degrees of sets that form a $\mathcal{K}$-pair.

Definition 1.1. *[Kalimullin] A pair of sets of natural numbers A and B is a $\mathcal{K}$-pair if there is a c.e. set W such that:*

$$A \times B \subseteq W \ \& \ \overline{A} \times \overline{B} \subseteq \overline{W}.$$

The notion of a $\mathcal{K}$-pair is a special case of a U-e-ideal, introduced and used by Kalimullin to prove the definability of the jump operation in the global structure $\mathcal{D}_e$. In [16] Kalimullin proves that the property of being a $\mathcal{K}$-pair is degree theoretic and first order definable in the global structure $\mathcal{D}_e$. A pair of sets form a $\mathcal{K}$-pair if and only if their degrees $\mathbf{a} = d_e(A)$ and $\mathbf{b} = d_e(B)$ satisfy the property:

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \rightleftharpoons \forall \mathbf{x}[(\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x}].$$

We will call a pair of enumeration degrees a $\mathcal{K}$-pair if they contain representatives which form a $\mathcal{K}$-pair in the sense of Definition 1.1.

$\mathcal{K}$-pairs have been proven useful for coding structures in $\mathcal{G}_e$, some of their main advantages lying in their properties: e.g. every $\mathcal{K}$-pair of nonzero degrees in $\mathcal{G}_e$ is a minimal pair of low enumeration degrees. It has been shown in [12] for instance, that using countable $\mathcal{K}$-systems, systems of nonzero e-degrees such that every pair of distinct degrees forms a $\mathcal{K}$-pair, that every countable distributive semi-lattice can be embedded below every nonzero Δ_2^0 e-degree. The local definability of $\mathcal{K}$-pairs is the first step in a larger project [14], aimed at showing that the theory of $\mathcal{G}_e$ is computably isomorphic to first order arithmetic, where $\mathcal{K}$-systems are used to code standard models of arithmetic in $\mathcal{G}_e$.

Kalimullin [16] has shown that if a pair of sets A and B do not form a $\mathcal{K}$-pair then there is a set C, computable from $A \oplus B \oplus K$, where K denotes the halting set, such that the degree $d_e(C)$ witnesses the fact that $d_e(A)$ and $d_e(B)$ do not satisfy the formula $\mathcal{K}$. Hence if A and B are Δ_2^0 enumeration degrees then C is also Δ_2^0 and the property "$\mathbf{a}$ and $\mathbf{b}$ form a $\mathcal{K}$-pair" is first order definable in the substructure of the Δ_2^0 enumeration degrees by the same formula, $\mathcal{K}$. If A and B are properly Σ_2^0 then the witness C is at best estimated as Δ_3^0, hence it is quite possible that there are fake pairs $(\mathbf{a}, \mathbf{b})$ of Σ_2^0 enumeration degrees, such that:

$$\mathcal{G}_e \models \mathcal{K}(\mathbf{a}, \mathbf{b}), \text{ but } \mathcal{D}_e \models \neg \mathcal{K}(\mathbf{a}, \mathbf{b}).$$

The key to the definability of $\mathcal{K}$-pairs lies in the cupping properties of $\mathcal{G}_e$. We say that a Σ_2^0 enumeration degree $\mathbf{u}$ is cuppable if there exists an incomplete $\mathbf{v} < \mathbf{0}_e'$ such that $\mathbf{u} \vee \mathbf{v} = \mathbf{0}_e'$. Cooper, Sorbi and Yi [9] prove that not every nonzero Σ_2^0 enumeration degree is cuppable, in contrast to the Δ_2^0 enumeration degrees, where for every nonzero degree one can find a total Δ_2^0 cupping partner. Soskova and Wu [21] prove furthermore that every nonzero Δ_2^0 enumeration degree is low-cuppable, i.e. cuppable by a low enumeration degree. In [13] a stronger version of Soskova and Wu's theorem is proved, which also reveals certain cupping properties of $\mathcal{K}$-pairs:

Theorem 1.1. [13] *For every nonzero Δ_2^0 degree $\mathbf{a}$ there is a $\mathcal{K}$-pair, $\{\mathbf{b}, \mathbf{c}\}$ of nonzero Δ_2^0 degrees, such that $\mathbf{a} \vee \mathbf{b} = \mathbf{b} \vee \mathbf{c} = \mathbf{0}_e'$.*

The final ingredient for the definability of $\mathcal{K}$-pairs in $\mathcal{G}_e$ is provided by the following new cupping property of the elements in $\mathcal{G}_e$, which we believe is of independent interest.

Theorem 1.2. *For every pair of Σ_2^0 enumeration degrees $\mathbf{u}$ and $\mathbf{v}$ with $\mathbf{u} \vee \mathbf{v} = \mathbf{0}_e'$, there exists a low Δ_2^0 enumeration degree $\mathbf{a} < \mathbf{0}_e'$, such that at least one of the following is true:*

    (1) $\mathbf{u} \vee \mathbf{a} = \mathbf{0}_e'$,
    (2) $\mathbf{v} \vee \mathbf{a} = \mathbf{0}_e'$.

This property reveals that the class of low-cuppable Σ_2^0 enumeration degrees contains at least half of the cuppable enumeration degrees. Whether or not there are cuppable Σ_2^0 degrees, which are not low-cuppable remains open.

Using Theorem 1.2 we show how one can give a local definition of a nonempty set of $\mathcal{K}$-pairs. Applying Theorem 1.1 we prove that every member of a $\mathcal{K}$-pair is bounded by some element from this set. Thus as an application of these two structural properties of $\mathcal{G}_e$ we obtain the desired definability result:

Theorem 1.3. *There is a first order formula $\mathcal{LK}$ in the language of $\mathcal{G}_e$ such that a pair of nonzero Σ_2^0 enumeration degrees $\mathbf{a}$ and $\mathbf{b}$ form a $\mathcal{K}$-pair if and only if:*

$$\mathcal{G}_e \models \mathcal{LK}(\mathbf{a}, \mathbf{b}).$$

The definability of $\mathcal{K}$-pairs allows us to give a first order definition of two further classes that have been of interest in the study of the local structure. A nonzero degree $\mathbf{a} \in \mathcal{G}_e$ is *downwards properly* Σ_2^0 if all nonzero degrees $\mathbf{b} \leq \mathbf{a}$ are properly Σ_2^0. For example every non-cuppable Σ_2^0 degree is necessarily downwards properly Σ_2^0 as every Δ_2^0 enumeration degree is cuppable. Another example is given by Cooper, Li, Sorbi and Yang [8], who show that there is a Σ_2^0 degree which does not bound a minimal pair, whereas every Δ_2^0 degree does.

The symmetric class of the *upwards properly* Σ_2^0 enumeration degrees contains the incomplete enumeration degrees $\mathbf{a} \in \mathcal{G}_e$ such that all incomplete degrees $\mathbf{b} \geq \mathbf{a}$ are properly Σ_2^0. Soskova [20] proves that there is an enumeration degree $\mathbf{a} < \mathbf{0}_e'$, such that no pair of incomplete degrees above it forms a splitting of $\mathbf{0}_e'$. This, combined with Arslanov and Sorbi's [2] result, that there is a splitting of $\mathbf{0}_e'$ above every incomplete Δ_2^0 enumeration degree, gives an example of an upwards properly Σ_2^0 degree. Bereznyuk, Coles and Sorbi [3], prove that there is an upwards properly Σ_2^0 degree above any incomplete member of $\mathcal{G}_e$. Cooper and Copestake [7] show furthermore that there are properly Σ_2^0 enumeration degrees that are incomparable with every nonzero, incomplete Δ_2^0 degree, and hence are both upwards and downwards properly Σ_2^0.

We show that these two classes are also first order definable in the local structure.

Theorem 1.4. *The following two classes of Σ_2^0 enumeration degrees are first order definable in $\mathcal{G}_e$:*

    (1) *The class of downwards properly Σ_2^0 enumeration degrees;*
    (2) *The class of upwards properly Σ_2^0 enumeration degrees.*

The revealed relationship between definability and information content in the Σ_2^0 enumeration degrees calls forth the search for other examples of this phenomenon. In view of the nature of the particular classes that are proved definable in this article, an important question that remains open is:

Question 1.4.1. *Is the class of the Δ_2^0 or the class of the total enumeration degrees enumeration first order definable in $\mathcal{G}_e$?*

2. Preliminaries

We assume that the reader is familiar with the notion of enumeration reducibility, and refer to Cooper [5] for a survey of basic results on the structure of the enumeration degrees and to Sorbi [19] for a survey of basic results on the local structure $\mathcal{G}_e$. For completeness we will nevertheless outline here basic definitions and properties of the enumeration degrees used in this article.

Definition 2.1. *A set A is* enumeration reducible *($\leq_e$) to a set B if there is a c.e. set Φ such that:*

$$A = \Phi(B) = \{n \mid \exists u(\langle n, u \rangle \in \Phi \ \& \ D_u \subseteq B)\},$$

where D_u denotes the finite set with code u under the standard coding of finite sets. We will refer to the c.e. set Φ as an enumeration operator *and its elements will be called* axioms.

A set A is *enumeration equivalent* ($\equiv_e$) to a set B if $A \leq_e B$ and $B \leq_e A$. The equivalence class of A under the relation $\equiv_e$ is the enumeration degree $d_e(A)$ of A. The structure of the enumeration degrees $\langle \mathcal{D}_e, \leq \rangle$ is the class of all enumeration degrees with relation $\leq$ defined by $d_e(A) \leq d_e(B)$ if and only if $A \leq_e B$. It has a least element $\mathbf{0}_e = d_e(\emptyset)$, the set of all c.e. sets. We can define a least upper bound operation, by setting $d_e(A) \vee d_e(B) = d_e(A \oplus B)$ and a jump operator $d_e(A)' = d_e(J_e(A))$. The enumeration jump of a set A, denoted by $J_e(A)$ is defined by Cooper [4] as $L_A \oplus \overline{L_A}$, where $L_A = \{ n \mid n \in \Phi_n(A)\}$.

Enumeration degrees which contain a set of the form $A \oplus \overline{A}$ are called *total enumeration degrees*. The interest in the class of the total enumeration degrees, arises from the fact that it is an isomorphic copy of the Turing degrees. The map ι, which sends a Turing degree $d_T(A)$ to the e-degree $d_e(A \oplus \overline{A})$, is an embedding, which preserves the order, the least upper bound and the jump operation.

We shall say that an e-degree is *quasi-minimal* if and only if it bounds no total degree, except for the least e-degree $\mathbf{0}_e$.

Finally we introduce one further piece of notation. In what follows we will often need to work with a set C reducible to the least upper bound of two other sets, say A and B. To keep notation simple we will consider the set C as being enumerated relative to two sources and write $C = \Phi(A, B)$, instead of $C = \Phi(A \oplus B)$. Naturally we will assume that an axiom of the operator Φ has the structure $\langle n, D_A, D_B \rangle$ and that it is valid if an only if $D_A \subseteq A$ and $D_B \subseteq B$.

Further notation and terminology used in this article are based on that of [6].

3. A local definition of $\mathcal{K}$-pairs

$\mathcal{K}$-pairs can be viewed as a generalization of the notion of a semi-recursive set, defined by Jockusch [15]. Recall that a set A is semi-recursive if it has a computable selector function $s_A : \mathbb{N} \times \mathbb{N}$ such that for all natural numbers n and m: $s_A(n, m) \in \{n, m\}$ and if $\{n, m\} \cap A \neq \emptyset$ then $s_A(n, m) \in A$. It is not difficult to see that if A is semi-recursive then A and $\overline{A}$ form a $\mathcal{K}$-pair. Indeed let s_A be the selector function for A and let

$$\overline{s_A}(n, m) = \begin{cases} n \text{ , if } s_A(n, m) = m \\ m \text{ , if } s_A(n, m) = n. \end{cases}$$

Now consider the c.e. set $W = \{(s_A(n, m), \overline{s_A}(n, m)) \mid n, m \in \mathbb{N}\}$ and notice that $A \times \overline{A} \subseteq W$ and $\overline{A} \times \overline{\overline{A}} = \overline{A} \times A \subseteq \overline{W}$.

Another simple example of a $\mathcal{K}$-pair is $\{W, A\}$, where A is any set of natural numbers and W is a c.e. set. This we shall consider as a trivial example and we will mainly be interested in nontrivial $\mathcal{K}$-pairs, ones consisting of two non-c.e sets. We shall say that a degree $\mathbf{a}$ is half of a $\mathcal{K}$-pair if there is a nonzero degree $\mathbf{b}$ such that $\{\mathbf{a}, \mathbf{b}\}$ forms a $\mathcal{K}$-pair of degrees. The degree $\mathbf{b}$ will be called a $\mathcal{K}$-partner for $\mathbf{a}$. Some basic properties of $\mathcal{K}$-pairs of degrees, collected from Kalimullin [16], are summarized in the following theorem.

Theorem 3.1 (Kallimullin). *Let $\mathcal{K}(\mathbf{a}, \mathbf{b})$ denote the formula with free variables $\mathbf{a}$ and $\mathbf{b}$, defined by:*

$$\forall \mathbf{x}[(\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x}].$$

(1) *A pair of degrees $\mathbf{a}$ and $\mathbf{b}$ form a $\mathcal{K}$-pair if and only if $\mathcal{D}_e \models \mathcal{K}(\mathbf{a}, \mathbf{b})$.*
(2) *A pair of Δ_2^0 degrees $\mathbf{a}$ and $\mathbf{b}$ form a $\mathcal{K}$-pair if and only if $\mathcal{G}_e \models \mathcal{K}(\mathbf{a}, \mathbf{b})$.*
(3) *Every half of a nontrivial $\mathcal{K}$-pair in $\mathcal{G}_e$ is quasi-minimal and low.*
(4) *The set of degrees which form a $\mathcal{K}$-pair with a fixed degree $d_e(A)$ is an ideal with upper bound $d_e(\overline{A})$.*
(5) *If $\mathbf{a}$ and $\mathbf{b}$ form a nontrivial $\mathcal{K}$-pair then $\mathbf{a}$ and $\mathbf{b}$ form a minimal pair, i.e. the only degree that is both below $\mathbf{a}$ and $\mathbf{b}$ is $\mathbf{0}_e$.*
(6) *Every nonzero Δ_2^0 enumeration degree bounds a nontrivial $\mathcal{K}$-pair.*

We add one additional property to the list, which motivates our interest in the cupping properties of the elements in $\mathcal{G}_e$.

Lemma 3.1. *Let $\mathbf{a}$ and $\mathbf{b}$ be Σ_2^0 enumeration degrees such that:*

$$\mathcal{G}_e \models K(\mathbf{a}, \mathbf{b}).$$

If $\mathbf{c}$ is a Σ_2^0 enumeration degree, such that $\mathbf{b} \vee \mathbf{c} = \mathbf{0}_e'$, then $\mathbf{a} \leq \mathbf{c}$.

Proof. By the property $\mathcal{K}(\mathbf{a}, \mathbf{b})$ applied to $\mathbf{c}$ we get:

$$(\mathbf{b} \vee \mathbf{c}) \wedge (\mathbf{a} \vee \mathbf{c}) = \mathbf{c}.$$

Replacing $(\mathbf{b} \vee \mathbf{c})$ with its equal $\mathbf{0}_e'$ we get:

$$\mathbf{0}_e' \wedge (\mathbf{a} \vee \mathbf{c}) = \mathbf{c}.$$

Now as $\mathbf{0}_e'$ is the largest element of $\mathcal{G}_e$ we get:

$$\mathbf{a} \vee \mathbf{c} = \mathbf{c} \text{ or equivalently } \mathbf{a} \leq \mathbf{c}.$$

$\square$

Jockusch [15] shows that for every set B there is a semi-recursive set $A \equiv_T B$ such that A and $\overline{A}$ are non-c.e. This, combined with the quasi-minimality of $\mathcal{K}$-pairs, proves that every total member of $\mathcal{G}_e$ can be represented as the least upper bound of the elements of a nontrivial $\mathcal{K}$-pair. In particular $\mathbf{0}_e'$ can be split by a nontrivial $\mathcal{K}$-pair of Δ_2^0 enumeration degrees.

We are now ready to give a first order definition of $\mathcal{K}$-pairs assuming Theorems 1.1 and 1.2. Consider the formula

$$\mathcal{L}(\mathbf{a}) \rightleftharpoons \mathbf{a} > \mathbf{0}_e \ \& \ \exists \mathbf{b} > \mathbf{0}_e (\mathbf{a} \vee \mathbf{b} = \mathbf{0}_e' \ \& \ \mathcal{K}(\mathbf{a}, \mathbf{b})).$$

From the argument above it follows that there are elements of $\mathcal{G}_e$ which satisfy this formula. We show that every element which satisfies this formula is in fact a half of a $\mathcal{K}$-pair.

Propostion 3.1. *If $\mathcal{G}_e \models \mathcal{L}(\mathbf{a})$ then $\mathbf{a}$ is half of a nontrivial $\mathcal{K}$-pair.*

Proof. Let $\mathbf{b} > \mathbf{0}_e$ be a degree such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$ & $\mathcal{K}(\mathbf{a}, \mathbf{b})$. Then $\{\mathbf{a}, \mathbf{b}\}$ is a splitting of $\mathbf{0}'_e$ and hence applying Theorem 1.2 we get a low Δ^0_2 degree $\mathbf{c}$ which cups $\mathbf{a}$ or $\mathbf{b}$.

Case 1: $\mathbf{b} \vee \mathbf{c} = \mathbf{0}'_e$. By Lemma 3.1 we get $\mathbf{a} \leq \mathbf{c}$. Now by the monotonicity of the enumeration jump it follows that $\mathbf{a}$ is low, hence Δ^0_2, and by the property that all nonzero Δ^0_2 are low-cuppable, $\mathbf{a}$ is as well low-cuppable.

Case 2: $\mathbf{a} \vee \mathbf{c} = \mathbf{0}'_e$. Then similarly $\mathbf{b} \leq \mathbf{c}$ and hence Δ^0_2 and low-cuppable.

Therefore if one of the degrees $\mathbf{a}$ and $\mathbf{b}$ is low-cuppable, then both are low-cuppable and both are Δ^0_2. Now applying part 2 of Theorem 3.1 we get that $\mathbf{a}$ and $\mathbf{b}$ form a $\mathcal{K}$-pair. $\square$

The set defined by the formula $\mathcal{L}$ is therefore a nonempty set of low enumeration degrees. It does not contain all halves of nontrivial $\mathcal{K}$-pairs. Let $\mathbf{c}$ be any total incomplete Δ^0_2 enumeration degree and let A be a semi-recursive set, such that $d_e(A \oplus \overline{A}) = \mathbf{c}$ and both A and $\overline{A}$ are not c.e. It follows from Theorem 3.1 that $\mathbf{a} = d_e(A)$ is half of a nontrivial $\mathcal{K}$-pair and $d_e(\overline{A}) = \overline{\mathbf{a}}$ is the largest element of the ideal of $\mathcal{K}$-partners for $\mathbf{a}$. Hence for every $\mathcal{K}$-partner $\mathbf{b}$ of $\mathbf{a}$, $\mathbf{a} \vee \mathbf{b} \leq \mathbf{a} \vee \overline{\mathbf{a}} = \mathbf{c} < \mathbf{0}'_e$ and $\mathbf{a}$ does not satisfy $\mathcal{L}$. Nevertheless the set $\mathcal{L}$ contains an upper bound to every half of a nontrivial $\mathcal{K}$-pair in $\mathcal{G}_e$.

Propostion 3.2. *If $\mathbf{a} \in \mathcal{G}_e$ is a half of a nontrivial $\mathcal{K}$-pair then there is a degree $\mathbf{c} \geq \mathbf{a}$ such that $\mathcal{G}_e \models \mathcal{L}(\mathbf{c})$.*

Proof. Let $\mathbf{b}$ be a nonzero $\mathcal{K}$-partner for $\mathbf{a}$. Then $\mathbf{b}$ is a Δ^0_2 enumeration degree and hence by Theorem 1.1 there is a nontrivial $\mathcal{K}$-pair $\{\mathbf{c}, \mathbf{d}\}$ such that $\mathbf{b} \vee \mathbf{c} = \mathbf{d} \vee \mathbf{c} = \mathbf{0}'_e$. Consider the degree $\mathbf{c}$. First of all $\mathbf{c}$ satisfies the formula $\mathcal{L}$ with $\mathbf{d}$ as witness for this. Secondly by Lemma 3.1 $\mathbf{a} \leq \mathbf{c}$. $\square$

On the other hand, suppose that there is a pair of Σ^0_2 enumeration degrees $\mathbf{a}$ and $\mathbf{b}$ which satisfy the formula $\mathcal{K}$ in $\mathcal{G}_e$ but are not a $\mathcal{K}$-pair. It follows by an argument similar to the one in the Proposition 3.1 that both $\mathbf{a}$ and $\mathbf{b}$ are not low-cuppable and hence are downwards properly Σ^0_2. As every member of the set defined by $\mathcal{L}$ is low and hence bounds only Δ^0_2 enumeration degrees it follows that both $\mathbf{a}$ and $\mathbf{b}$ are incomparable (and even form a minimal pair) with every element which satisfies $\mathcal{L}$.

To finalize the proof of Theorem 1.3 we set $\mathcal{LK}$ to be the formula:

$$\mathcal{LK}(\mathbf{a}, \mathbf{b}) \rightleftharpoons \mathcal{K}(\mathbf{a}, \mathbf{b}) \ \& \ \mathbf{a} > \mathbf{0}_e \ \& \ \mathbf{b} > \mathbf{0}_e \ \& \ \exists \mathbf{c}(\mathbf{c} \geq \mathbf{a} \ \& \ \mathcal{L}(\mathbf{c})).$$

Now we can easily prove Theorem 1.4 as well. The last property in Preposition 3.1 shows that every Δ^0_2 enumeration degree bounds a nontrivial $\mathcal{K}$-pair. As every $\mathcal{K}$-pair consists of Δ^0_2 enumeration degrees, it follows that a degree is downwards properly Σ^0_2 if and only it bounds no nontrivial $\mathcal{K}$-pair. Thus a degree $\mathbf{a}$ is downwards properly Σ^0_2 if and only if:

$$\mathcal{G}_e \models \forall \mathbf{b}, \mathbf{c}[(\mathbf{b} \leq \mathbf{a} \ \& \ \mathbf{c} \leq \mathbf{a}) \Rightarrow \neg \mathcal{LK}(\mathbf{b}, \mathbf{c})].$$

To prove the second part of this theorem, recall that every total enumeration degree can be represented as the least upper bound of the elements of a $\mathcal{K}$-pair. The least upper bound of the elements of every $\mathcal{K}$-pair, on the other hand, is a Δ^0_2 degree. The last ingredient comes from a theorem of Arslanov, Cooper and

Kalimullin [1] (Theorem 7), which states that for every Δ_2^0 enumeration degree $\mathbf{a} < \mathbf{0}_e'$ there is a total enumeration degree $\mathbf{b}$ such that $\mathbf{a} \leq \mathbf{b} < \mathbf{0}_e'$. From all this it follows that a degree $\mathbf{a}$ is upwards properly Σ_2^0 if and only if no incomplete degree above it can be represented as the least upper bound of the elements of a $\mathcal{K}$-pair, i.e. if:

$$\mathcal{G}_e \models \forall \mathbf{c}, \mathbf{d}(\mathcal{LK}(\mathbf{c}, \mathbf{d}) \ \& \ \mathbf{a} \leq \mathbf{c} \vee \mathbf{d} \Rightarrow \mathbf{c} \vee \mathbf{d} = \mathbf{0}_e').$$

4. Cupping properties of $0_e'$-splittings

We start this section with a very general description of the idea behind the construction for the proof of Theorem 1.2. We then proceed to formalizing this idea, giving more intuition as we progress.

4.1. **General idea.** The construction is inspired by the non-splitting technique introduced in [20]. There it is shown that there is a Σ_2^0 e-degree $\mathbf{a} < \mathbf{0}_e'$, such that no pair of Σ_2^0 degrees $\mathbf{u}, \mathbf{v}$ above $\mathbf{a}$ splits $\mathbf{0}_e'$. An equivalent way to formulate this is as follows.

Theorem 4.1 ([20]). *There exists a Σ_2^0 enumeration degree $\mathbf{a} < \mathbf{0}_e'$, such that for every pair of Σ_2^0 enumeration degrees $\mathbf{u}$ and $\mathbf{v}$ with $\mathbf{u} \vee \mathbf{v} = \mathbf{0}_e'$ at least one of the following is true*

$$\mathbf{u} \vee \mathbf{a} = \mathbf{0}_e' \ \text{ or } \ \mathbf{v} \vee \mathbf{a} = \mathbf{0}_e'$$

In the construction of a non-splitting degree, a set A and an auxiliary set E are constructed to meet the following requirements:

(1) For every enumeration operator Ψ:

$$\mathcal{N}_\Psi : E \neq \Psi(A).$$

(2) For all triples of an enumeration operator Θ and Σ_2^0 sets U and V:

$$\mathcal{P}_{\Theta, U, V} : E = \Theta(U, V) \Rightarrow (\exists \Gamma, \Lambda)[\overline{K} = \Gamma(U, A) \vee \overline{K} = \Lambda(V, A)]$$

Here $\overline{K}$ is a Π_1^0 member of $\mathbf{0}_e'$.

The $\mathcal{N}$-requirements guarantee that $d_e(A)$ is nonzero. Now if $U \oplus V \equiv_e \overline{K}$ then there is an operator Θ such that $\Theta(U, V) = E$ and hence the requirement $\mathcal{P}_{\Theta, U, V}$ ensures that $d_e(A)$ cups at least one of the degrees $d_e(U)$ or $d_e(V)$ to $\mathbf{0}_e'$.

Hence one incomplete degree is a cupping partner to at least half of the cuppable enumeration degrees. There is no hope that this particular non-splitting degree can be constructed as a low enumeration degree, as by Arslanov and Sorbi [2] there is a splitting of $\mathbf{0}_e'$ above every incomplete Δ_2^0 enumeration degree. Our task is however much less demanding. Given a particular pair of degrees which splits $\mathbf{0}_e'$ we have to show that at least one of them is low-cuppable.

Let $\mathbf{u}$ and $\mathbf{v}$ be two given Σ_2^0 enumeration degrees with least upper bound $\mathbf{0}_e'$. Fix two representatives $U \in \mathbf{u}$ and $V \in \mathbf{v}$.

Definition 4.1. *A set A is 1-generic if for every c.e. set W of finite binary strings there is an initial segment of the characteristic function of A, $\tau \subseteq A$, such that:*

$$\tau \in W \vee \forall \mu \supseteq \tau(\mu \notin W).$$

Copestake [10] proves that the degree of every 1-generic Δ_2^0 set is low. As every 1-generic set is non c.e., our main plan is to construct a 1-*generic* Δ_2^0 set A, such that $d_e(A)$ cups $\mathbf{u}$ to $\mathbf{0}_e'$. If this plan fails we turn to a backup plan: construct a 1-generic Δ_2^0 sets B, such that $d_e(B)$ cups $\mathbf{v}$ to $\mathbf{0}_e'$. So fix a computable listing of all c.e. sets W_i, viewed as sets of finite binary strings. We need to satisfy one of two groups of requirements:

(1) **The main requirements.** There is a Δ_2^0 set A, which satisfies $\mathcal{C}$ and $\mathcal{G}_i$ for all $i < \omega$, where:

$$\mathcal{C}: \quad \exists\Gamma(\Gamma(U, A) = \overline{K});$$

$$\mathcal{G}_i: \quad \exists\tau \subseteq A(\tau \in W_i \vee \forall\mu \supseteq \tau(\mu \notin W_i)).$$

(2) **The backup requirements.** There is a Δ_2^0 set B, which satisfies $\check{\mathcal{C}}$ and $\check{\mathcal{G}}_i$ for all $i < \omega$, where:

$$\check{\mathcal{C}}: \quad \exists\Lambda(\Lambda(V, B) = \overline{K})$$

$$\check{\mathcal{G}}_i: \quad \exists\tau \subseteq B(\tau \in W_i \vee \forall\mu \supseteq \tau(\mu \notin W_i)).$$

Following the non-splitting construction we will again make use of an auxiliary Π_1^0 set E, meant to help us use the fact that $U \oplus V$ is in the largest possible Σ_2^0 enumeration degree. Even though we are constructing this set, we know that there is an enumeration operator Θ such that $\Theta(U \oplus V) = E$. Using the recursion theorem we may assume that we are given this operator in advance. We will use this operator to predict changes in the approximations to the sets U and V: an extraction of an element e from the set E will ultimately lead to an extraction of elements out of the set $U \oplus V$. Now we turn to our main strategy: construct a 1-generic Δ_2^0 set A and an operator Γ such that $\Gamma(U \oplus A) = \overline{K}$. Here $\overline{K}$ is a fixed Π_1^0 member of $\mathbf{0}_e'$. To construct Γ we have a main enumeration strategy which watches the approximation to $\overline{K}$ and enumerates axioms for elements currently in $\overline{K}$ using finite subsets of the current approximations to U and A. When an element exits the approximation to $\overline{K}$ it invalidates perviously enumerated axioms by extracting numbers from A. To meet a main genericity requirement, $\mathcal{G}_i$, we have a corresponding main genericity strategy, which tries to find an initial segment of A, either in the set W_i, or which cannot be extended to an element of W_i. The strategy to construct Γ is in conflict with the strategies to make A 1-generic. Extracting elements from A, in order to rectify the enumeration $\Gamma(U \oplus A)$ prevents us from restraining A on certain initial segments. To resolve this conflict we try to provoke an extraction from the set U, using the set E. An extraction from the set U will allow us to rectify Γ without changing the approximation to A. If every genericity strategy after finitely many unsuccessful attempt eventually manages to successfully provoke an extraction from the set U, then it is successful, and ultimately all main requirements will be satisfied. Otherwise there will be a least genericity strategy, which fails infinitely often to provoke an extraction from the set U, as every attempt ends in an extraction from the set V. In this case this particular genericity strategy activates the backup strategy: to construct a second 1-generic Δ_2^0 set B and an operator Λ such that $\Lambda(V \oplus B) = \overline{K}$. The backup strategies, both enumeration and genericity act in a similar way to their main counterparts. The only difference is that every time a backup genericity strategy tries to provoke a change in V, it times its actions with the main genericity strategy, which is responsible for its activation. This ensures that the attempt is successful for the backup strategy.

With this general plan in mind we start to formalize the intuitive description of the strategies. We start by selecting approximations to the given sets U and V.

4.2. **Approximations.** We will use good approximations to the given sets. The notion of a good approximation to a Σ_2^0 sets is first used by Jockusch [15] and by Cooper [4](Σ_2^0 approximations with infinitely many *thin* stages). Later on Lachlan and Shore [17] formalize this notion and prove that every n-c.e.a set has one.

We fix a good Σ_2^0 approximation $\{(U \oplus V)^{\{s\}}\}_{s<\omega}$ to the set $U \oplus V$. A good Σ_2^0 approximation is one, which has the following two properties:

$\mathcal{G}$. There are infinitely many *good* stages s such that $(U \oplus V)^{\{s\}} \subseteq U \oplus V$.

Σ_2^0. For all n there exists a stage s such that at all stages $t > s$ we have $(U \oplus V) \restriction n \subseteq (U \oplus V)^{\{s\}}$.

Denote by $G_{U \oplus V}$ the set of good stages in the approximation to $U \oplus V$. We use the following property of good approximations proved in [17]: for every enumeration operator Θ with standard Σ_1^0 approximation $\{\Theta^{\{s\}}\}_{s<\omega}$,

$$(4.1) \qquad \lim_{s \in G_{U \oplus V}} \Theta^{\{s\}}((U \oplus V)^{\{s\}}) = \Theta(U \oplus V).$$

Denote $\Theta^{\{s\}}((U \oplus V)^{\{s\}})$ by $\Theta(U \oplus V)^{\{s\}}$. As noted above we will be constructing a Π_1^0 approximation to a set E and using the recursion theorem we will assume that we are given in advance an operator Θ such that $\Theta(U \oplus V) = E$. Denote by $l(\Theta(U \oplus V), E, s)$ the maximal number n such that $\Theta(U \oplus V)^{\{s\}} \restriction n = E^{\{s\}} \restriction n$. Then as the sets $\Theta(U \oplus V)$ and E are equal:

$$\lim_{s \in G_{U \oplus V}} l(\Theta(U \oplus V), E, s) = \infty.$$

Stages at which $l(\Theta(U \oplus V), E, s) > \max\{l(\Theta(U \oplus V), E, t) \mid t < s\}$ will be called expansionary stages. Thus as $\Theta(U \oplus V) = E$, there are infinitely many expansionary stages for the operator Θ.

One additional consequence of the equality between the two sets will be used. An element of the set E must eventually be permanently enumerated in the set $\Theta(U \oplus V)$. Hence it will use some finite amount of positive information from the two sets U and V. We introduce the e-degree version of a use function.

Let F be a finite set. The *age* of this set with respect to the approximation to $U \oplus V$ measured at stage s is the number:

$$a(F, s) = \begin{cases} s - \mu\, t[\forall r \in [t, s](F \subseteq (U \oplus V)^{\{r\}})] + 1, & \text{if } F \subseteq (U \oplus V)^{\{s\}}; \\ 0 & \text{otherwise.} \end{cases}$$

In other words the age of a finite set is the number of consecutive stages ending in the current stage s at which the set F is a subset of the approximation to $U \oplus V$. For instance the age of the empty set $a(\emptyset, s) = s + 1$ for all s and if $F \not\subseteq U \oplus V^{\{s\}}$ then $a(F, s) = 0$.

Let $Ax = \langle e, D_U \oplus D_V \rangle \in \Theta^{\{s\}}$ be an axiom for e in Θ. The *age* of this axiom at stage s is the number

$$a(Ax, s) = a(D_U \oplus D_V, s).$$

Thus invalid axioms have age 0 at stage s and valid axioms have age equal to the number of consecutive stages at which they have been valid, ending in the current

stage. Denote by $ax(e, s)$ the finite set, such that $\langle e, ax(e, s)\rangle$ is the oldest axiom for e in Θ at stage s, i.e. of greatest age. (If there are more than one valid axioms of maximal age, choose the one with least code).

The use of an element e measured at stage s is the set

$$\theta(e, s) = \bigcup_{i \leq e,\ i \in \Theta(U \oplus V)^{\{s\}}} ax(i, s),$$

the collection of the finite sets which form the second half of the oldest valid axioms for all elements that currently appear in the set $\Theta(U, V)$. As $\Theta(U, V) = E$ it follows tat for every $e \in E$ there will be a stage s such that at all stages $t \geq s$ the use of e will remain unchanged, i.e. $\theta(e, s) = \theta(e, t)$. This will be used in the *Honestification* module described below.

4.3. **The tree of strategies.** The construction will be in stages. At every stage s we construct a finite path of length less than or equal to s, $\delta^{\{s\}}$, through a tree of strategies, defined below, approximating the so called true path - a leftmost infinite path of strategies visited at infinitely many stages. If $\gamma \subseteq \delta^{\{s\}}$ then we shall say that γ is *visited* or *activated* at stage s and s will be called a γ-true stage. Every strategy will have outcomes representing different possible ways in which the corresponding requirement might be satisfied. The outcomes of each strategy are ordered linearly. Denote by O the collection of all outcomes. The tree T can be viewed as a computable function with domain the set of finite strings of outcomes, $O^{<\omega}$, and range the set of strategies.

The tree of strategies T contains strategies of four types: the main enumeration strategy, main genericity strategies, backup enumeration strategies and backup genericity strategies. The root of the tree is assigned the main enumeration strategy. We will denote this strategy by α. This strategy initiates the construction of a set A and an operator Γ and is successful if $\Gamma(U \oplus A) = \overline{K}$. The strategy has two outcomes $\mathsf{e} <_L \mathsf{b}$. Outcome e represents the fact that there are infinitely many expansionary stages. The node $\alpha\hat{\ }\mathsf{b}$ is a leaf in the tree T. The node $\alpha\hat{\ }\mathsf{e}$ is assigned the first main genericity strategy $\beta(0)$. Outcome b represents the fact that $l(\Theta(U \oplus V), E, s)$ is bounded. By our choice of Θ this cannot be the true outcome of α. We nevertheless leave this outcome on the tree. The effect of this outcome will be that any other strategy is activated only on expansionary stages.

The main genericity strategy $\beta = \beta(i)$ tries to ensure that A satisfies the i-th genericity requirement $\mathcal{G}_i$. It has outcomes ∞, h, w and f_n for every n arranged as follows:

$$\infty <_L \cdots <_L \mathsf{f}_n <_L \ldots \mathsf{f}_1 <_L \mathsf{f}_0 <_L \mathsf{h} <_L \mathsf{w}.$$

Outcome ∞ represents the fact that β has been unsuccessful infinitely often to secure a witness $\tau \in W_i$ as an initial segment of A, but has provided sufficient conditions for the backup strategies to succeed. The node $\beta\hat{\ }\infty$ is assigned the backup enumeration strategy $\check{\alpha}(\beta(i))$. The outcome f_n represents the fact that β has been successful in securing a witness $\tau \in W_i$ as an initial segment of A on its n-th attempt and outcome w the fact that β has found a witness $\tau \subseteq A$ which has no extension in the set W_i. In both cases, $o \in \{\mathsf{f}_n \mid n < \omega\}$ and $o = \mathsf{w}$, the strategy has successfully satisfied the i-th genericity requirement and $\beta\hat{\ }o$ is assigned the next genericity requirement $\beta(i + 1)$.

Finally outcome h signifies that the strategy β has found an element $e \in E$ whose use does not stabilize at any stage, i.e. for every s there is a stage $t > s$ such that $\theta(e, s) \neq \theta(e, t)$. Again this is an outcome that would contradict our choice of Θ. It cannot be the true outcome of the strategy, but it could still seem like the correct outcome on finitely many occasions. We leave it on the tree as it will play a role in the initialization process. The node $\beta \hat{\ } h$ is therefore a leaf in the tree T.

The backup enumeration strategy $\check{\alpha} = \check{\alpha}(\beta)$, activated by β, initiates the construction of a set $B_{\check{\alpha}}$, and an operator $\Lambda_{\check{\alpha}}$. The strategy is successful if ultimately $\Lambda_{\check{\alpha}}(V \oplus B_{\check{\alpha}}) = \overline{K}$. This strategy has only one outcome e. The node e is assigned the first backup genericity strategy $\check{\beta}(\check{\alpha}, \beta, 0)$.

Finally the backup genericity strategy $\check{\beta} = \check{\beta}(\check{\alpha}, \beta, i)$, working with respect to $\check{\alpha}$ and coordinating actions with β, ensures that $B_{\check{\alpha}}$ satisfies the i-th genericity requirement $\mathcal{G}_i(B_{\check{\alpha}})$. It has three outcomes arranged as follows:

$$\mathsf{f} <_L \mathsf{h} <_L \mathsf{w}.$$

Outcome f represents the fact that $\check{\beta}$ has been successful in securing a witness $\tau \in W_i$ as an initial segment of $B_{\check{\alpha}}$. Outcome w represents the fact that $\check{\beta}$ has found an initial segment $\tau \subseteq B_{\check{\alpha}}$ such that $\forall \mu \subseteq \tau (\mu \notin W_i)$. Both nodes $\check{\beta} \hat{\ } f$ and $\check{\beta} \hat{\ } w$ are assigned the next genericity strategy $\check{\beta}(\check{\alpha}, \beta, i + 1)$. Outcome h just as in the main genericity strategy signifies the fact that $\check{\beta}$ has found an element $\check{e} \in E$ with unstable use. This again cannot be the true outcome of this strategy, or else we could argue that $\Theta(U, V) \neq E$, but it could still look like the true outcome any finite number of stages. The node $\check{\beta} \hat{\ } h$ is a leaf in the tree T.

The ordering of the outcomes induces a standard linear ordering of the nodes, the finite strings in the domain of T, namely: $\gamma \leq \sigma$ if $\gamma \subseteq \sigma$ (γ is an initial segment of σ) or if there exists ρ such that $\rho \hat{\ } o_1 \subseteq \gamma$, $\rho \hat{\ } o_2 \subseteq \sigma$ and $o_1 <_L o_2$. In the latter case we will also write $\gamma <_L \sigma$. If $\gamma < \sigma$ we shall say that γ has higher priority than σ. An infinite path in the tree T will be a function $f \subseteq T$ with domain a maximal linearly ordered infinite subset of the domain of T. We will abuse notation and denote with $f \upharpoonright n$ both the node of length n in the domain of f and the strategy assigned to it.

The next proposition shows that we have two types of infinite paths in T, ones that have a strategy for every main requirement and ones that have a strategy for every backup requirement.

Propostion 4.1. *Suppose f is an infinite path in the tree T. Then one of the following is true:*

(1) *For every $i > 0$ a main genericity strategy of type $\beta(i)$ is assigned to the node $f \upharpoonright i$.*

(2) *There is a strategy $\beta \subseteq f$ such that $\check{\alpha}(\beta)$ is assigned to $f \upharpoonright |\beta| + 1$ and for every i a backup strategy $\check{\beta}(\check{\alpha}, \beta, i)$ is assigned to the node $f \upharpoonright (|\beta| + i + 1)$.*

Proof. Suppose that (1) is not true and let β be the largest node of type main genericity strategy such that $\beta \subseteq f$. By our choice of β and the fact that f is infinite it follows that $\check{\alpha}(\beta) = \beta \hat{\ } \infty \subseteq f$ and $\check{\beta}(\check{\alpha}, \beta, 0) = \check{\alpha} \hat{\ } e \subseteq f$. Now inductively if $\check{\beta}(\check{\alpha}, \beta, i) \subseteq f$ then as f is infinite either $\check{\beta}(\check{\alpha}, \beta, i) \hat{\ } f \subseteq f$ or $\check{\beta}(\check{\alpha}, \beta, i) \hat{\ } w \subseteq f$ and in both cases this is a $\check{\beta}(\check{\alpha}, \beta, i + 1)$ strategy. $\square$

4.4. **Strategies and parameters.** In this section we shall give more intuition about how strategies are designed to work and define their parameters.

4.4.1. *The main enumeration strategy.* As noted above, α denotes the main enumeration strategy.

The strategy α monitors the length of agreement $l(\Theta(U \oplus V), E, s)$ at every stage s. If s is not expansionary then α has outcome b. If s is an expansionary stage then α has outcome e.

At every stage s α monitors the construction of the set A and the operator Γ, so that ultimately $\Gamma(U \oplus A) = \overline{K}$. The axioms in Γ will have a particular format: to every natural number n we will assign an A-marker $a(n)$ and a U-marker $u(n)$; the axiom enumerated in Γ at stage s for n will be of the form $\langle n, (U^{\{s\}} \restriction u(n) + 1) \oplus (A^{\{s\}} \restriction a(n) + 1) \rangle$

The markers will be defined by the main genericity strategies. The only job of α is to ensure that the constructed operator is correct. So at stage s it will consider all elements $n < s$ and correct $\Gamma(U, A)$ by enumerating axioms in Γ for elements $n \in \overline{K}^{\{s\}} \setminus \Gamma(U \oplus A)^{\{s\}}$. In this way we can be sure that if $n \in \overline{K}$ then α will consider it at infinitely many *good* stages in the approximation to U and will eventually enumerate an axiom, which remains permanently valid. To invalidate an axiom it is enough to consider it at infinitely many stages, regardless of whether they are good or not. So at expansionary stages the strategy will extract from A the already defined A-markers that appear in valid axioms for elements $n \in \Gamma(U \oplus A)^{\{s\}} \setminus \overline{K}^{\{s\}}$.

The *parameters for* α are hence A and Γ, both with initial value $\emptyset$; the markers $a(n)$ and $u(n)$ for every natural number n, also called the α-markers for n, initially undefined.

4.4.2. *The main genericity strategy.* At every stage s there will be at most one copy of the i-th main genericity strategy which is not in initial state. Let $\beta = \beta(i)$ be the i-th main genericity strategy. Recall that the strategy β has to ensure that there is a finite binary string $\tau \subseteq A$ such that $\tau \in W_i$ or no extension $\mu \supseteq \tau$ is in the set W_i. The strategy β has to overcome the difficulty set by the higher priority strategy α which is extracting markers from A in the rectification process. This is why the simple genericity strategy: *select a witness $\tau \subseteq A$, wait until (if ever) an extension $\mu \supseteq \tau$ enters W_i, restrain μ as an initial segment of A*, will not work. The strategy to satisfy $\mathcal{G}_i$ is a more complex version of this strategy.

The strategy β will have a threshold d_β. The value of this threshold will always be the i-th element of $\overline{K}$. We cannot guess in advance this number but, as $\overline{K}$ is an infinite set, approximated by its standard Π_1^0 approximation, after finitely many wrong guesses we will eventually pick the right value for the threshold.

The strategy β is responsible for defining values for the parameters $a(d_\beta)$ and $u(d_\beta)$. The first marker $a(d_\beta)$ that it defines is denoted as a_β^0 and plays the real role of the threshold, the element, below which β can safely assume that A is correct on all elements and will not be further modified. The values of the A-markers are always selected to be larger than the values of the markers defined by higher priority strategies. Note that the way, in which the axioms are defined by the strategy α, ensures that every axiom enumerated in $\Gamma^{\{s\}}$ for elements $n > d_\beta$ is an extension of the valid axiom for d_β at stage s. Thus by extracting the marker $a(d_\beta)^{\{s\}}$ from the set A the strategy β can invalidate all axioms for all elements $n > d_\beta$ valid at stage s.

Assuming that higher priority main genericity strategies have finished with their actions, and α has finished correcting Γ for elements $n < d_\beta$ at stage s_0, the

strategy β can safely assume that $\tau_0 = A^{\{s_0\}} \restriction a(d_\beta) + 1$ is a good candidate for a first witness.

If $\tau_0 \notin W_i$ then the strategy β is successful and needs no further actions. The outcome is w and the next genericity strategy is activated. If however there is an extension $\mu_0 \supseteq \tau_0$ such that $\mu_0 \in W_i$, then the strategy β is now in a difficult position, namely β cannot restrain A on elements a such that $|\tau| < a \le |\mu|$, without injuring α. This is where the set E comes into play. We will select an element e_0 currently in the constructed set E called an agitator and arrange things so that every valid axiom for d_β in Γ, extends the use of e_0. The process of making this arrangement will be called *honestification*.

To do this we wait for a large enough stage s such that the use $\theta(e_0, s)$ seems stable at stages $t > s$. Every time we see that the current stage does not meet this description, i.e. the use has changed since the previous β-true stage, we we must forcefully invalidate all previously enumerated axioms for d_β and reset the value of the parameter $u(d_\beta)$. The use of e_0 will however eventually become stable, as otherwise we can argue that $\Theta(U \oplus V) \ne E$, contradicting our choice of Θ.

After finitely many iterations of honestification, at a stage s, say, we will have achieved our goal: the use of e_0 has stopped changing and all valid axioms for d_β in Γ extend $\theta(e_0, s)$. We shall say that Γ is honest at d_β at stage s. Suppose also that we have found an extension $\mu_0 \supset \tau_0$ in the c.e. set W_i. The strategy β will now *attack* by extracting e_0 from E. It will have outcome ∞ at this stage. At every stage $s^+ > s$, $\theta(e_0, s) \not\subseteq (U \oplus V)^{\{s^+\}}$. If this is because of a permanent extraction from the set U, i.e at all further stages $t > s(\ \theta(e_0, s) \not\subseteq U^{\{t\}} \oplus \mathbb{N})$ then all axioms enumerated in Γ for elements $n \ge d_\beta$ are invalid at all furthers stages and the strategy β can successfully restrain $\mu_0 \subseteq A$ with no injury to α. The strategy has outcome f_0 at all stages $t > s$ while $\theta(e_0, s) \not\subseteq U^{\{t\}} \oplus \mathbb{N}$.

If the extraction disappears at stage s_1 (in this case $\theta(e_0, s) \not\subseteq \mathbb{N} \oplus V^{\{s_1\}}$) the strategy will evaluate this first cycle as unsuccessful. It will extract the marker $a(d_\beta)$, thereby preserving its work from injury by the strategy α. It will activate the backup strategy $\check{\alpha}(\beta)$ below outcome ∞. It will then start a new cycle with a new larger agitator e_1 and witness $\tau_1 \supseteq \mu_0^*$, where μ_0^* is the string μ_0 inverted at only one position $a(d_\beta)$. At the end of every cycle the strategy will record in a parameter Wit_β information about previous attacks. After every attack the strategy will go back an re-evaluate previous attacks. Outcome ∞ will be visited only if there is further evidence that all previous cycles are unsuccessful.

The *parameters for $\beta(i)$* are: the threshold d_β , always assigned at stage s the i-th element of $\overline{K}^{\{s\}}$, with first marker defined by β a_β^0 initially undefined; the current agitator e_β and witness τ_β initially undefined; the list of witnesses Wit_β initially empty.

4.4.3. *The backup enumeration strategy.* The strategy $\check{\alpha} = \check{\alpha}(\beta(i))$ is similar to the main enumeration strategy. It initiates the construction of a set $B_{\check{\alpha}}$ and an operator $\Lambda_{\check{\alpha}}$, so that ultimately $\Lambda_{\check{\alpha}}(V \oplus B_{\check{\alpha}}) = \overline{K}$.

Note that $\check{\alpha}$ is visited in two different situations: at the beginning of an attack by β and after the end of an unsuccessful attack. Only in the second case can we be sure that the necessary extractions from the approximation of V have been secured. At such stages, at which β does not start an attack, called *peaceful stages*, $\check{\alpha}$ will correct the enumeration $\Lambda_{\check{\alpha}}(V \oplus B_{\check{\alpha}})^{\{s\}}$ by extracting elements from B. As $\check{\alpha}$ is not

visited at every stage, we cannot assume that it will be visited at infinitely many good stages for the approximation to V. The strategy $\check{\alpha}$ will therefore work with a delayed approximation to the set V. Fix a stage s and let $s^- < s$ be the previous visit of $\check{\alpha}$ ($s = 0$ if $\check{\alpha}$ has not been visited yet). Then $V_{\check{\alpha}}^{\{s\}} = \bigcap_{s^- < t \leq s} V^{\{t\}}$. It is not difficult to see that $\{V_{\check{\alpha}}^{\{s\}}\}_{s<\omega}$ is also a good Σ_2^0 approximation to V. The backup genericity strategies will as well assign to every element n markers $b_{\check{\alpha}}(n)$ and $v_{\check{\alpha}}(n)$. If $\check{\alpha}$ enumerates an axiom for n at stage s, it is of the form: $\langle n, (V_{\check{\alpha}}^{\{s\}} \restriction v_{\check{\alpha}}(n) + 1) \oplus (B_{\check{\alpha}}^{\{s\}} \restriction b_{\check{\alpha}}(n) + 1)\rangle$.

The *parameters for* $\check{\alpha}(\beta)$ are hence $B_{\check{\alpha}}$ and $\Lambda_{\check{\alpha}}$, both with initial values $\emptyset$; markers $b_{\check{\alpha}}(n)$ and $v_{\check{\alpha}}(n)$ for very natural number n, initially undefined.

4.4.4. *The backup genericity strategy.* The i-th backup strategy $\check{\beta}(\check{\alpha}, \beta, i)$ ensures that $B_{\check{\alpha}}$ satisfies the i-th genericity requirement in a similar way to the main genericity strategy. It has a threshold $d_{\check{\beta}}$ - the i-th element of $\overline{K}$, and is responsible for defining the markers $b_{\check{\alpha}}(d_{\check{\beta}})$ and $v_{\check{\alpha}}(d_{\check{\beta}})$. The first $B_{\check{\alpha}}$-marker that $\check{\beta}$ defines again plays a special role and is denoted by $b_{\check{\beta}}^0$. The strategy has an agitator $e_{\check{\beta}}$ for which it ensures that the operator $\Lambda_{\check{\alpha}}$ is honest at $d_{\check{\beta}}$. The strategy selects a witness $\tau_{\check{\beta}} = B_{\check{\alpha}} \restriction b_{\check{\alpha}}(d_{\check{\alpha}}) + 1$ and starts searching for an extension of $\tau_{\check{\beta}}$ in the set W_i. If there is no such extension then the strategy is successful and has outcome w. Otherwise it has found an extension $\mu \subseteq \tau_{\check{\beta}}$ in the set W_i and now would like to force a change in the approximation to V in order to be able to secure $\mu \subseteq B_{\check{\alpha}}$. To do so the strategy $\check{\beta}$ will *time* its attack with one of the attacks of the strategy β. Instead of attacking immediately, it will wait for a stage s at which β is also attacking. As every new cycle of β comes with a new larger agitator e_β, at stage s we have $e_{\check{\beta}}^{\{s\}} < e_\beta^{\{s\}}$, hence $\theta(e_{\check{\beta}}^{\{s\}}, s) \subseteq \theta(e_\beta^{\{s\}}, s)$. If both $e_{\check{\beta}}$ and e_β are extracted during a joint attack by $\check{\beta}$ and β at stage s then it will be sufficient for β to look at the changes resulting in $\theta(e_{\check{\beta}}, s) \not\subseteq (U \oplus V)^{\{t\}}$ at further stages $t > s$, when evaluating this attack. In this way whenever the backup strategies are activated at two consecutive stages $t > t^- > s$ of the second type (at stages t^- and t the strategy β has just evaluated an attack as unsuccessful) we have that $\theta(e_{\check{\beta}}, s_1) \not\subseteq \mathbb{N} \oplus (\bigcap_{t^- < r \leq t} V^{\{r\}})$ and so without injury to $\check{\alpha}$ the strategy $\check{\beta}$ can secure μ as an initial segment to $B_{\check{\alpha}}$ and have outcome f at all further stages.

The *parameters for* $\check{\beta}(\check{\alpha}, \beta, i)$ are: the threshold $d_{\check{\beta}}$, always assigned at stage s the i-th element of $\overline{K}^{\{s\}}$, with first $B_{\check{\alpha}}$-marker $b_{\check{\beta}}^0$; the current agitator $e_{\check{\beta}}$ and witness $\tau_{\check{\beta}}$ initially undefined.

4.5. **The construction.** The construction is in stages. At every expansionary stage s we construct a finite path of length less than or equal to s, $\delta^{\{s\}}$. The intention is that there will be a true path - a leftmost path of strategies visited at infinitely many stages, along which all strategies are successful.

At the start of the construction all nodes are initialized, $E^{\{0\}} = \mathbb{N}$ and $\delta^{\{0\}} = \emptyset$. At stage $s > 0$ we construct the $E^{\{s\}}$ from its previous value by allowing active strategies to extract elements from $E^{\{s-1\}}$ and the finite string $\delta^{\{s\}}$ inductively in steps. $\delta^{\{s\}} \restriction 0$ is always the root of the tree, the main enumeration strategy α. We obtain $\delta^{\{s\}} \restriction (n + 1)$ by activating the strategy $\delta^{\{s\}} \restriction n$ and allowing it to select an outcome o, then $\delta^{\{s\}} \restriction (n + 1) = \delta^{\{s\}} \restriction n\hat{\ }o$. This process continues until we have defined a string $\delta^{\{s\}}$ of length s or until we have reached a leaf in the tree

of strategies. At the end of stage s we initialize all strategies $\gamma > \delta^{\{s\}}$. When we initialize a strategy γ- it's parameters are set to their initial values. Otherwise the parameters inherit their values from the previous stage at which γ was visited and we will not indicate a stage when referring to the current values of the parameters.

Suppose we have defined $\delta^{\{s\}} \upharpoonright n$ and $n < s$. We have four cases depending on the type of strategy assigned to $\delta^{\{s\}} \upharpoonright n$.

Case 1: The strategy $\delta^{\{s\}} \upharpoonright n = \alpha$ is the main enumeration strategy. The strategy proceeds as follows:

(1) For all $n < s$ such that $n \in \overline{K}^{\{s\}} \setminus \Gamma(U^{\{s\}} \oplus A^{\{s\}})$ and both $a(n)$ and $u(n)$ are defined then enumerate in Γ the axiom

$$\langle n, (U^{\{s\}} \upharpoonright u(n) + 1) \oplus (A \upharpoonright a(n) + 1)\rangle.$$

(2) If s is not expansionary

$$l(\Theta(U \oplus V), E, s) \leq \max_{t < s} l(\Theta(U \oplus V), E, t)$$

then let the outcome be b.

(3) If s is expansionary then for all $n < s$ such that $n \in \Gamma(U^{\{s\}} \oplus A) \setminus \overline{K}^{\{s\}}$ find all valid axioms, say $\langle n, D_u \oplus D_a \rangle \in \Gamma$, extract the largest element of D_a from A. (Note that we are changing the value of the parameter A.) Let the outcome be $o = $ e.

Case 2: The strategy $\delta^{\{s\}} \upharpoonright n = \beta(i)$ is a main genericity strategy. At stage s the strategy first passes through *Check*. Let s^- be the previous stage at which β was visited.

- **Check:**
  Let d_β be the i-th element of $\overline{K}^{\{s\}}$. If the strategy is in initial state or $d_\beta^{\{s^-\}} \neq d_\beta^{\{s\}}$, i.e there is an element $m \leq d_\beta^{\{s^-\}}$ such that $m \in \overline{K}^{\{s^-\}} \setminus \overline{K}^{\{s\}}$ then go to step 1 of *Initialization*. If at stage s the strategy α extracts an element $a < a_\beta^0$ then initialize all of β's parameters and go to step 2 of *Initialization*. Otherwise proceed to the submodule indicated at the previous β-true stage s^-.
- **Initialization:**
  (1) Define a new A-marker $a(d_\beta)$ as a fresh number, larger than any number that has so far been used in the construction, and enumerate it in the set A. Set $a_\beta^0 = a(d_\beta)$ and make all A- and U-markers for elements $n > d_\beta$ undefined.
  (2) Initialize all lower priority strategies.
  (3) Define a new agitator $e_\beta \in E^{\{s\}}$, as a fresh number.
  (4) If $e_\beta \geq l(\Theta(U \oplus V), E, s)$ then end this substage with outcome h and return to this step at the next β-true stage. Otherwise proceed to the next step.
  (5) Extract $a(d_\beta)$ from the set A and define a fresh value $a(d_\beta)$ and enumerate it in A. Define $u(d_\beta)$ to be a number larger than the maximal number in $\theta(e_\beta, s)$. Make all A- and U-markers for elements $n > d_\beta$ undefined. Let $\tau_\beta = A \upharpoonright a(d_\beta) + 1$. End this substage with outcome h. Proceed to *Honestification* at the next stage.

- **Honestification:** If $\theta(e_\beta, s) \neq \theta(e_\beta, s^-)$ or if there is a stage t, such that $s^- < t \leq s$ and $\theta(e_\beta, s) \nsubseteq (U \oplus V)^{\{t\}}$ then extract $a(d_\beta)$ from the set A. Define a fresh value for the marker $a(d_\beta)$ and enumerate it in A. Define a new value for the marker $u(d_\beta)$ larger than the maximal number in $\theta(e_\beta, s)$. Make all A- and U-markers for elements $n > d_\beta$ undefined. Let $\tau_\beta = A \restriction a(d_\beta) + 1$. End this substage with outcome h. Return to *Honestification* at the next stage.

  Otherwise go to *Waiting* .

- **Waiting:** If there is a finite string $\mu \supseteq \tau_\beta$ such that $\mu \in W_i^{\{s\}}$ then proceed to *Attack*. Otherwise let the outcome be $o = \mathsf{w}$. Return to Honestification at the next stage.

- **Attack:**
  (1) Let $\mu \supseteq \tau_\beta$ be a string such that $\mu \in W_i^{\{s\}}$. Set $a_\mu = a(d_\beta)$ and μ^* to be the string μ modified in bit a_μ so that $\mu^*(a_\mu) = 0$. Enumerate a new entry in the list Wit_β namely:

  $$\langle \mu, \mu^*, a_\mu, \theta(e_\beta, s) \rangle.$$

  (2) Extract the agitator e_β from the set $E^{\{s\}}$ and for all a such that $a_\beta^0 \leq a \leq |\mu|$ set $A(a) = \mu^*(a)$. (Note that we are modifying the parameter A so that the current marker of the threshold $a_\mu = a(d_\beta)$ is extracted from the set.) Define a new value for the marker $a(d_\beta)$ as a fresh number, larger than the length of the string μ, $|\mu|$, and enumerate it in A. Make all A- and U-markers for elements $n > d_\beta$ undefined.

  (3) Let the outcome be $(o = \infty)$. At the next true stage go to *Result*.

- **Result:** Let $\bar{e}$ be the least element that was extracted during s^-, the stage of the attack. Note that $\theta(\bar{e}, s^-) \subseteq \theta(e_\beta, s^-)$. If $e_\beta \neq \bar{e}$ then modify the fourth component of the last entry in the list Wit_β, making it: $\langle \mu, \mu^*, a_\mu, \theta(\bar{e}, s^-) \rangle$.

  Scan all entries in the list Wit_β in the order in which they are enumerated in the list from first to last.

  Suppose β is examining the n-th entry $Wit_\beta[n] = \langle \mu_n, \mu_n^*, a_n, U_n \oplus V_n, s_n \rangle$. For all a such that $a_\beta^0 \leq a \leq |\mu_n|$ set $A(a) = \mu_n(a)$.
  (1) If at all stages t, such that $s^- < t \leq s$, $U_n \nsubseteq U^{\{t\}}$ then let the outcome be $o = \mathsf{f}_n$. Return to this sub-step at the next true stage.
  (2) Otherwise there is a stage t, such that $s^- < t \leq s$ and $V_n \nsubseteq V^{\{t\}}$. For all a such that $a_\beta^0 \leq a \leq |\mu_n|$, set $A(a) = \mu_n^*(a)$. (This is necessary because the strategy α might have acted at this stage to invalidate an axiom for an element $m > d_\beta$, which extends $\langle m, U_n \oplus \{a_n\} \rangle$.) We say that $Wit_\beta[n]$ is unsuccessful.
  (3) If all entries are scanned and all are unsuccessful then end this substage with outcome $o = \infty$. At the next stage return to step 3 of *Initialization*, choosing a new agitator.

Case 3. The strategy $\delta^{\{s\}} \restriction n = \breve{\alpha}(\beta(i))$ is a backup enumeration strategy. Let s^- be the previous visit of $\breve{\alpha}$. Let $V_{\breve{\alpha}}^{\{s\}} = \bigcap_{s^- < t \leq s} V^{\{t\}}$. The strategy proceeds as follows:

(1) For all $n < s$, such that $n \in \overline{K}^{\{s\}} \setminus \Lambda_{\check{\alpha}}(V_{\check{\alpha}}^{\{s\}} \oplus B_{\check{\alpha}})$ and both $b_{\check{\alpha}}(n)$ and $v_{\check{\alpha}}(n)$ are defined, enumerate in $\Lambda_{\check{\alpha}}$ the axiom

$$\langle n, (V_{\check{\alpha}}^{\{s\}} \upharpoonright v_{\check{\alpha}}(n) + 1) \oplus (B_{\check{\alpha}} \upharpoonright b_{\check{\alpha}}(n) + 1)\rangle.$$

(2) If at stage s the strategy β **does not attack**, i.e. the visit is peaceful, then for all $n < s$ such that $n \in \Lambda_{\check{\alpha}}(V_{\check{\alpha}}^{\{s\}} \oplus B_{\check{\alpha}}) \setminus \overline{K}^{\{s\}}$ find all valid axioms, say $\langle n, D_v \oplus D_b\rangle \in \Lambda_{\check{\alpha}}$, extract the largest element of D_b from $B_{\check{\alpha}}$. Let the outcome be $o = \mathsf{e}$.

Case 4. The strategy $\delta^{\{s\}} \upharpoonright n = \check{\beta}(\check{\alpha}, \check{\beta}, i)$ is a backup genericity strategy.

- **Check:** Let $d_{\check{\beta}}$ be the i-th element of $\overline{K}^{\{s\}}$. If the strategy is in initial state or if $d_{\check{\beta}}^{\{s^-\}} \neq d_{\check{\beta}}^{\{s\}}$ then go to step 1 of *Initialization*. If at stage s the strategy $\check{\alpha}$ extracts an element $b < b_{\check{\beta}}^0$ then initialize all of $\check{\beta}$'s parameters and go to step 2 of *Initialization*. Otherwise proceed to the submodule indicated at the previous $\check{\beta}$-true stage s^-.

- **Initialization:**
  (1) Define a new $B_{\check{\alpha}}$-marker $b_{\check{\alpha}}(d_{\check{\beta}})$ as a fresh number, larger than any number that has so far been used in the construction, and enumerate it in the set $B_{\check{\alpha}}$. Let $b_{\check{\beta}}^0 = b_{\check{\alpha}}(d_{\check{\beta}})$ and make all $B_{\check{\alpha}}$- and $V_{\check{\alpha}}$-markers for elements $n > d_{\check{\beta}}$ undefined.
  (2) Initialize all lower priority strategies.
  (3) Define a new agitator $e_{\check{\beta}} \in E^{\{s\}}$, as a fresh number.
  (4) If $e_{\check{\beta}} \geq l(\Theta_e(U \oplus V), E, s)$ then end this substage with outcome h and return to this step at the next $\check{\beta}$-true stage. Otherwise proceed to the next step.
  (5) Extract $b_{\check{\alpha}}(d_{\check{\beta}})$ from the set $B_{\check{\alpha}}$ and define a fresh value $b_{\check{\alpha}}(d_{\check{\beta}})$ and enumerate it in $B_{\check{\alpha}}$. Set $v_{\check{\alpha}}(d_{\check{\beta}})$ to be a number larger than $\max(\theta(e_{\check{\beta}}, s))$. Make all $B_{\check{\alpha}}$- and $V_{\check{\alpha}}$-markers for elements $n > d_{\check{\beta}}$ undefined. Set $\tau_{\check{\beta}} = B_{\check{\alpha}} \upharpoonright b_{\check{\alpha}}(d_{\check{\beta}}) + 1$. End this substage with outcome h. Proceed to *Honestification* at the next stage.

- **Honestification:** If $\theta(e_{\check{\beta}}, s) \neq \theta(e_{\check{\beta}}, s^-)$ or if there is a stage t, such that $s^- < t \leq s$ and $\theta(e_{\check{\beta}}, s) \not\subseteq (U \oplus V)^{\{t\}}$ then extract $b_{\check{\alpha}}(d_{\check{\beta}})$ from the set $B_{\check{\alpha}}$. Define a fresh value for the marker $b_{\check{\alpha}}(d_{\check{\beta}})$ and enumerate it in $B_{\check{\alpha}}$. Define a new value for the marker $v_{\check{\alpha}}(d_{\check{\beta}})$ larger than the maximal number in $\theta(e_{\check{\beta}}, s)$. Make all $B_{\check{\alpha}}$- and $V_{\check{\alpha}}$-markers for elements $n > d_{\check{\beta}}$ undefined. Set $\tau_{\check{\beta}} = B_{\check{\alpha}} \upharpoonright b_{\check{\alpha}}(d_{\check{\beta}}) + 1$. End this substage with outcome h. Return to *Honestification* at the next stage.
  Otherwise go to *Waiting*.

- **Waiting:** If there is a finite string $\mu \supseteq \tau_{\check{\beta}}$ such that $\mu \in W_i^{\{s\}}$ then proceed to *Attack*. Otherwise let the outcome be $o = \mathsf{w}$. Return to *Honestification* at the next stage.

- **Attack:**
  (1) If at stage s the strategy β dose not perform an attack then let the outcome by $o = \mathsf{w}$, return to *Honestification* at the next stage. Otherwise proceed to the next step. (Note this is how $\check{\beta}$ *times* its attack with β.)

(2) Let $\mu \supseteq \tau_{\check{\beta}}$ be a string such that $\mu \in W_i^{\{s\}}$. For all b such that $\left|\tau_{\check{\beta}}\right| < b < |\mu|$, set $B_{\check{\alpha}}(b) = \mu(b)$. Define a new fresh value for the marker $b_{\check{\alpha}}(d_{\check{\beta}})$, a number larger than the length of the string μ, and enumerate it in $B_{\check{\alpha}}$. Make all $B_{\check{\alpha}}$- and $V_{\check{\alpha}}$-markers for elements $n > d_{\check{\beta}}$ undefined.

(3) Extract $e_{\check{\beta}}$ from the set $E^{\{s\}}$ and go to *Result*.

- **Result:** Let the outcome be $o = \mathsf{f}$. Return to this step at the next true stage.

4.6. **Verification of the construction.** The tree of strategies T has two features which make it nontrivial to prove that there is an infinite true path in the construction. Some nodes have infinitely many outcomes and some nodes are leaves. To start off we prove that the true path exists even though some nodes in the tree have infinitely many outcomes and contains at least one node different from the root, i.e. is of length at least 2. Later on we shall see that the true path is infinite.

Propostion 4.2. *There exists a path f in the tree of strategies T such that:*
1. *For all $\beta \subseteq f$ there is a stage s such that at all $t > s$, $(\delta^{\{t\}} \geq \beta)$.*
2. *For all $\beta \subseteq f$ there exist infinitely many stages s such that $\beta \subseteq \delta^{\{s\}}$.*

Proof. We prove that every strategy, visited at infinitely many stages has a leftmost infinite outcome, i.e. a leftmost outcome that it has at infinitely many true stages, also called the *true outcome*, which is not a leaf. Then f is defined inductively by $f(0) = \emptyset$ and $f(n + 1)$ is the true outcome of $f \restriction n$, if $f \restriction n$ is not a leaf and $f = f \restriction n$, otherwise.

The main enumeration strategy has true outcome e, as by our choice of Θ and the property 4.1 there are infinitely many expansionary stages. The backup enumeration strategies and backup genericity strategies have only finitely many outcomes. the leftmost one visited at infinitely many stages is their true outcome.

So suppose β is a main genericity strategy, visited at infinitely many stages. If β has outcome ∞ at infinitely many stages then this is the true outcome of β. Otherwise there is a stage s such that β does not have outcome ∞ at all $t > s$. It follows from the construction that no new entries are enumerated into the list Wit_β after stage s and hence the only possible outcomes for β at stages $t > s$ are finitely many: w, h, and f_n where $n \leq \left|Wit_\beta^{\{s\}}\right|$, and the leftmost one of them visited at infinitely many stages is β's true outcome. $\square$

We know now that there is at least one main genericity strategy along the true path, namely $\beta(0) = \alpha\char`\^\mathsf{e}$. We shall next investigate some properties of main genericity strategies along the true path.

Propostion 4.3. *Let β be a main genericity strategy on the true path. Then:*
1. *There is a stage s_β^0 such that: β is not initialized at stages $t > s_\beta^0$, β is the only main genericity strategy of type (i) accessible at stages $t > s_\beta^0$ and β is visited at every $\alpha\char`\^\mathsf{e}$-true stage $t > s_\beta^0$.*
2. *There is a stage s_β^d such that at stages $t > s_\beta^d$ the value of d_β does not change. At stage s_β^d the final value of the marker a_β^0 is defined. All α-markers for the element d_β are defined by β at stages $t \geq s_\beta^d$.*

(3) *There is a stage $s_\beta^c \geq s_\beta^d$ such that at stages $t > s_\beta^c$ Check does not send β to* Initialization. *At stages $t > s_\beta^c$ the parameter A is not modified on elements $a < a_\beta^0$.*

(4) *After stage s_β^c every time β changes the value of the marker $a(d_\beta)$ all main genericity strategies $\beta(j)$, where $j > i$, that are accessible at stages $t > s_\beta^c$, i.e. for which there is a true stage $t > s_\beta^c$, are in initial state.*

(5) *If β's true outcome is w or f_n for some natural number n then there is a stage s_β^e such that β does not modify any parameters and has its true outcome at every true stage $t > s_\beta^e$.*

Proof. Assume inductively that the statement is true for main genericity strategies along the true path of higher priority than $\beta = \beta(i)$. It follows by the definition of the tree that all such strategies have outcome w or f_n for some n. By the induction hypothesis (claim 5) and the fact that $\beta \subseteq f$ there is a stage s_β^0 such that β is not initialized after stage s_β^0 and such that at stages $t > s_\beta^0$ the parameter A is not modified by main genericity strategies of higher priority than β. Furthermore it follows that the markers $a(n)$ and $u(n)$ for the first $i - 1$ elements of $\overline{K}$ do not change as the only accessible strategies of type $\beta(j)$, where $j \leq i$ are the ones that are initial segments of β. In other words if $s > s_\beta^0$ is an $\alpha\hat{}e$-true stage then s is a β-true stage. Hence the only strategies of lower priority than β that are accessible at stages $t > s_0$ are strategies which extend β.

Let $s_\beta^d \geq s_\beta^0$ be such that $\overline{K}^{\{s_\beta^d\}}$ correctly approximates the first i elements of $\overline{K}$. Then after stage s_β^d the value of the threshold d_β does not change and a_β^0 receives its final value, i.e. $(a_\beta^0)^{\{t\}} = (a_\beta^0)^{\{s_\beta^d\}} = a_\beta^0$ for all $t \geq s_\beta^d$. As β is the only strategy of type (i) accessible after stage s_β^d, d_β receives α-markers only from β after stage s_β^d.

After stage s_β^d the strategy α will extract finitely many numbers $a < a_\beta^0$. And every time such an element is extracted it will not be reenumerated back in A. This follows from the fact that at stages $t > s_\beta^d$ strategies of higher priority β do not modify A and accessible strategies of lower priority $\gamma \supseteq \beta$ are initialized at stage s_β^d and modify A only on elements larger than $a_\gamma^0 \geq a_\beta^0$. Hence there is a least stage s_β^c such that at stages $t > s_\beta^c$ the strategy α does not extract numbers less than a_β^0 from A and hence $A_{\upharpoonright} a_\beta^0$ is not modified at stages $t > s_\beta^c$. At stage s_β^c the strategy β performs step 2 of Initialization for the last time and all lower priority strategies are in initial state.

The following is a diagram which shows the way in which β can change its outcome at consecutive true stages:

$$\mathsf{h} \Leftarrow \infty \Leftarrow \cdots \Leftarrow f_1 \Leftarrow f_0 \Leftarrow \infty \Leftarrow (\mathsf{h} \Leftrightarrow \mathsf{w})$$

The strategy β changes the value of the marker $a(d_\beta)$ at stage $s > s_\beta^c$ only when it has outcome h during *Initialization* or *Honestification* and when it has outcome ∞ during *Attack* at stage s. In the second case all main genericity strategies are initialized at stage s. In the first case let s^- be the largest stage $s_\beta^c < s^- < s$ such that β has outcome ∞ if there is such state and $s^- = s_\beta^c$ otherwise. Then strategies extending outcomes f_i for some natural number i are in initial state at stage s^- and are not accessible at stages $t \in [s^-, s]$. Strategies extending w are initialized at stage s.

If β has true outcome $o \in \{\mathsf{w}, \mathsf{f}_n \mid n < \omega\}$, then from the diagram it follows that there is a stage s_β^e such that β has outcome o at all stages $t > s_\beta^e$. From the construction it follows that β does not modify any parameters . $\square$

The properties listed above are true for the backup genericity strategies along f.

Propostion 4.4. *Let $\check{\beta}$ be a backup genericity strategy on the true path. Then:*

(1) *There is a stage $s_{\check{\beta}}^0$ such that: $\check{\beta}$ is not initialized at stages $t > s_{\check{\beta}}^0$, $\check{\beta}$ is the only backup genericity strategy of type $(\check{\alpha}, \beta, i)$ accessible at stages $t > s_{\check{\beta}}^0$ and $\check{\beta}$ is visited at every $\check{\alpha}$-true stage $t > s_{\check{\beta}}^0$.*

(2) *There is a stage $s_{\check{\beta}}^d$ such that at stages $t > s_{\check{\beta}}^d$ the value of $d_{\check{\beta}}$ does not change. At stage $s_{\check{\beta}}^d$ the limit value of the marker $b_{\check{\beta}}^0$ is defined. All $\check{\alpha}$-markers for the element $d_{\check{\beta}}$ are defined by $\check{\beta}$ at stages $t \geq s_{\check{\beta}}^d$.*

(3) *There is a stage $s_{\check{\beta}}^c > s_{\check{\beta}}^d$ such that at stages $t > s_{\check{\beta}}^c$ Check does not send $\check{\beta}$ to Initialization. At stages $t > s_{\check{\beta}}^c$ the value of $B_{\check{\alpha}}$ is not modified below $b_{\check{\beta}}^0$.*

(4) *After stage $s_{\check{\beta}}^c$ every time $\check{\beta}$ changes the value of the marker $b_{\check{\alpha}}(d_{\check{\beta}})$ all backup genericity strategies $\check{\beta}(\check{\alpha}, \beta, j)$, where $j > i$, that are accessible at stages $t > s_{\check{\beta}}^c$, i.e. for which there is a true stage $t > s_{\check{\beta}}^c$, are in initial state.*

(5) *If $\check{\beta}$'s true outcome is w or f then there is a stage $s_{\check{\beta}}^e$ such that $\check{\beta}$ does not modify any parameters and has its true outcome at true every stage $t > s_{\check{\beta}}^e$.*

Proof. The proof is carried out in the same way as the proof of Proposition 4.3. $\square$

Finally we are ready to prove that the true path is infinite.

Proposition 4.5. *If $\beta \subseteq f$ is a main genericity strategy then it's true outcome is not h. Similarly if $\check{\beta} \subseteq f$ is a backup genericity strategy then it's true outcome is not h. Hence the true path is infinite.*

Proof. Suppose towards a contradiction that $\beta \subseteq f$ has true outcome h and let $s \geq s_\beta^c$ be a stage such that at stages $t > s$ the strategy β is not initialized and does not have outcome ∞. Then at stages $t > s$ the strategy β has a fixed agitator $e_\beta \in E$. As $E = \Theta_e(U \oplus V)$ then $\lim_t \theta(e_\beta) = \theta(e_\beta)$ exists and $\theta(e_\beta) \subseteq U \oplus V$. Let $s_{lim} > s$ be a β-true stage such that at all stages $t > s_{lim}$ we have $\theta(e_\beta, t) = \theta(e_\beta)$ and $\theta(e_\beta) \subseteq (U \oplus V)^{\{t\}}$. It follows that β cannot have outcome h at stages $t > s_{lim}$ contradicting the fact that h is β's true outcome.

The second statement is proved in a similar way. $\square$

Proposition 4.6. *Let $\beta \subseteq f$ be a main genericity strategy and let d_β be the limit value of β's threshold attained at stage s_β^d. If τ_β is defined as the current witness of β at stage $s \geq s_\beta^c$ then for all $a \geq a_\beta^0$ such that a is a previous value of the A-marker of the threshold d_β, $\tau_\beta(a) = 0$.*

Proof. We prove this with induction on the stage s. Suppose that the statement is true for values of the witness τ_β defined before stage s. Recall that $\tau_\beta^{\{s\}} = A^{\{s\}} \upharpoonright a(d_\beta)$. If a is an old marker of the threshold d_β, $a < a(d_\beta)^{\{s\}}$ and it is sufficient to prove that $a \notin A^{\{s\}}$.

Suppose towards a contradiction that a is an old marker defined before and cancelled before or on stage s and $a \in A^{\{s\}}$. As every time β cancels an old value

of the A-marker of the threshold, it extracts this value from A, this could only be possible if a strategy later on at stage $s_a < s$ re-enumerates a in A and a remains in the set A at all stages $t \in [s_a, s]$. By Proposition 4.3 the only strategy that can re-enumerate a in the set A is the strategy β and by construction this is only possible if at stage s_a the strategy β starts evaluating $Wit_\beta[n]^{\{s_a\}}$ for some n, and $\mu_n(a) = 1$. But μ_n is defined as an extension of a previous value τ of the witness τ_β and by induction if $a < |\tau|$ then $\tau(a) = 0$. It follows that $|\tau| \le a < |\mu_n|$ and by construction the only possibility is that $a = a_n$, as the next value of the A-marker for d_β is defined as a number larger than $|\mu|$. So at stage s_a the strategy β has outcome f_n and is evaluating the result of its most recent attack. At stage s a new value for τ_β is defined, so β must have evaluated it most recent attack as unsuccessful. This means that at a stage in the interval (s_a, s) the strategy β evaluates $Wit_\beta[n]$ as unsuccessful and extracts $a_n = a$ from the set A, contradicting our choice of stage s_a. It follows that the assumption is wrong and $a \notin A^{\{s\}}$. $\quad\square$

Propostion 4.7. *Let $\beta \subseteq f$ be a main genericity strategy and let d_β be the limit value of β's threshold attained at stage s_β^d. If an axiom $\langle x, D_u \oplus D_a \rangle$ is enumerated in $\Gamma^{\{s\}}$ for an element $x \notin \overline{K}$ after stage s_β^d then one of the following holds:*

(1) *$s \le s_\beta^c$. Then the axiom is invalid at all stages $t > s_\beta^c$.*
(2) *D_a contains a marker a of the threshold d_β which is cancelled as current at stage s^+ at which β defines the next value of the A-marker of the threshold. Then $a \notin A^{\{t\}}$ at all $t > s^+$.*
(3) *D_a contains a marker a of the threshold d_β which is cancelled as current at stage s^+ and at stage s^+, $a = a_n$ becomes a component of the n-th entry in the list Wit_β, $\langle \mu_n, \mu_n^*, a_n, U_n \oplus V_n \rangle$ and in this case $U_n \subseteq D_u$.*
(4) *D_a contains the final value of the A-marker of the threshold d_β.*

Proof. As $s > s_\beta^d$ and $x \in \overline{K}^{\{s\}} \setminus \overline{K}$ it follows that $x > d_\beta$. By the fact that every time a new value for the markers of d_β are defined, the markers for x are cancelled and the format of the axioms enumerated in Γ it follows that D_a contains the current marker $a = a(d_\beta)^{\{s\}}$ which is defined by β by Proposition 4.3.

If $s \le s_\beta^c$ then at stage s_β^c the strategy α extracts an element $a_0 < a_\beta^0$ such that $a_0 \in A^{\{s\}}$, hence $a_0 \in D_a$, which is never re-enumerated in the set A. It follows that the axiom is invalid at all stages $t > s_\beta^c$. So suppose that $s > s_\beta^c$

If $a = a(d_\beta)^{\{t\}}$ at all $t > s$ then case 4 is true. Otherwise let s^+ be the stage at which $a(d_\beta)$ is changed. If $a \in A^{\{t\}}$ at some $t > s^+$. Then as in Proposition 4.6 only β can enumerate a in the set A during *Result* if for some n, $\mu_n(a) = 1$. This is only possible if $a_n = a$.

So if $a \ne a_n$ for any member if the list Wit_β then $a \notin A^{\{t\}}$ at all stages $t > s^+$.

Finally suppose $a = a_n$ for some n. Then s^+ is the stage of the n-th attack after stage s_β^c. Let s^- be the stage at which $a(d_\beta)$ received the value a. Then at this stage $u(d_\beta)$ is set to a number larger than the maximal number of $\theta(e_\beta^{\{s^-\}}, s^-)$ and all α-marker for elements $n > d_\beta$ are undefined. At stages $t \in (s^-, s^+]$ the strategy β does not change the value of $a(d_\beta)$ and hence has outcome w if visited. It follows that $s \in (s^-, s^+]$ and $e_\beta^{\{s\}} = e_\beta^{\{s^-\}} = e_\beta$. On the other hand $\theta(e_\beta, s^+) = \theta_{e_\beta, s} \subseteq (U \oplus V)^{\{s\}}$. Hence $\theta(e_\beta, s^+) \subseteq D_u \oplus \mathbb{N}$. Finally $U_n \oplus V_n$ is defined as the value of

$\theta(\bar{e}, s^+)$, where $\bar{e} \leq e$. As by the definition of the use function $\theta(\bar{e}, s^+) \subseteq \theta(e, s^+)$, we can conclude that $U_n \subseteq D_u$.

As by Proposition 4.6 at all stages $t \geq s^+$ if $a_n \in A^{\{t\}}$ then β has outcome f_n at stage t and hence $U_n \nsubseteq U^{\{t\}}$, the axiom $\langle x, D_u \oplus D_a \rangle$ is invalid at all stages $t \geq s_n$. $\hfill\square$

Corollary 4.1. *If $\beta = \beta(i) \subseteq f$ has true outcome w or f_n for some natural number n then β successfully satisfies $\mathcal{G}_i$.*

Proof. Suppose that β has outcome $o = \mathsf{w}$ or $o = \mathsf{f}_n$ for some natural number n at all stages $t \geq s_\beta^e$, where s_β^e is a least such stage. Then by Proposition 4.3 at stage s_β^e all lower priority main genericity strategies $\gamma \supseteq \beta\hat{\ }o$ are in initial state at stage s^e and will not modify $A^{\{t\}}$ on numbers $a < a(d_\beta)^{\{t\}} < (a_\gamma^0)^{\{t\}}$ at stages $t \geq s_\beta^e$. Higher priority main genericity strategies do not modify A at all.

Suppose that β has outcome w at all stages $t \geq s_\beta^e$. Then the final value of the witness τ_β is defined at stage s_β^h, the previous β-true stage before s_β^e. Note that s_β^h is the last stage at which β has outcome h and $\tau_\beta = A^{\{s_\beta^h\}} \upharpoonright a(d_\beta)^{\{s_\beta^h\}}$. The final value of the witness τ_β does not belong to the set W_i, otherwise after stage s_β^e, β would have outcome ∞. To see that $\tau_\beta \subseteq A$ we show that α does not modify $A \upharpoonright |\tau|$ at stages $t > s_\beta^h$. Indeed, the only case in which α would need to change A after stage s_h on a number less than $|\mu|$ is when it sees a valid axiom for and element $x \notin \overline{K}^{\{t\}}$ which was enumerated before stage s_h. By Proposition 4.7 it follows that Case 1, 2 and 4 cannot apply to this axiom. Case 3 does not apply as well as by Proposition 4.6 for all entries in the list Wit_β, $\tau_\beta(a_n) = 0$ and hence $A^{\{s_h\}}(a_n) = 0$. As by assumption β does not enter Result after stage s_β^h, no strategy including β can re-enumerate these markers back in $A^{\{t\}}$ at all $t > s_\beta^h$. As there are no more choices for the assumed axiom, it follows that t does not exists and α does not modify $A^{\{t\}}$ on numbers $a < |\tau|$ at stages $t > s_\beta^h$.

Now suppose that β has outcome f_n. Then $\mu_n \subseteq W_i$ and at stage s_β^e the strategy β starts evaluating $Wit_\beta[n]$ and sets $\mu_n \subseteq A^{\{s_\beta^e\}}$. In this case as well it is easy to see that μ_n will remain an initial segment of A at all further stages as axioms for elements $x \notin \overline{K}$ with maximal A-marker less than $|\mu_n|$ are invalid at all stages $t > s_\beta^e$. They cannot be Case 4 axioms, and Case 1 and 2 axioms are obviously not valid at any stage $t > s_\beta^e$. Case 4 axioms which contain markers a_m for $m < n$ are invalid as $\mu_n(a_m) = 0$. Indeed μ_n is defined as an extension of a previous witness τ_β which by Proposition 4.6 has this property. Finally if the axiom is of the form $\langle x, D_u \oplus D_a \rangle$ and D_a contains a_n then $U_n \subseteq D_u$. However at all stages $t > s_\beta^e$, $U_n \nsubseteq U^{\{t\}}$ hence in this case as well α does not modify $A^{\{t\}}$ on numbers $a \leq |\mu_n|$, hence $\mu_n \subseteq A$. $\hfill\square$

Propostion 4.8. *Let $\check{\beta} = \check{\beta}(\check{\alpha}, \check{\beta}, i) \subseteq f$ and let $d_{\check{\beta}}$ be the limit value of $\check{\beta}$'s threshold attained at stage $s_{\check{\beta}}^d$. If an axiom $\langle x, D_v \oplus D_b \rangle$ is enumerated in $\Lambda^{\{s\}}$ for an element $n \notin \overline{K}$ after stage $s_{\check{\beta}}^d$ then one of the following holds:*

(1) $s \leq s_{\check{\beta}}^c$. *Then the axiom is invalid at all stages $t > s_{\check{\beta}}^c$.*

(2) D_b *contains a marker b of the threshold which is extracted at stage s^+ at which $\check{\beta}$ defines the next value of the $B_{\check{\alpha}}$-marker, and is never reenumerated in $B_{\check{\alpha}}$.*

    (3) D_v contains a subset V_n which eventually becomes a component of the n-th entry in the list Wit_β, $\langle \mu_n, \mu_n^*, a_n, U_n \oplus V_n \rangle$.

    (4) D_a contains the final value of the A-marker of the threshold d_β.

Proof. Part one is proved just as Part 1 of Proposition 4.7, so assume that $s > s_{\check\beta}^c$.

We note that $s_{\check\beta}^c > s_\beta^c$ as every time β is restarted during *Check*, $\check\beta$ is initialized.

As $s > s_{\check\beta}^c$ and $x \in \overline{K}^{\{s\}} \setminus \overline{K}$ it follows that $x > d_{\check\beta}$. By the fact that every time new values for the markers of $d_{\check\beta}$ are defined, the $\check\alpha$-markers for n are cancelled and the format of the axioms enumerated in $\Lambda_{\check\alpha}$ it follows that D_b contains the current marker $b = b_{\check\alpha}(d_{\check\beta})^{\{s\}}$ which is defined by $\check\beta$ by Proposition 4.4. If $b = b_{\check\alpha}(d_{\check\beta})^{\{t\}}$ at all $t > s$ then case 4 is true. Otherwise let s^+ be the stage at which $b_{\check\alpha}(d_{\check\beta})$ is changed. If this is during Initialization or Honestification then $b \notin B_{\check\alpha}^{\{t\}}$ at all $t > s^+$.

Suppose that s^+ is the stage at which $\check\beta$ performs an *Attack* and times it with the n-th Attack of β. Then at stage s^+ the strategy $\check\beta$ extracts its agitator $e_{\check\beta}$. On the next β-true stage the strategy β evaluates the result of its n-th attack and enumerates as the fourth component $U_n \oplus V_n = \theta(e_{\check\beta}, s^+)$.

Let s^- be the stage at which $b(\check\alpha)(d_{\check\beta})$ received the value b. Then at this stage $v_{\check\alpha}(d_{\check\beta})$ is set to a number larger than the maximal number of $\theta(e_{\check\beta}^{\{s^-\}}, s^-)$ and all $\check\alpha$-marker for elements $n > d_{\check\beta}$ are undefined. At stages $t \in (s^-, s^+]$ the strategy $\check\beta$ does not change the value of $b_{\check\alpha}(d_{\check\beta})$ and hence has outcome w if visited. It follows that $s \in (s^-, s^+]$ and $e_{\check\beta}^{\{s\}} = e_{\check\beta}^{\{s^-\}} = e_{\check\beta}^{\{s^+\}} = e_{\check\beta}$ and $\theta(e_{\check\beta}, s^-) = \theta_{e_{\check\beta}, s} = \theta_{e_{\check\beta}, s^+} = U_n \oplus V_n \subseteq \bigcap_{t \in [s^-, s]} (U \oplus V)^{\{t\}}$. Hence $V_n \subseteq D_v$. $\qquad\square$

Corollary 4.2. *If $\check\beta = \check\beta(\check\alpha, \beta, i) \subseteq f$ has true outcome w or f then β successfully satisfies $\mathcal{G}_i(A)$.*

Proof. Suppose that $\check\beta$ has outcome $o = $ w or $o = $ f at all stages $t \geq s_{\check\beta}^e$, where $s_{\check\beta}^e$ is a least such stage. Then by Proposition 4.4 at stage $s_{\check\beta}^e$ all lower priority backup genericity strategies $\gamma \supseteq \check\beta\hat{\;}o$ are in initial state at stage $s_{\check\beta}^e$ and will not modify $B_{\check\alpha}^{\{t\}}$ on numbers $b \leq b_{\check\alpha}(d_{\check\beta})^{\{t\}} < (b_\gamma^0)^{\{t\}}$ at stages $t \geq s_{\check\beta}^e$. Higher priority strategies do not modify $B_{\check\beta}$ at all at stages $t \geq s_{\check\beta}^e$.

We prove that $\check\alpha$ does not modify $B_{\check\alpha}^{\{t\}}$ on numbers $b \leq b_{\check\alpha}(d_\beta)^{\{t\}}$ at stages $t > s_\beta^e$. Suppose towards a contradiction that it does. Then at a stage $t > s_{\check\beta}^e$, $\check\alpha$ invalidates a valid at stage t axiom, $\langle x, D_v \oplus D_a \rangle$, for an element $x \notin \overline{K}^{\{t\}}$. By Proposition 4.8 this must be a case 3 axiom and D_v contains a subset V_n which at a stage $s < s_{\check\beta}^e$ becomes a component of the n-th entry in the list Wit_β, $\langle \mu_n, \mu_n^*, a_n, U_n \oplus V_n \rangle$. By construction at stage $t > s_{\check\beta}^e$ the strategy β has outcome ∞ after evaluating $Wit_\beta[n]$ as unsuccessful, hence $V_n \not\subseteq V_{\check\alpha}^{\{t\}}$, contradicting the assumption that $\langle x, D_v \oplus D_a \rangle$ is valid at stage t.

If $\check\beta$ has outcome w at all stages $t \geq s_{\check\beta}^e$. Then $\tau_{\check\beta} \subseteq B_{\check\alpha}$ and $\tau_{\check\beta} \notin W_i$. Indeed if $\tau_{\check\beta} \in W_i^{\{t\}}$ and $t_1 < t_2$ are two consecutive $\check\beta$-true stages after t then at least one of them is a stage of an attack by the main genericity strategy β, hence $\check\beta$ would attack and have outcome h $<_L$ w, contradicting the choice of $s_{\check\beta}^e$.

If $\check{\beta}$ has outcome f then at stage $s_{\check{\beta}}^e$ it has found an $\mu \in W_i$, ensured $\mu \subseteq B_{\check{\alpha}}^{\{s_{\check{\beta}}^e\}}$ and defined the final value of $b_{\check{\alpha}}(d_{\check{\beta}}) > |\mu|$. It follows that $\mu \subseteq B_{\check{\alpha}}$. $\qquad\square$

Lemma 4.1. *There is a Δ_2^0 1-generic set C such that $U \oplus C \equiv_e \overline{K}$ or $V \oplus C \equiv_e \overline{K}$.*

Proof. By proposition 4.1 we have two cases for the true path f:

Case 1: For every i a strategy of type $\beta(i)$ is assigned to the node $f \upharpoonright i$. We will prove that in this case $C = A$ is the required set.

It follows from the construction and Part (2) of Proposition 4.3 that the limit values of the markers $a_i^0 = a_{\beta(i)}^0$ exist for every $\beta(i) \subseteq f$ and form an unbounded increasing sequence. Furthermore by Part (3) of Proposition 4.3 for every i there is a stages $s_i = s_{\beta(\alpha,i)}^c$, such that at all stages $t > s_i$, $A^{\{s_i\}} \upharpoonright A_i = A^{\{t\}} \upharpoonright A_i$ and hence the set A is Δ_2^0.

For every number n if $n \notin \overline{K}$ then the actions of the main enumeration strategy α ensure that at infinitely many stages s (the $\alpha\char`^\mathsf{e}$-true stages after the extraction of n from $\overline{K}$) $n \notin (\Gamma(U \oplus A))^{\{s\}}$, hence $n \notin \Gamma(U \oplus A)$.

Let n be the i-th element of $\overline{K}$. Then $n = d_\beta$ at all stages $t > s_\beta^d$, where $\beta = \beta(\alpha, i) \subseteq f$. Hence at all stages $t > s_\beta^d$ the α-markers for n are defined. By our assumption Proposition 4.5 and our assumption on f, either $o = \mathsf{f}_k$ for some k or $o = \mathsf{w}$ is the true outcome for β and there is a stage s_β^e such that the α-markers for n, $u(n)$ and $a(n)$ do not change after stage s_β^e. Let $s > s_{\beta\char`^o}^c$ be a U-true stage such that at all $t > s$ $U \upharpoonright u(n) + 1 \subseteq U^{\{t\}}$. Then at stage s, α ensures that there is a valid axiom for n in Γ, say $\langle n, U_n \oplus A_n \rangle$. By our choice of stage s $U_n \subseteq U^{\{t\}}$ and $A_n \subseteq A^{\{t\}}$ at all stages $t > s$, hence $n \in \Gamma(U \oplus A)$.

Finally by Corollary 4.1 it follows that A is 1-generic.

Case 2: There is a strategy $\beta \subseteq f$ such that $\check{\alpha}(\beta)$ is assigned to $f \upharpoonright |\beta| + 1$ and for every i a backup strategy $\check{\beta}(\check{\alpha}, \beta, i)$ is assigned to the node $f \upharpoonright |\beta| + i + 1$. We will prove that in this case $C = B_{\check{\alpha}}$ is the required set.

That $B_{\check{\alpha}}$ is Δ_2^0 is proved as in the first case using the limit values of the markers $b_i^0 = b_{\check{\beta}(\check{\alpha},\beta,i)}^0$ for every $\check{\beta} \subseteq \check{\alpha} \subseteq f$ and by Part (3) of Proposition 4.4.

To prove that $\Lambda_{\check{\alpha}}(V \oplus B_{\check{\alpha}})$ we observe that there are infinitely many stages s (the $\check{\alpha}$-true stages after the extraction of n from $\overline{K}$ at which β does not attack) at which α ensures that there are no valid axioms for elements $n < s$, $n \notin \overline{K}^{\{s\}}$. If n is the i-th element of $\overline{K}$ then $n = d_{\check{\beta}}$ at all stages $t > s_{\check{\beta}}^d$, where $\check{\beta} = \check{\beta}(\check{\alpha}, \beta, i)$ and the values of its $\check{\alpha}$-markers will eventually reach a limit. Hence at a $\check{\alpha}$ stage s at which $V \upharpoonright v_{\check{\alpha}}(n) + 1 \subseteq V_{\check{\alpha}}^{\{t\}} \subseteq V$ the strategy $\check{\alpha}$ ensures that there is a valid axiom for n in $\Lambda_{\check{\alpha}}$ at all stages $t > s$, hence $n \in \Lambda_{\check{\alpha}}(V \oplus B_{\check{\alpha}})$.

Finally by Corollary 4.2 it follows that $B_{\check{\alpha}}$ is 1-generic. $\qquad\square$

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