Introducing a New Characterization of the Low_n Degrees
A look inside Kenneth Harris' PhD Thesis
Part 2

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Revision

Definition

A set A has the uniform escape property (UEP) if there is a partial computable function $h(e, x)$ such that whenever ϕ_e^A is total, then $h(e, x)$ is total and escapes domination from ϕ_e^A, i.e:

$$(\exists^\infty x)[\phi_e^A(x) \leq h(e, x)].$$

Theorem

A is low if and only if A has UEP.

Tot and Fin

Definition

$Tot = \{e | \phi_e \text{ is total }\}$.

- *Tot* is Π_2:

$$Tot(e) \Leftrightarrow (\forall x)(\exists s)[\phi_{e,s}(x) \downarrow].$$

- *Tot* is Π_2 - complete.
  Let A be a Π_2- set with Π_2 index e, then from the (SQNF):

$$A(x) \Leftrightarrow W_{g(e,x)} = \omega \Leftrightarrow g(e, x) \in Tot.$$

Tot and Fin

Definition

$Fin = \{e | dom(\phi_e) \text{ is finite }\}$.

- Fin is Σ_2:

$$Fin(e) \Leftrightarrow (\exists x)(\forall y > x)(\forall s)[\phi_{e,s}(y)\uparrow].$$

- Fin is Σ_2 - complete.
  Let A be a Σ_2- set with Σ_2 index e, hence $\overline{A}$ has Π_2 index e then from the (SQNF):

$$A(x) \Leftrightarrow \neg\overline{A}(x) \Leftrightarrow$$

$$W_{g(e,x)} \text{ is finite } \Leftrightarrow g(e,x) \in Fin.$$

UEP $\Rightarrow$ low

Let A be a set that has the uniform escape property.
We will prove that:

- $A' \leq_m Tot$, hence $A' \in \Pi_2$
- $A' \leq_m Fin$, hence $A' \in \Sigma_2$
- Hence $A' \in \Delta_2$ and $A' \leq_T \emptyset'$

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UEP $\Rightarrow$ low

- A' is c.e. in A, hence there is some index a such that:

$$A' = W_a^A$$

- $A'(x) \Leftrightarrow (\forall^\infty s)[x \in W_{a,s}^A]$.
- $\neg A'(x) \Leftrightarrow (\forall s)[x \notin W_{a,s}^A]$
- We will find a total computable function k such that $A'(x) \Leftrightarrow k(x) \in Tot$.
- We will find a total computable function l such that $A'(x) \Leftrightarrow l(x) \in Fin$

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UEP $\Rightarrow$ low (Tot)

The Plan

For every x we will define an A-function $\phi^A_{i(x)}$ such that:

- $x \in A'$ then $\phi^A_{i(x)}(n)$ is total and hence $h(i(x), n)$ is total.
- $x \notin A'$ then if $h(i(x), n)$ were total, it would ne dominated by $\phi^A_{i(x)}(n)$.

Then $h(i(x), n) = \phi_{k(x)}(n)$ by the S^m_n theorem.

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UEP $\Rightarrow$ low- Tot

The fixed point theorem:

Theorem

For all sets $A \subseteq \omega$ and all x, y if $f(x, y)$ is an A-computable function, then there is a computable $i(x)$ such that $\phi^A_{i(x)} = \phi^A_{f(x,i(x))}$

- We are going to define an A-computable function $F(x, y, n)$, where y will keep the place of the index.
- Then we will use the relativized S^m_n- theorem to get a computable function s such that $F(x, y, n) = \phi^A_{s(x,y)}(n)$.
- By the fixed point theorem we will have that $\phi^A_{s(x,i(x))} = \phi^A_{i(x)}$.

UEP $\Rightarrow$ low- Tot

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Attack and Extend

We will define an A-computable function $F(x, y, n, t)$ by primitive recursion on t. For each t and all $x, y < t$, we will extend F in exactly one of the following ways:

For all $n < t$ if $F(x, y, n, t-1)\downarrow$, then let $F(x, y, n, t) = F(x, y, n, t-1)$, otherwise:

- $E(x, y, t)$: define $F(x, y, n, t) = 0$.
- $A(x, y, t)$: if $h(y, n)[t]\downarrow$, then define $F(x, y, n, t) = h(y, n) + 1$. Otherwise leave $F(x, y, n, t)\uparrow$.

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Definition of F

- If $x, y < t$ and $x \in W_{a,t}$ then $E(x, y, t)$.
- If $x, y < t$ and $x \notin W_{a,t}$ then $A(x, y, t)$.

Define:

$$F(x, y, n) = m \Leftrightarrow (\exists t) F(x, y, n, t) \downarrow = m$$

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Justification of F

Lemma

Fix x and y. If $(\exists^\infty t)E(x,y,t)$, then $F(x,y,n)$ is total. If $(\forall^\infty t)A(x,y,t)$ and $h(y,n)$ is total, then $F(x,y,n)$ dominates $h(y,n)$.

Proof.

(1) Fix n and let $t > n$ be such that $E(x,y,t)$. Then $F(x,y,n,t)\downarrow$ hence $F(x,y,n)\downarrow$.

(2) Let t_0 be such that $(\forall t > t_0)A(x,y,t)$. Let $n > t_0$, then there is a first stage $t_1 > n$ such that $h(y,n)[t_1]\downarrow$, then $F(x,y,n,t_1)\downarrow > h(y,n)$

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Proof of Tot

Let $\phi_{k(x)}(n) = h(i(x), n)$.

Lemma

$$A'(x) \Leftrightarrow k(x) \in Tot$$

Proof.

1. Suppose $A'(x)$.
   - Then $(\exists^\infty t)[x \in W^A_{a,t}]$.
   - Hence $(\exists^\infty t)E(x, i(x), t)$,
   - And $\phi^A_{i(x)} = F(x, i(x), n)$ is total.

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Let $\phi_{k(x)}(n) = h(i(x), n)$.

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Proof.

2. Suppose $\neg A'(x)$.
   - Then $(\forall t)[x \notin W^A_{a,t}]$
   - Hence $(\forall^\infty t)A(x, i(x), t)$.
   - If we assume that $h(i(x), n)$ is total, then $\phi_{i(x)} = F(x, i(x), n)$ would dominate $h(i(x), n)$, contradicting *UEP*.
   - Hence $k(x) \notin Tot$.

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Fin

To prove that $A' <_m Fin$ we change the definition of $F(x, y, n, t)$:

- If $x, y < t$ and $x \notin W_{a,t}$ then $E(x, y, t)$.
- If $x, y < t$ and $x \in W_{a,t}$ then $A(x, y, t)$.

Define:

$$F(x, y, n) = m \Leftrightarrow (\exists t) F(x, y, n, y) \downarrow = m$$

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Proof of Fin

Consider the function $h'(x,n)$

$$h'(x,n) = \begin{cases} h(i(x),n) & \text{if } (\forall m \leq n) h(i(x),m)\downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

- $h(i(x),n)$ is total $\Leftrightarrow$ $h'(x,n)$ is total.
- If $h(i(x),n)$ is not total, then $dom(h'(x,n))$ is finite.

Proof of Fin

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- If $h(i(x), n)$ is not total, then $dom(h'(x, n))$ is finite.

Proof of Fin

Let $\phi_{I(x)}(n) = h'(x, n)$.

Lemma

$$A'(x) \Leftrightarrow I(x) \in Fin$$

Proof.

1. Suppose $A'(x)$.
   - Then $(\forall^\infty t)[x \in W^A_{a,t}]$.
   - Hence $(\forall^\infty t)A(x, i(x), t)$.
   - If we assume that $h(i(x), n)$ is total, then $\phi_{i(x)} = F(x, i(x), n)$ would dominate $h(i(x), n)$, contradicting UEP.
   - Hence $dom(h'(x, n))$ is finite and $I(x) \in Fin$.

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   - If we assume that $h(i(x), n)$ is total, then $\phi_{i(x)} = F(x, i(x), n)$ would dominate $h(i(x), n)$, contradicting *UEP*.
   - Hence $dom(h'(x, n))$ is finite and $I(x) \in Fin$.

□

Proof of Fin

Let $\phi_{l(x)}(n) = h'(x, n)$.

Lemma

$$A'(x) \Leftrightarrow l(x) \in Fin$$

Proof.

1. Suppose $A'(x)$.
   - Then $(\forall^\infty t)[x \in W_{a,t}^A]$.
   - Hence $(\forall^\infty t)A(x, i(x), t)$.
   - If we assume that $h(i(x), n)$ is total, then $\phi_{i(x)} = F(x, i(x), n)$ would dominate $h(i(x), n)$, contradicting UEP.
   - Hence $dom(h'(x, n))$ is finite and $l(x) \in Fin$.

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Proof of Fin

Let $\phi_{I(x)}(n) = h'(x, n)$.

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   - If we assume that $h(i(x), n)$ is total, then $\phi_{i(x)} = F(x, i(x), n)$ would dominate $h(i(x), n)$, contradicting *UEP*.
   - Hence $dom(h'(x, n))$ is finite and $I(x) \in Fin$.

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Proof of Fin

Let $\phi_{l(x)}(n) = h'(x, n)$.

Lemma

$$A'(x) \Leftrightarrow l(x) \in Fin$$

Proof.

1. Suppose $A'(x)$.
    - Then $(\forall^\infty t)[x \in W^A_{a,t}]$.
    - Hence $(\forall^\infty t)A(x, i(x), t)$.
    - If we assume that $h(i(x), n)$ is total, then $\phi_{i(x)} = F(x, i(x), n)$ would dominate $h(i(x), n)$, contradicting *UEP*.
    - Hence $dom(h'(x, n))$ is finite and $l(x) \in Fin$.

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Proof of Fin

Let $\phi_{l(x)}(n) = h'(x, n)$.

Lemma

$$A'(x) \Leftrightarrow l(x) \in Fin$$

Proof.

2. Suppose $\neg A'(x)$.
   - Then $(\forall t)[x \notin W^A_{a,t}]$.
   - Hence $(\forall t)E(x, i(x), t)$ and $\phi^A_{i(x)} = F(x, i(x), n)$ is total.
   - Hence $h(i(x), n) = \phi_{k(x)}(n)$ is total.
   - Then $dom(h'(x, n))$ is not finite, $l(x) \notin Fin$.

□

Proof of Fin

Let $\phi_{l(x)}(n) = h'(x, n)$.

Lemma

$$A'(x) \Leftrightarrow l(x) \in Fin$$

Proof.

2. Suppose $\neg A'(x)$.
   - Then $(\forall t)[x \notin W^A_{a,t}]$.
   - Hence $(\forall t)E(x, i(x), t)$ and $\phi^A_{i(x)} = F(x, i(x), n)$ is total.
   - Hence $h(i(x), n) = \phi_{K(x)}(n)$ is total.
   - Then $dom(h'(x, n))$ is not finite, $l(x) \notin Fin$.

□

Proof of Fin

Let $\phi_{I(x)}(n) = h'(x, n)$.

Lemma

$$A'(x) \Leftrightarrow I(x) \in Fin$$

Proof.

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   - Then $(\forall t)[x \notin W^A_{a,t}]$.
   - Hence $(\forall t)E(x, i(x), t)$ and $\phi^A_{i(x)} = F(x, i(x), n)$ is total.
   - Hence $h(i(x), n) = \phi_{K(x)}(n)$ is total.
   - Then $dom(h'(x, n))$ is not finite, $I(x) \notin Fin$.

□

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Lemma

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□

General Case

Definition

A set A has the property $n - UEP$ if there is a partial computable function $h(e, y_1, \ldots y_{n-1}, n)$, such that:

$$(Q_1 y_{n-1})(Q_2 y_{n-1}) \ldots [\phi_e^A(\langle \overline{y}, n \rangle) \text{ - total}]$$

then

$$(Q_1 y_{n-1})(Q_2 y_{n-1}) \ldots [h(e, \overline{y}, n)\text{- total and escapes } \phi_e^A(\langle \overline{y}, n \rangle)]$$

Where $Q_1, Q_2 \in \{\forall^\infty, \exists^\infty\}$.
For odd n: alternate $\exists^\infty \forall^\infty$.
For even n: alternate $\forall^\infty \exists^\infty$

General (SQNF) Theorem

Theorem

For $n \geq 1$

1. *There exists a computable function g such that for any $A \in \Sigma_{2n+1}$ with index e.*

$$A(x) \Leftrightarrow (\forall^\infty y_{2n-1})(\forall y_{2n-2}) \dots (\forall^\infty y_1)[W_{g(e,x,\overline{y})} = \omega]$$

$$\neg A(x) \Leftrightarrow (\forall y_{2n-1})(\forall^\infty y_{2n-2}) \dots (\forall y_1)[W_{g(e,x,\overline{y})} \text{ is finite}]$$

2. *There exists a computable function g such that for any $A \in \Pi_{2n}$ with index e.*

$$A(x) \Leftrightarrow (\forall y_{2n-2})(\forall^\infty y_{2n-3}) \dots (\forall^\infty y_1)[W_{g(e,x,\overline{y})} = \omega]$$

$$\neg A(x) \Leftrightarrow (\forall^\infty y_{2n-2})(\forall y_{2n-3}) \dots (\forall y_1)[W_{g(e,x,\overline{y})} \text{ is finite}]$$

General (SQNF) Theorem

Corollary

1. *There exists a computable function f such that for any $A \in \Sigma_{2n+1}$ with index e.*

$$A(x) \Leftrightarrow (\forall^{\infty} y_{2n-1})(\forall y_{2n-2}) \dots (\forall^{\infty} y_1)(\forall z)[\langle \overline{y}, z \rangle \in W_{f(e,x)}]$$

$$\neg A(x) \Leftrightarrow (\forall y_{2n-1})(\forall^{\infty} y_{2n-2}) \dots (\forall y_1)(\forall^{\infty} z)[\langle \overline{y}, z \rangle \notin W_{f(e,x)}]$$

2. *There exists a computable function f such that for any $A \in \Pi_{2n}$ with index e.*

$$A(x) \Leftrightarrow (\forall y_{2n-2})(\forall^{\infty} y_{2n-3}) \dots (\forall^{\infty} y_1)(\forall z)[\langle \overline{y}, z \rangle \in W_{f(e,x)}]$$

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low_n implies $n - Uep$

Case A is low_{2n-1}.

- Define $V^A(e, \bar{(}y)) \Leftrightarrow (\exists^\infty x)(\exists s)[\phi^A_{e,s}(\langle \bar{y}, x \rangle) \downarrow < s \wedge x \notin W_{g(u,e,\bar{y}),s}]$
- Here u is a Π_{2n} index of the relation $U^A(e) \leftrightarrow (\exists^\infty)(\forall^\infty) \dots (\forall^\infty) V^A(e, \bar{y})$.
- Prove: $(\exists^\infty)(\forall^\infty) \dots (\forall^\infty)[\phi^A_e(\langle \bar{y}, x, \rangle)$ - is total] $\Rightarrow U^A(e)$.

low_n implies $n - Uep$

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$n - UEP$ implies low_n

- If n is odd, we produce computable functions k and l

$$x \in A^{(n)} \Leftrightarrow \exists^\infty \forall^\infty \dots \phi_{k(x,\overline{y})} \text{ is total.}$$

$$x \in A^{(n)} \Leftrightarrow \forall^\infty \exists^\infty \dots dom(\phi_{l(x,\overline{y})}) \text{ is finite.}$$

- If n is odd, we produce computable functions k and l

$$x \in \overline{A^{(n)}} \Leftrightarrow \forall^\infty \exists^\infty \dots \phi_{k(x,\overline{y})} \text{ is total.}$$

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