The Local Structure of the Enumeration Degrees

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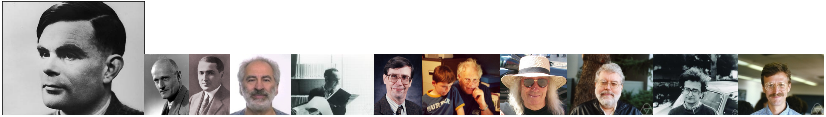
Outline

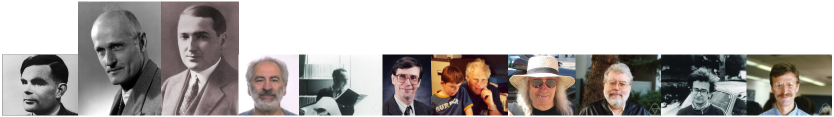
- Historical background (Ambos Spies, Fejer: *Degrees of unsolvability*).
- Main definition: Enumeration reducibility, Enumeration degrees, etc.
- Properties of the local structure.
- The priority method using a tree of strategies.



The beginning

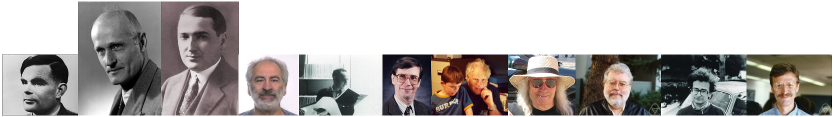
A.Turing, 1936: On computable numbers, with an application to the Entscheidungsproblem, *Proc. Lond. Math. Soc.*, II. Ser., **42**, 230—265.

- Solves the *Entscheidungs* problem.
- Introduces relativized computation with oracle Turing machines.



Degree Theory

S. C. Kleene, 1936: General recursive functions of natural numbers.
S. C. Kleene, 1943: Recursive predicates and quantifiers.
E. L. Post, 1944: Recursively enumerable sets of positive integers and their decision problems.
E. L. Post, 1948: Degrees of recursive unsolvability: preliminary report.
S. C. Kleene, 1952: *Introduction to Metamathematics*.
S. C. Kleene and E. L. Post, 1954: The upper semi-lattice of degrees of recursive unsolvability.



The computably enumerable degrees

- Natural problems from other parts of mathematics.
- Post's theorem: The degrees below $0'$ are exactly the Δ^0_2 Turing degrees.
- Post's Problem: Are there intermediate c.e. Turing degrees?
- Friedberg and Mučnik 1956-7: The priority method, the hallmark of the field.

Structural properties

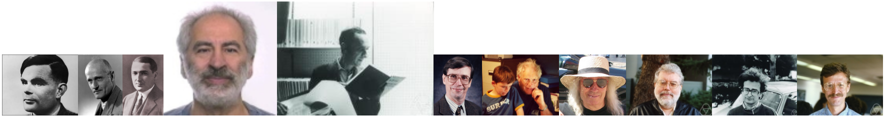
Let $\langle \mathcal{A}, \mathbf{0}, \mathbf{1}, <, \vee \rangle$ be an upper semi-lattice.

Definition

If $\mathbf{a} \vee \mathbf{b} = \mathbf{c}$ and $\mathbf{a}, \mathbf{b} < \mathbf{c}$ then we shall say that **a** cups **b** to **c**.
We shall also say that the pair $(\mathbf{a}, \mathbf{b})$ is a splitting of **c**.
In the special case when $\mathbf{c} = \mathbf{1}$, we shall simply say that **a** cups **b**.

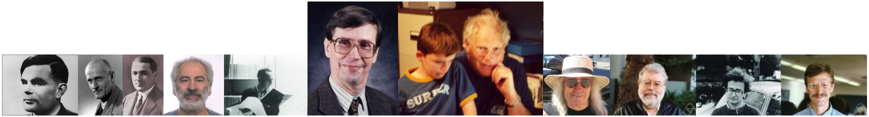
Definition

If $\mathbf{a} \wedge \mathbf{b} = \mathbf{c}$ and $\mathbf{a}, \mathbf{b} > \mathbf{c}$ then we shall say that **a** caps **b** to **c**.
We shall also say that **a** and **b** form a minimal pair above **c**.
In the special case when $\mathbf{c} = \mathbf{0}$, we shall simply say that **a** caps **b** and that $(\mathbf{a}, \mathbf{b})$ is a minimal pair.



Further advancements

- Infinite injury priority method.
- Sack's Density (1963) and Splitting (1964) theorems.
- Shoenfield's conjecture 1965: The c.e. Turing degrees are a decidable dense homogeneous partial order, reminiscent of the rational numbers.
- Lachlan and Yates 1966: There are minimal pairs of c.e. Turing degrees.



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Complications

- $0'''$ priority method.
- Lachlan's Non-diamond theorem (1966).
- Cooper and Yates' Non-cuppable theorem (1973).
- Lachlan's Non-bounding theorem (1979).

The monster paper

- Density and splitting cannot be combined, Lachlan's Non-splitting theorem(1975):
  There is a pair of c.e. Turing degrees $\mathbf{a} < \mathbf{b}$ such that $\mathbf{b}$ cannot be split in the c.e. Turing degrees above $\mathbf{a}$.
- First use of a tree of strategies.

The monster paper: consequences

- Harrington's non-splitting theorem (1980): There is a c. e. degree $\mathbf{a} < \mathbf{0}'$ such that no pair of c.e. degrees $\mathbf{b}, \mathbf{c} \geq \mathbf{a}$ split $\mathbf{0}'$.
- Harrington and Shelah (1982): The theory of the c.e. Turing degrees is not decidable.
- Harrington and Slaman: The theory of first order arithmetic can be interpreted in the theory of the c.e. Turing degrees.

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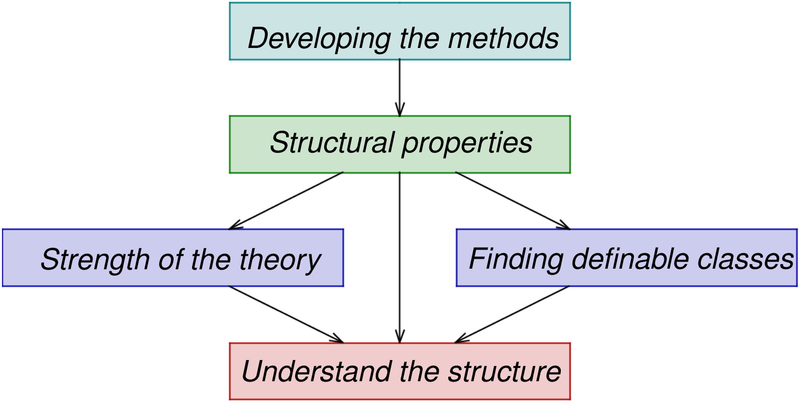
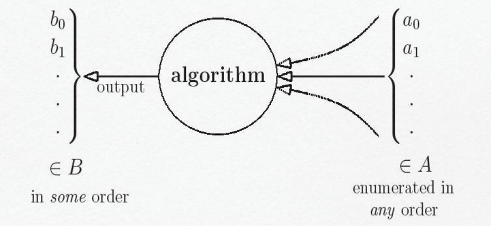
Underlying idea



Alternative approaches to formalizing information content

- Algorithmic randomness.
  - Low for random degrees: zooming into the Turing structure
- Strong reducibilities: restricted computation.
  - Many-one reducibility, Truth table reducibility.
  - Computational complexity: $P = ?NP$.

Enumeration reducibility



Definition (Friedberg, Rogers 1959)

A set B is *enumeration reducible* ($\leq_e$) to a set A if there is a c.e. set Φ (e-operator) such that:

$$n \in B \Leftrightarrow \exists u(\langle n, u \rangle \in \Phi \wedge D_u \subseteq A),$$

where D_u denotes the finite set with code u under the standard coding of finite sets.

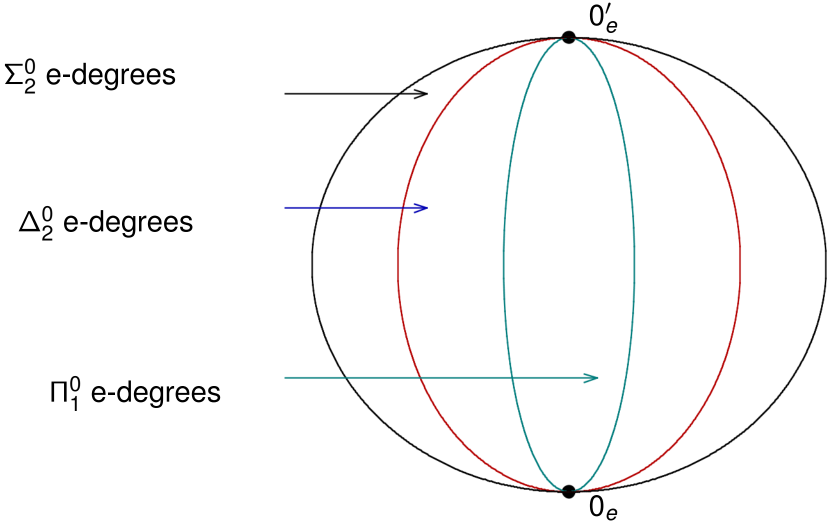
Enumeration Degrees

"... enumeration reducibility is *the* fundamental, general concept of relative computability in as much as the nature of the computable universe is intimately bound up with the set of enumeration operators."

Cooper, 1990

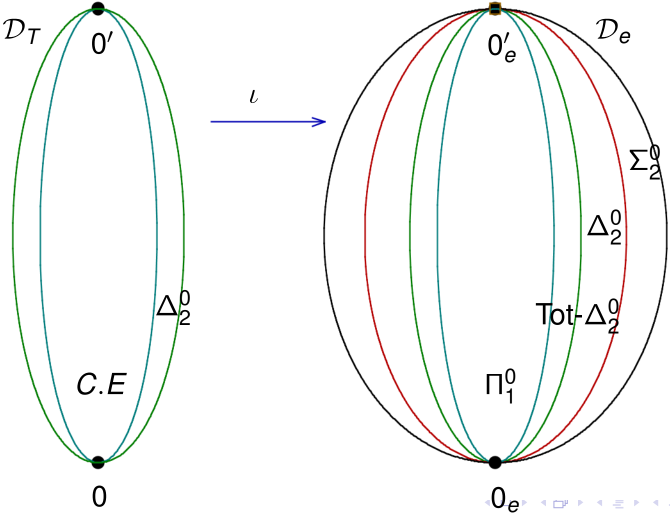
- Enumeration equivalence: $A \equiv_e B \Leftrightarrow A \leq_e B \wedge B \leq_e A$.
- Enumeration degree: $d_e(A) = \{\ B |\ A \equiv_e B\}$.
- Least upper bound: $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- Jump operator: $d_e(A)' = d_e(\overline{K_A} \oplus A)$.
- Upper semi-lattice with jump: $\langle \mathcal{D}_e, 0_e, \leq, \cup, ' \rangle$.

The local structure of the enumeration degrees $\mathcal{D}_e(\leq 0'_e)$



Transferring results from the Turing degrees

Natural embedding: $\iota(d_t(A)) = d_e(A \oplus \overline{A})$.



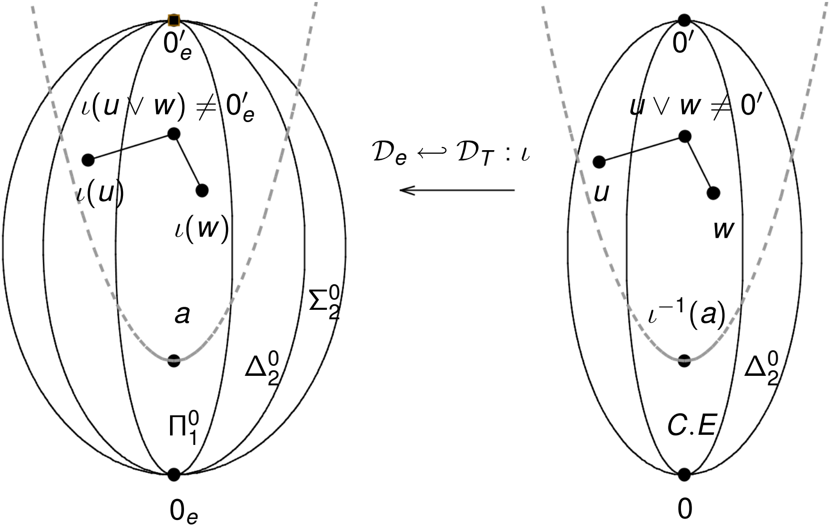
Theorem (S, Cooper)

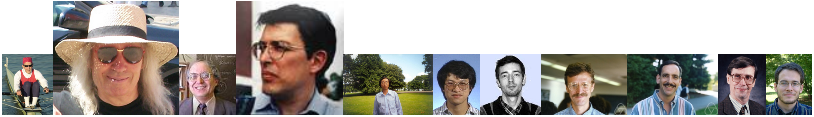
There exists a Π^0_1 enumeration degree $\mathbf{a} < \mathbf{0}'_e$ such that there exists no nontrivial splitting of $\mathbf{0}'_e$ by a pair of a Π^0_1 enumeration degree and a Σ^0_2 enumeration degree both above $\mathbf{a}$.

Corollary (Extending Harrington's Non-splitting Theorem)

There exists a computably enumerable degree $\mathbf{a} < \mathbf{0}'$ such that there is no nontrivial splitting of $\mathbf{0}'$ by a pair of a c.e. degree and a Δ^0_2 degree both above $\mathbf{a}$.

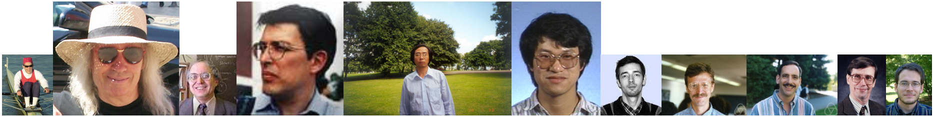
Putting words into actions





Zooming in: The Δ^0_2 enumeration degrees

- Cooper, Sorbi, Yi (1996): Every nonzero Δ^0_2 enumeration degree is cuppable.
- Cooper, Sorbi, Li, Yang (2006): Every nonzero Δ^0_2 enumeration degree bounds a minimal pair.
- Arslanov, Sorbi (1999): There is a Δ^0_2 splitting of $0'_e$ above each incomplete Δ^0_2 enumeration degree.
- Arslanov, Sorbi, Kalimullin (2001): The Δ^0_2 enumeration degrees are dense.



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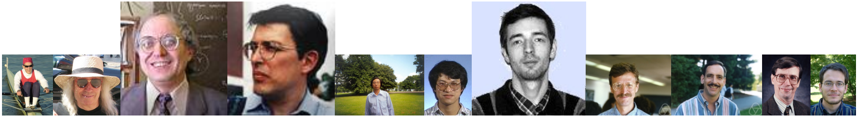

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Zooming out: The Σ^0_2 enumeration degrees

- Cooper (1984): The Σ^0_2 enumeration degrees are dense.
- Cooper, Sorbi, Yi (1996): There is a non-cuppable nonzero Σ^0_2 enumeration degree.
- Cooper, Sorbi, Li, Yang (2006): There is a nonzero Σ^0_2 enumeration degree that does not bound a minimal pair.
- An ideal of properly Σ^0_2 enumeration degrees and 0_e:
  $\mathcal{I} = \{\, a |\ a > 0_e \Rightarrow (\forall x, y \leq a)$
  $[0_e < x \wedge 0_e < y \Rightarrow (\exists d)[d \leq x \wedge d \leq y \wedge d \neq 0_e]]\}$.

Definition

A set A is 1-generic if for every c.e. set W there exists a finite string $\lambda \subset \chi_A$ such that:

$$\lambda \in W \vee (\forall \mu \supseteq \lambda)(\mu \notin W).$$

Degrees of 1-generic sets are called 1-generic degrees.

Theorem (S)

There exists a 1-generic Σ^0_2 enumeration degree **a** *that does not bound a minimal pair in the semi-lattice of the enumeration degrees.*

Completing the picture

Theorem (S)

There is a Σ^0_2 enumeration degree $\mathbf{a} < \mathbf{0}'_\mathbf{e}$ such that $\mathbf{0}'_\mathbf{e}$ cannot be split in the enumeration degrees above the degree $\mathbf{a}$.

A filter of properly Σ^0_2 enumeration degrees and $0'_e$:
$\mathcal{F} = \{\ a |\ a < 0'_e \Rightarrow (\forall u, v)$
$[a \leq u < 0'_e \wedge a \leq v < 0'_e \Rightarrow u \vee v \neq 0'_e]\}$.

Things are not so simple

- Slaman, Woodin (1997): The theory of the Σ^0_2 enumeration degrees is undecidable.
- Cooper's conjecture: The structures of the Σ^0_2 e-degrees and the c.e. Turing degrees are elementary equivalent.
- Ahmad: The diamond can be embedded in the Σ^0_2 enumeration degrees.
- Ahmad and Lachlan: Non-splittable degrees exist.
- Kent 2005: The theory of the Δ^0_2 enumeration degrees is undecidable.

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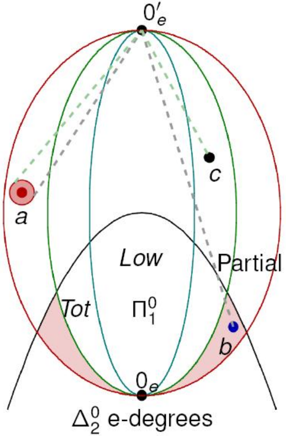
 Cupping properties of the Δ^0_2 enumeration degrees

Theorem (S, Wu)

Every nonzero Δ^0_2 enumeration degree can be cupped by a partial low Δ^0_2 enumeration degree.

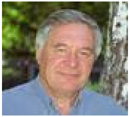


Reaching the first limit

Theorem (S)

Let $\{\mathbf{a}_i\}_{i<\omega}$ be a Δ^0_2-computably enumerable sequence of enumeration degrees. There exists a nonzero Δ^0_2 enumeration degree $\mathbf{b}$ such that for every $i < \omega$ if $\mathbf{a}_i$ is incomplete then $\mathbf{a}_i \vee \mathbf{b} \neq 0'_e$.

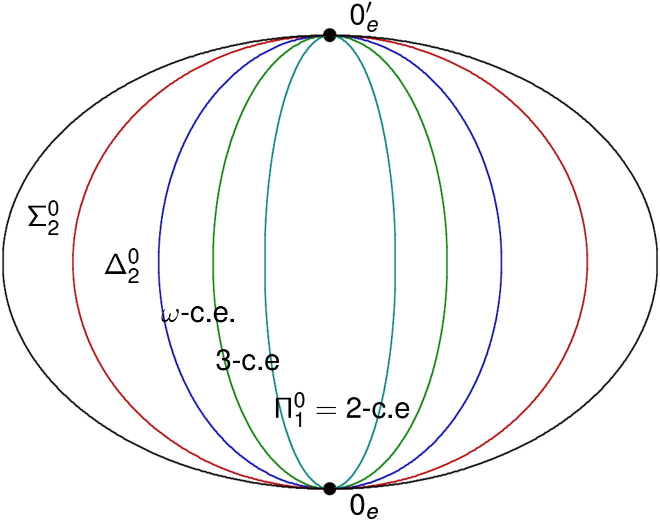
Here a class $\{\mathbf{a}_i\}_{i<\omega}$ of Δ^0_2 enumeration degrees is Δ^0_2-computably enumerable if there is a computable sequence of Δ^0_2 approximations $\{A_i[s]\}_{i,s<\omega}$ to representatives A_i of every degree $\mathbf{a}_i$ in the class.



Definition (Ershov)

1. A set A is n-c.e. if there is a computable function f such that for each x, $f(x,0)=0$, $|\{s+1 \mid f(x,s) \neq f(x,s+1)\}| \leq n$ and $A(x) = \lim_s f(x,s)$.
2. A is ω-c.e. if there are two computable functions $f(x,s), g(x)$ such that for all x, $f(x,0)=0$, $|\{s+1 \mid f(x,s) \neq f(x,s+1)\}| \leq g(x)$ and $\lim_s f(x,s) = A(x)$.
3. A degree **a** is n-c.e.(ω-c.e.) if it contains a n-c.e.(ω-c.e.) set.

The Difference Hierracy



Consequences

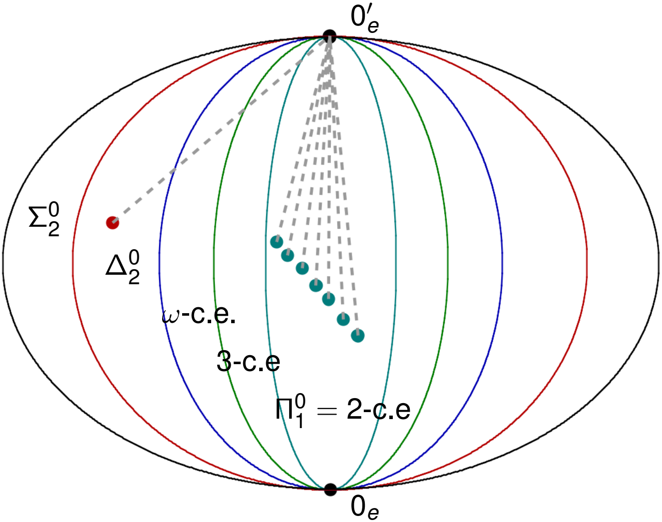
Corollary

There exists a nonzero Δ^0_2 enumeration degree that cannot be cupped by any incomplete ω-c.e. degree.

Theorem (S, Wu)

For every nonzero ω-c.e. enumeration degree $\mathbf{a}$ *there exists an incomplete 3-c.e. enumeration degree* $\mathbf{b}$ *that cups* $\mathbf{a}$.

Cupping classes of enumeration degrees



(Cooper, Seetapun and Li): There exists a single incomplete Δ^0_2 Turing degree that cups every nonzero c.e. Turing degree.

The second limitation

For any larger subclass, which contains the nonzero 3-c.e enumeration degrees this cannot be done as:

Theorem (S)

Let $\mathbf{a}$ *be an incomplete* Σ^0_2 *enumeration degree. There exists a nonzero* 3*-c.e. enumeration degree* $\mathbf{b}$ *such that* $\mathbf{a} \vee \mathbf{b} \neq \mathbf{0}'_e$.

The n-c.e. enumeration degrees are far from simple

An analog of Lachlan's non-splitting theorem for every class of n-c.e. enumeration degrees, where $n \geq 3$.

Theorem (Arslanov, Cooper, Kalimullin, S)

There exists a pair of a Π^0_1 enumeration degree $\mathbf{a}$ and a 3-c.e. enumeration degree $\mathbf{b} < \mathbf{a}$ such that $\mathbf{a}$ cannot be split by a pair of enumeration degrees above $\mathbf{b}$.

Questions

1. What is the exact theoretical complexity of any of the classes we considered?
2. Can we define a smaller class within a larger class?
3. What is the precise role of genericity in the enumeration degrees?
4. What is the mathematical reason for the similarities and differences between the local structures $\mathcal{D}_T(\leq 0')$ and $\mathcal{D}_e(\leq 0'_e)$?

The priority method

- Friedberg and Mučnik 1956-7. Solution to Post's problem.
- The main method used in $\mathcal{D}_T(\leq 0')$ and $\mathcal{D}_e(\leq 0'_e)$.
- Construction of representatives of degrees in the local structure.

Theorem

There exist incomparable Π^0_1 enumeration degrees.

Step 1: Formalizing the requirements

We shall construct two Π^0_1 sets A and B so that ultimately $d_e(A)$ and $d_e(B)$ are incomparable.
The sets A and B should be incomparable: $A \nleq_e B$ and $B \nleq_e A$.
Let $\{\Phi_e\}_{e<\omega}$ be a computable enumeration of all c.e. sets:

1. $\mathcal{P}_e : A \neq \Phi_e^B$;
2. $\mathcal{Q}_e : B \neq \Phi_e^A$,

Approximations: The computable content of the constructions

Definition

A Π^0_1 approximation to a set A is a computable sequence of cofinite sets $\{A[s]\}_{s<\omega}$, such that:

- $A[0] = \mathbb{N}$.
- $n \notin A[s] \Rightarrow (\forall t \geq s)[n \notin A[t]]$.
- $n \in A \Leftrightarrow (\forall t)[n \in A[t]]$.

The construction runs in stages:

- At every stage s we only have finite (computable) information to the given sets: $\Phi_e[s]$.
- We construct $A[s]$ and $B[s]$ based on the finite amount of information given.

Step 2: Designing the basic modules

To every requirement we associate a finite set of instructions:

- Similar requirements have similar basic modules.
- Actions:
  - Modify own parameters;
  - Modify the approximations to the constructed sets;
  - Impose restrictions.
- If executed infinitely many times, guarantee satisfaction of the corresponding requirement.

Basic module for $\mathcal{P}_e$

$$\mathcal{P}_e : A \neq \Phi_e^B.$$

1. If the witness x_e is not selected, then let x_e be a fresh number, one that has not appeared in the construction so far.
2. If $x_e \notin \Phi_e^B[s]$ then do nothing.
3. If $x_e \in \Phi_e^B[s]$ then there is an axiom $\langle x_e, D \rangle \in \Phi_e[s]$ with $D \subseteq B[s]$. Extract x_e from $A[s]$ and restrain D in B.

Step 3: Identifying the outcomes

- More than one possible method for satisfying a requirement.
- Strategies choose their method with respect to the current situation and the methods chosen by other strategies.
- The choice of a particular method corresponds to an outcome.
- Two outcomes for $\mathcal{P}_e$:
  - Wait forever for $x_e \in \Phi_e^B$: Outcome w.
  - At some stage x enters Φ_e^B: Outcome f.

Conflicts

$\mathcal{P}_e$-strategy

1. If the witness x_e is not selected, then let x_e be a fresh number, one that has not appeared in the construction so far.
2. If $x_e \notin \Phi_e^B[s]$ then do nothing.
3. If $x_e \in \Phi_e^B[s]$ via axiom $\langle x_e, D \rangle$ then extract x_e from $A[s]$ and restrain D in B.

$\mathcal{Q}_j$-strategy

1. If the witness x_j is not selected, then let x_j be a fresh number, one that has not appeared in the construction so far.
2. If $x_j \notin \Phi_j^A[s]$ then do nothing.
3. If $x_j \in \Phi_j^A[s]$ via axiom $\langle x_j, F \rangle$ then extract x_j from $B[s]$ and restrain F in A.

Resolving the conflicts: Priority ordering

- We order the set of requirements $\mathcal{R}$ linearly:
  $$\mathcal{P}_0 < \mathcal{Q}_0 < \mathcal{P}_1 < \mathcal{Q}_1 < \mathcal{P}_2 \ldots$$
- Requirements in earlier positions have higher priority.
- Lower priority requirements respect the restrictions imposed by higher priority requirements.
- They assume that the method chose by higher priority strategies is final.
- If they are wrong - we say that they are injured. An injured requirement is initialized and starts work from the beginning under the changed assumptions.

Step 4: The tree of strategies

- Injury appears when a strategy decides to change its method (outcome).
- We order the set of outcomes $O = \{w, f\}$ linearly:

$$f <_L w$$

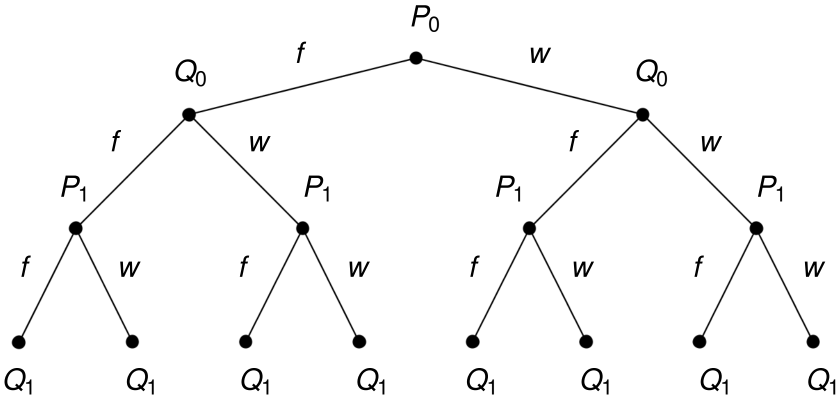
The set $O^{\leq\omega}$ has an induced lexicographical order $<$.

Definition

The tree of strategies is a computable function T with domain $D(T)$ a downwards closed subset of $O^{<\omega}$ and range $R(T) = \mathcal{R}$, such that:

- For every *path* $f \subseteq D(T)$ we have $R(T \restriction f) = \mathcal{R}$.
- Higher priority requirements are assigned to nodes at higher levels of the tree.

Step 4: The tree of strategies



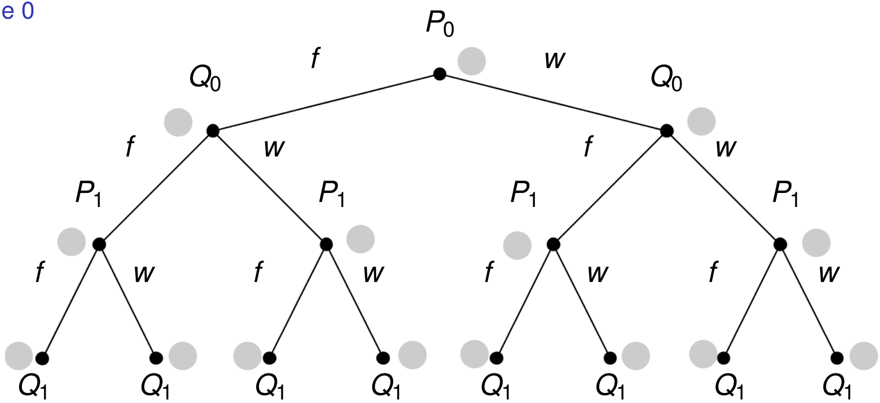
Each node on the tree has its own instance of a strategy associated with it.

The construction

- At stage 0 all nodes are initialized.
- At each stage $s > 0$ we construct a finite path $\delta[s]$ of length s through the domain of T starting at the root of the tree.
- Nodes $\alpha \subseteq \delta[s]$ are activated at stage s.
  - They run their instance of the basic module.
  - Select an outcome o.
  - Initialize lower priority requirements which have not predicted the outcome correctly.
- The next node visited at stage s will be $\alpha\hat{\ }o$.

Visualizing the construction

Stage 0



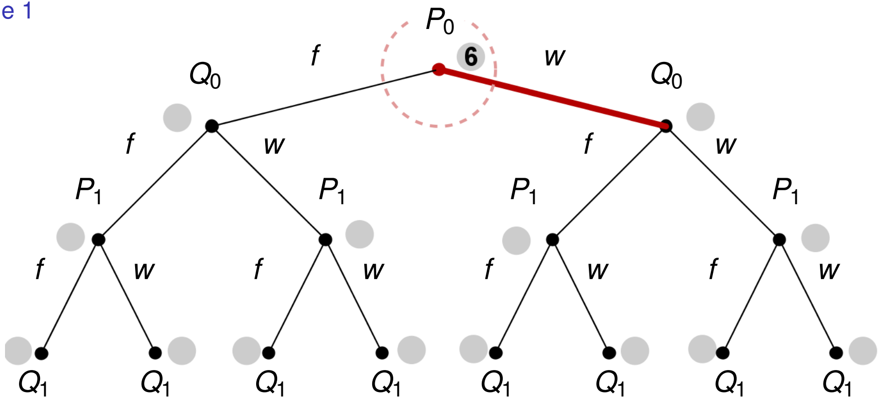
$A = \mathbb{N}$
$\Phi_0[0] = \emptyset$

$B = \mathbb{N}$

All nodes are initialized.

Visualizing the construction

Stage 1



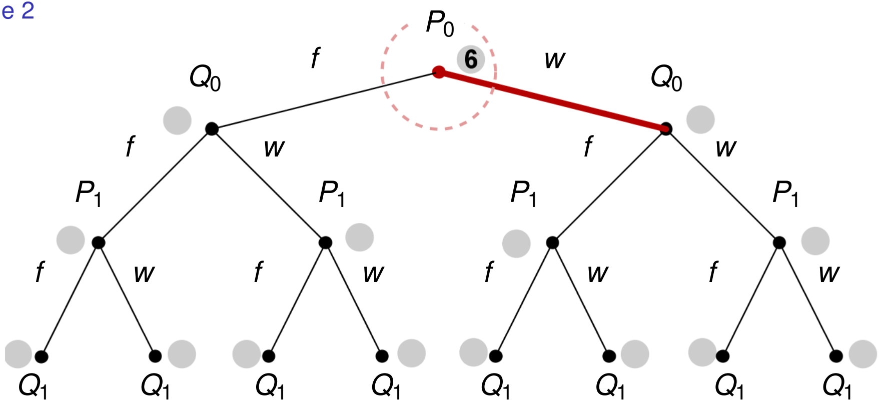
$A = \mathbb{N}$ $\qquad B = \mathbb{N}$

$\Phi_0[1] = \{\langle 2, \{3, 5\}\rangle\}$

Visit node $\emptyset$. Select a witness for node $\emptyset$: $x_\emptyset = 6$.
Check if $x_\emptyset \in \Phi_0^B[1]$. The answer is no, outcome is w. End stage 1.

Visualizing the construction

Stage 2



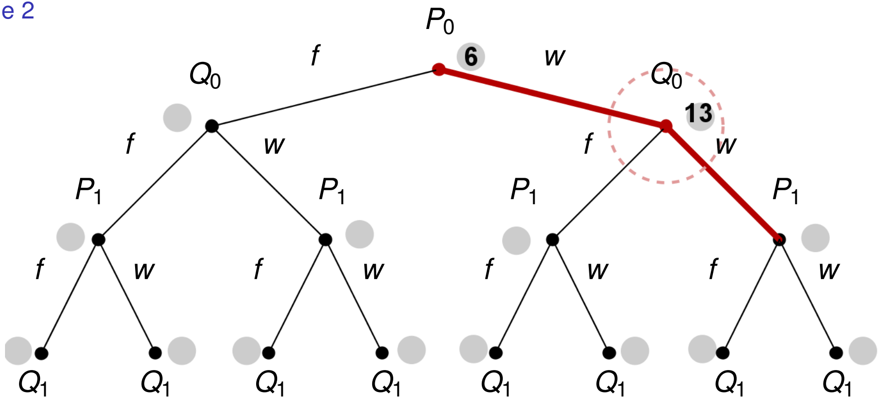
$A = \mathbb{N}$ $\qquad B = \mathbb{N}$

$\Phi_0[2] = \{\langle 2, \{3, 5\}\rangle, \langle 11, \{1, 12\}\rangle\}$

Visit node $\emptyset$. Check if $x_\emptyset \in \Phi_0^B[2]$. The answer is no, outcome is w.

Visualizing the construction

Stage 2



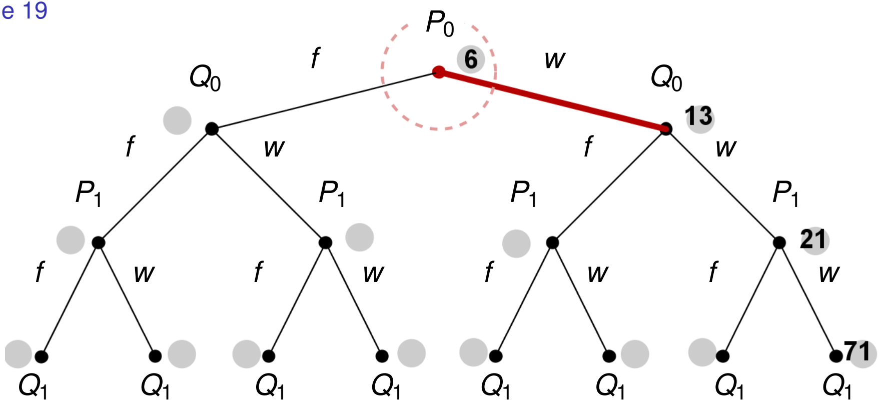
$A = \mathbb{N}$ $\quad$ $B = \mathbb{N}$

$\Phi_0[2] = \{\langle 2, \{3, 5\}\rangle, \langle 11, \{1, 12\}\rangle\}$

Visit node w. Select a witness x_w for the node w.
Check if $x_w \in \Phi_0^A[2]$. The answer is no, outcome is w. End stage 2.

Visualizing the construction

Stage 19



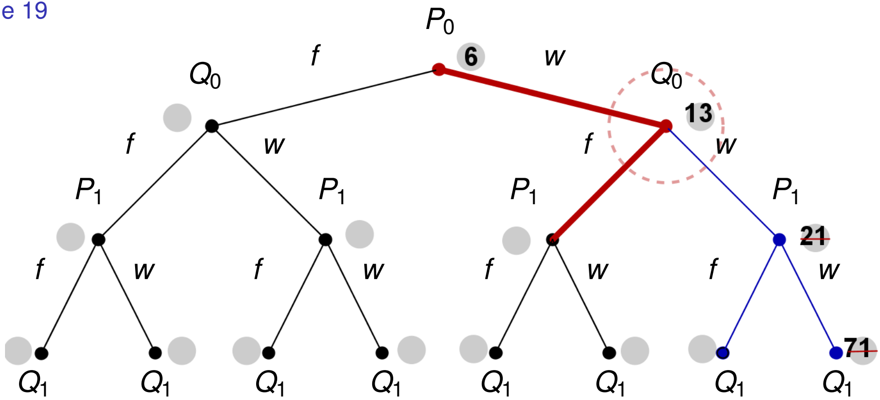
$A = \mathbb{N}$ $\qquad B = \mathbb{N}$

$\Phi_0[19] = \{\langle 2, \{3, 5\}\rangle, \langle 11, \{1, 17\}\rangle, \langle 13, \{2, 21, 88\}\rangle \ldots\}$

Visit node $\emptyset$. Visit node $\emptyset$. Check if $x_\emptyset \in \Phi_0^B[19]$. The answer is no, outcome is w.

Visualizing the construction

Stage 19



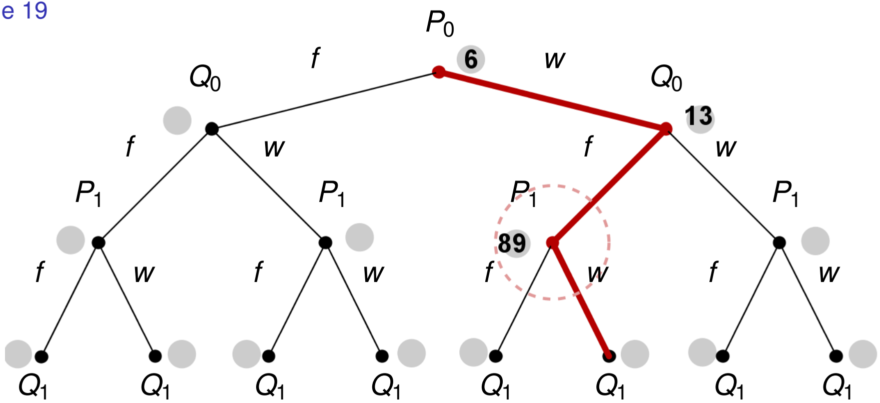
$A = \mathbb{N}$ $\qquad\qquad$ $B = \mathbb{N} \setminus \{13\}$

$\Phi_0[19] = \{\langle 2, \{3, 5\}\rangle, \langle 11, \{1, 17\}\rangle, \langle 13, \{2, 21, 88\}\rangle \ldots\}$

Visit node w. Check if $x_w \in \Phi_0^A[19]$. The answer is Yes. Extract x_w from B, restrain $\{2, 21, 88\}$ in A. The outcome is f. Initialize all nodes to the right of f.

Visualizing the construction

Stage 19



$A = \mathbb{N}$ $\qquad B = \mathbb{N} \setminus \{13\}$

$\Phi_0[19] = \{\langle 2, \{3,5\}\rangle, \langle 11, \{1,17\}\rangle, \langle 13, \{2,21,88\}\rangle \dots\}$

$\Phi_1[19] = \{\langle 1, \{2,4\}\rangle, \langle 10, \{5,16\}\rangle, \dots\}$

Visit node wf. Select a witness x_{wf}. Check if $x_{wf} \in \Phi_1^B[19]$. The answer is no, outcome is w.

The key point: The true path

Lemma (True path lemma)

There exists an infinite path h in the tree of strategies, called the true path, with the following properties:

1. $(\forall n)(\exists^\infty s)[\ h \upharpoonright n \subseteq \delta[s]\ ]$;
2. $(\forall n)(\exists s_i(n))(\forall s > s_i(n))[\ h \upharpoonright n$ *is not initialized at stage* $s\ ]$.

In our case: the leftmost path of nodes visited infinitely often is the true path.
Every node along the true path satisfies its requirement.

Thank you!