The Strongest Nonsplitting Theorem

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1 Introduction

Sacks [14] showed that every computably enumerable (c.e.) degree $\geq \mathbf{0}$ has a c.e. splitting. Hence, relativising, every c.e. degree has a Δ_2 splitting above each proper predecessor (by 'splitting' we understand 'nontrivial splitting'). Arslanov [1] showed that $\mathbf{0}'$ has a d.c.e. splitting above each c.e. $\mathbf{a} < \mathbf{0}'$. On the other hand, Lachlan [9] proved the existence of a c.e. $\mathbf{a} > \mathbf{0}$ which has no c.e. splitting above some proper c.e. predecessor, and Harrington [8] showed that one could take $\mathbf{a} = \mathbf{0}'$. Splitting and nonsplitting techniques have had a number of consequences for definability and elementary equivalence in the degrees below $\mathbf{0}'$.

Heterogeneous splittings are best considered in the context of cupping and noncupping. Posner and Robinson [13] showed that every nonzero Δ_2 degree can be nontrivially cupped to $\mathbf{0}'$, and Arslanov [1] showed that every c.e. degree $> \mathbf{0}$ can be d.c.e. cupped to $\mathbf{0}'$ (and hence since every d.c.e., or even n-c.e., degree has a nonzero c.e. predecessor, every n-c.e. degree $> \mathbf{0}$ is d.c.e. cuppable.) Cooper [2] and Yates (see Miller [11]) showed the existence of degrees noncuppable in the c.e. degrees. Moreover, the search for relative cupping results was drastically limited by Cooper [3], and Slaman and Steel [15] (see also Downey [7]), who showed that there is a nonzero c.e. degree $\mathbf{a}$ below which even Δ_2 cupping of c.e. degrees fails.

We prove below what appears to be the strongest possible of such nonsplitting and noncupping results.

Theorem 1. *There exists a computably enumerable degree* $\mathbf{a} < \mathbf{0}'$ *such that there exists no nontrivial cuppings of c.e. degrees in the* Δ_2 *degrees above* $\mathbf{a}$.

In fact, if we consider the extended structure of the enumeration degrees, Theorem 1 is a corollary of the even stronger result:

Theorem 2. *There exists a* Π_1 *e-degree* $\mathbf{a} < \mathbf{0}'_e$ *such that there exist no nontrivial cuppings of* Π_1 *e-degrees in the* Δ_2 *e-degrees above* $\mathbf{a}$.

This would appear to be the first example of a structural feature of the Turing degrees obtained via a proof in the wider context of the enumeration degrees (rather than the other way round).

Notation and terminology below is based on that of [5].

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* The first author was partially supported by the Marie Curie Early Training Site MATHOGAPS (MEST-CT-2004-504029).

** The second author was supported by EPSRC grant No. GR /S28730/01, and by the NSFC Grand International Joint Project, No. 60310213, *New Directions in the Theory and Applications of Models of Computation.*

2 Requirements and Strategies

We assume a standard listing of all quadruples (Ψ, Θ, U, W) of enumeration operators Ψ and Θ, Σ_2 sets U and c.e. sets W. We will construct Π_1 sets A and E to satisfy the corresponding list of requirements:

$$\begin{aligned} N_\Psi: &\quad E \neq \Psi^A, \\ P_{\Theta,U,W}: &\quad E = \Theta^{U,\overline{W}} \wedge U \in \Delta_2 \Rightarrow (\exists \Gamma, \Lambda)[\overline{K} = \Gamma^{U,A} \vee \overline{K} = \Lambda^{\overline{W},A}], \end{aligned}$$

where $\Gamma^{U,A}$, for example, denotes an e-operator enumerating relative to the data enumerated from two sources U and A. We describe the basic strategy to satisfy these requirements, only using $U \in \Delta_2$ for satisfying P in the case of a successful Γ-strategy.

2.1 The naive N_Ψ-Strategy

Select a witness x for N_Ψ and wait for $x \in \Psi^A$. Then extract x from E while restraining each $y \in A \restriction use(\Psi, A, x)$ (the use function $use(\Psi, A, x)$ is defined in the usual way by $use(\Psi, A, x) = \mu y[x \in \Psi^{A \restriction y}]$).

2.2 The naive P_Θ-Strategy

Definition 1. *Let Φ be an enumeration operator and A a set. We will consider a generalised use function φ defined as follows:*

$$\varphi(x) = \max \left\{ use(\Phi, A, y) | (y \leq x) \wedge (y \in \Phi^A) \right\}$$

We progressively try to rectify Γ_Θ at each stage by ensuring that $z \in \overline{K} \Leftrightarrow z \in \Gamma^{U,A}$ for each z below $l(E, \Theta^{U,\overline{W}})$. The definition of the enumeration operator Γ involves axioms with two types of markers $u(y) \in U$ and $\gamma(y) \in A$ - the generalised use functions for the operator Γ. Given a suitable choice of a marker $\gamma(y) \in A$ when $y \in \overline{K}$, Γ can be rectified via A-extraction.

2.3 N_Ψ below P_Θ

In combining these two strategies the A-restraint for N_Ψ following the extraction of x from E conflicts with the need to rectify Γ_Θ. We try to resolve this by choosing a threshold d for N_Ψ, and try to achieve $\gamma(z') > use(\Psi, A, x)$ for all $z' \geqslant d$ at a stage previous to the imposition of the restraint. We try to maintain $\theta(x) < u(d)$, in the hope that after we extract x from E, each return of $l(E, \Theta^{U,\overline{W}})$ will produce an extraction from $U \restriction \theta(x)$ which can be used to avoid an A-extraction in moving $\gamma(d)$.

In the event that some such attempt to satisfy N_Ψ ends with a $\overline{W} \restriction \theta(x)$-change, then we must implement the $\Lambda_{\Theta,\Psi}$-strategy which is designed to allow lower priority N-requirements to work below the Γ_Θ-activity, using the $\overline{W} \restriction \theta(x)$-changes thrown up by Γ_Θ to move $\Lambda_{\Theta,\Psi}$-markers. Each time we progress the $\Lambda_{\Theta,\Psi}$-strategy, we cancel the current witness of (N_Ψ, Γ), and if this happens infinitely often, N_Ψ might not be satisfied. This means that N_Ψ must be accompanied by an immediately succeeding copy (N_Ψ, Λ), say, designed to take advantage of the improved strategy for N_Ψ without any other $P_{\Theta'}$ intervening between (N_Ψ, Γ) and (N_Ψ, Λ).

2.4 The Approximations

During the construction we approximate the given sets at each stage. We need to choose these approximating sequences very carefully. Consider a $P_{\Theta,U,W}$ requirement.

Definition 2. *We inductively say that a stage $s+1$ is P_Θ-expansionary if and only if $l(E[s+1], \Theta^{U,\overline{W}}[s+1])$ attains a greater value at stage $s+1$ than at any previous P_Θ-expansionary stage.*

If the length of agreement is bounded, $P_{\Theta,U,W}$ is trivially satisfied and we do not have to act on its behalf. The strategies act only on $P_{\Theta,U,W}$- expansionary stages. It is essential that if $P_{\Theta,U,W}$ turns out to be equal to E we have infinitely many expansionary stages. We will work with a good approximating sequence to $U \oplus \overline{W}$ (basically one with sufficient thin stages, in the sense of Cooper [4]) as defined in [10]:

Consider a Δ_2 approximating sequence $\{U'_s\}$ to U and the standard approximating sequence $\{W'_s\}$ to the c.e. set W. We define $\left\{\overline{W}^*_s\right\}$ to be the Δ_2 approximating sequence to $\overline{W}$ constructed in the following way: $\overline{W}^*_s = \overline{W'_s} \restriction s$. Joining the two Δ_2 approximating sequences, we get $\left\{U'_s \oplus \overline{W}^*_s\right\}$ – that is, a Δ_2 approximating sequence to $U \oplus \overline{W}$. It follows from [10] that we can construct a good approximating sequence to $U \oplus \overline{W}$ in the following way: $(U \oplus \overline{W})_s = U'_s \oplus \overline{W}^*_s \restriction (\mu n[U'_s \oplus \overline{W}^*_s(n) \neq U'_{s+1} \oplus \overline{W}^*_{s+1}(n)])$. The resulting good approximating sequence has the following properties:

1. $\forall n \exists s (U \oplus \overline{W} \restriction n \subseteq (U \oplus \overline{W})_s \subseteq U \oplus \overline{W})$. Such stages are called good stages and hence there are infinitely many of them.
2. $\forall n \exists s_0 \forall s > s_0 (U \oplus \overline{W} \restriction n = (U \oplus \overline{W})_s \restriction n)$
3. If G is the set of all good stages of the approximation, then $\forall n \exists s_0 \forall s > s_0 (s \in G \Rightarrow \Theta_s((U \oplus \overline{W})_s) \restriction n = \Theta(U \oplus \overline{W}) \restriction n)$. This is also a result from [10].

From these properties and the fact that E is a Π_1 set, we can conclude that if $\Theta(U \oplus \overline{W}) = E$ then there are infinitely many expansionary stages.

We will use more information about the sequence $\left\{(U \oplus \overline{W})_s\right\}$ – it will be considered as a pair $((U \oplus \overline{W})_s, ap_s) = (U'_s \oplus \overline{W}^*_s \restriction (\mu n[U'_s \oplus \overline{W}^*_s(n) \neq U'_{s+1} \oplus \overline{W}^*_{s+1}(n)]), \mu n[U'_s \oplus \overline{W}^*_s(n) \neq U'_{s+1} \oplus \overline{W}^*_{s+1}(n)])$.

We will modify the definition of expansionary stages to incorporate the trueness of the approximations.

Definition 3. *We inductively say that a stage $s+1$ is P_Θ-expansionary if and only if $l(E[s+1], \Theta^{U,\overline{W}}[s+1])$ attains a greater value at stage $s+1$ than at any previous P_Θ-expansionary stage and $ap_{s+1} > l(E[s+1], \Theta^{U,\overline{W}}[s+1])$.*

Note that if U is a properly Σ_2 set, then we can still obtain a modified approximation to it in the way described above, but will not need to satisfy its requirement P in that case.

2.5 The Basic Module for one P_Θ- and one N_Ψ- requirement

The (P_Θ, Γ)-strategy tries to maintain the equality between $\overline{K}$ and $\Gamma^{U,A}$ at expansionary stages. It scans elements $n < l(\Theta^{U,\overline{W}}, E)$ fixing their axioms as appropriate.

Every strategy works below a right boundary R, assuming that as the stages grow the right boundary will grow unboundedly.

(P_Θ, Γ) builds an operator Γ by defining marker $u_s(n)$ and $\gamma_s(n)$ and corresponding axioms for elements $n \in \overline{K}$ of the form $\langle n, U_s \upharpoonright u_s(n), A_s \upharpoonright \gamma_s(n)\rangle$ at stage s.

It may happen that the two strategies (P_Θ, Γ) and (P_Θ, Λ) influence each other by extracting markers from A. In order to prevent that we define two nonintersecting infinite computable sets A_G and A_L for the possible values of A-markers for (P_Θ, Γ) and (P_Θ, Λ) respectively. Each time (P_Θ, Γ) defines a new marker for some n, it defines $\gamma(n)$ big (bigger than any number that appeared in the construction until now) and $\gamma(n) \in A_G$.

Each time (P_Θ, Λ) defines a new marker for some n, it defines $\lambda(n)$ big and $\lambda(n) \in A_L$.

We will describe the modules in a more general way, so that we can use them later in the construction involving all requirements.

The (P_Θ, Γ) - strategy

1. Wait for an expansionary stage. $(o = l)$
2. Choose $n < l(\Theta^{U,\overline{W}}, E)$ in turn $(n = 0, 1, \dots)$ and perform the following actions:
   - If $u(n)\uparrow$, then define it anew as $u(n) = u(n-1)+1$ (if $n = 0$, then define $u(n) = 1$). If $u(n)$ is defined, but $ap_s < u(n)$ skip to the next element.
   - If $n \in \overline{K}$:
     - If $\gamma(n)\uparrow$, then define it anew, and define an axiom $\langle n, (U \upharpoonright u(n) + 1, A \upharpoonright \gamma(n)+1)\rangle \in \Gamma$.
     - If $\gamma(n)\downarrow$, but $\Gamma^{(U,A)}(n) = 0$ (due to a change in U or in A), then enumerate the old axiom in a special parameter $Old(n)$ – this being a collection of all axioms that might later on turn out to be valid. The element enumerated in $Old(n)$ is of the form $(\gamma(n), \langle n, (U \upharpoonright u(n)+1, A \upharpoonright \gamma(n)+1)\rangle)$ – the pair of the old marker and old axiom. Then define $\gamma(n)$ anew and define an axiom $\langle n, (U \upharpoonright u(n)+1, A \upharpoonright \gamma(n)+1)\rangle \in \Gamma$.

- If $n \notin \overline{K}$, but $n \in \Gamma^{(U,A)}$ then look through all axioms defined for n in $Old(n)$ and extract the $\gamma(n)$ markers for any axiom that is valid.

Module for (N_Ψ, Γ) The basic module acts only at P_Θ- expansionary stages. If there are only finitely many expansionary stages, then P_Θ is trivially satisfied and N_Ψ moves to a truer path through the tree of outcomes.

At the beginning of each stage we check if the thresholds are correct, i.e. if $\overline{K} \upharpoonright d$ has not changed since the last true stage. If so we initialize all strategies below this one and start from initialization.

- **Initialization**
  1. If a threshold has not yet been defined or is cancelled, choose a new threshold d bigger than any defined until now.
  2. If a witness has not yet been defined or is cancelled, choose a new witness $x > d$, $x \in E$.
  3. Wait for $x < l(E, \Theta^{U,\overline{W}})$. $(o = w)$
  4. Extract all markers $\gamma(d)$ – old and new – and empty the list $Old(n)$ for $n \geq d$. Define $u(d)$ anew, bigger than $\theta(x)$. This gives us control over any axiom enumerated in Γ for the elements we are monitoring.
  5. For every element $y \leq x$, $y \in E$, enumerate into the list $Axioms$ the current valid axiom from Θ that has been valid longest.
- **Honestification**

  Scan the list $Axioms$. If for any element $y \leq x$, $y \in E$, the listed axiom is not valid anymore, then update the list $Axioms$, let (o = h) and
  1. Extract $\gamma(d)$ from A for all markers of axioms – the current one and the old ones. Empty the list $Old(d)$. Redefine $u(d) = \max(\theta(x), u(d)) + 1$.
  2. Cancel all markers $u(n)$ for $n > d$ and $n \in \overline{K}$. Empty the list $Old(n)$. Notice that the extraction of all markers $\gamma(d)$ guarantees that the old axioms for elements $n > d$ will never again be valid. Hence at the next expansionary stage $u(n)$ and $\gamma(n)$ will be defined anew, bigger than $\theta(x)$. This ensures the following property: for all elements $z \geq d$, $z \in \overline{K}$, the U-parts of axioms both old and new in Γ include the U-parts of all axioms listed in $Axioms$ for elements $y \leq x$, $y \in E$.

  Otherwise go to:
- **Waiting**

  If Γ is honest, i.e. $u(d) > \theta(x)$, and all the axioms enumerated in $Axioms$ have remained unchanged since the last stage, then wait for $x \in \Psi^A$ with $use(\Psi, A, x) < R$, returning at each successive stage to Honestification (o = w).
- **Attack**
  1. If $x \in \Psi^A$ and $u(d) > \theta(x)$, then extract x from E and restrain A on $use(\Psi, A, x)$. (o = g)
  2. Wait until the length of agreement has returned and $ap_s > u(d)$.

3. Result –
Let $x' \leq x$ be the least element that has been extracted from E during the stage of the Attack. When the length of agreement returns $x' \notin \Theta^{U,\overline{W}}$. Hence all axioms for x' in Θ are not valid, in particular the one enumerated in $Axioms$, say $\langle x', U_{x'}, \overline{W}_{x'} \rangle$.
If $\overline{W}_{x'} \subset \overline{W}_s$ then the attack is successful and the activity at (P_Θ, Γ) lifts the γ-markers of all elements greater than d above the restraint to maintain $A \upharpoonright \psi(x)$. Note that this change affects not only the current axiom, but also all axioms enumerated in Old, because we insured that all possibly valid axioms in Γ – old and current – contain as a subset $U_{x'}$. If the change in U_x is permanent, then this will lead to success for N_Ψ. Otherwise the attack is unsuccessful, and we are forced to capriciously destroy Γ by extracting markers $\gamma(d)$ from A, and to start over with a bigger witness.
4. Successful attack: Then all valid axioms in Γ for $n \geq d$ have $\gamma(n) > use(\Psi, A, x)$. (o = f) Return to Result at the next stage. Note that the Σ_2-nature of the set U can trick us into believing that an attack is successful, whereas in fact it later turns out not to be. This is why we keep monitoring the witness and trying to prove that the attack is unsuccessful.
5. Unsuccessful attack: Extract all γ- markers $\gamma(d)$ from A for both the current and the old axioms. Empty $Old(n)$ for $n \geq d$. Remove the restraint on A. Cancel the current witness x. Return to Initialization at the next stage (choosing a new big enough witness) (o = g).

Analysis Of Outcomes: P_Θ has two possible outcomes:

[l] – there is a stage after which $l(\Theta^{U,\overline{W}}, E)$ remains bounded by its previous expansionary value say L. Then P_Θ is trivially satisfied as if U is Δ_2, then $\Theta^{U,\overline{W}} \neq E$. In this case we implement a simple 'Friedberg- Muchnik' strategy for N_Ψ working below $(R = \infty)$.

[e] – infinitely many expansionary stages, on which (N_Ψ, Γ) acts:

The possible outcomes of the (N_Ψ, Γ)- strategy are:

[w] – There is an infinite wait at Waiting for $\Psi^A(x) = 1$. Then N_Ψ is satisfied because $E(x) = 1 \neq \Psi^A(x)$ and the Γ_Θ-strategy remains intact. Successive strategies work below $R = \infty$)

[f] – There is a stage after which the last attack remains successful. At sufficiently large stages $\overline{K} \upharpoonright d$ has its final value. So there is no injury to the outcomes below f, $\Psi^A(x) = 1$, N_Ψ is satisfied, leaving the Γ_Θ- strategy intact. Successive strategies work below $(R = \infty)$

[h] – There are infinitely many occurrences of Honestification, precluding an occurrence of Attack. Then there is a permanent witness x, which has unbounded $lim_{sup}\theta(x)$. This means that $\Theta^{U,V}(y) = 0$ for some $y \leq x, y \in E$. Thus P_Θ is again satisfied. In this case we also implement a simple Friedberg–Muchnik strategy for N_Ψ working below $(R = \gamma(d))$.

[g] – We implement the unsuccessful attack step infinitely often. As anticipated, we must activate the $\Lambda_{\Theta,\Psi}$-strategy for P_Θ. N_Ψ is not satisfied, but we have a copy of N_Ψ designed to take advantage of the switch of strategies for P_Θ below N_Ψ. It works below $(R = x)$.

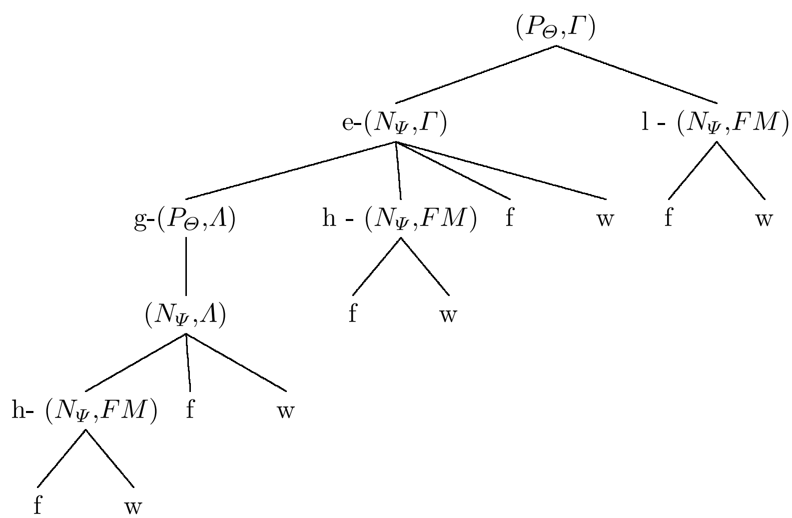
Module for the strategies below outcome *g*: Notice that the outcome g is visited in two different cases – at the beginning of an attack and when the attack turns out to be unsuccessful. The first case starts a nonactive stage for the subtree below g, allowing the other N-strategies to synchronize their attacks. The second case starts an active stage for the strategies in the subtree below g.

The (P_Θ, Λ) acts only on active stages in a similar but less complicated way than (P_Θ, Γ). Namely it does not have a list *Old* as any change in $\overline{W} \upharpoonright ap_s$ is permanent.

The (N_Ψ, Λ)-strategy is again similar to the (N_Ψ, Γ)-strategy. It has its own threshold $\widehat{d} > d$ and witness $\widehat{x} < x$. It has the Initialization, Honestification and Waiting modules which it executes on active stages. The corresponding outcomes are h and w.

It attacks on nonactive stages. The next stage at which this strategy is accessible after an attack will be an active stage – an unsuccessful attack for (N_Ψ, Γ). Note that the least element extracted during the attack is $x' \leq \widehat{x} < x$. So we have a $\overline{W} \upharpoonright \theta(\widehat{x})$ - change. Hence there will be no unsuccessful attacks and no outcome g, but only successful attacks and outcome f.

The tree of outcomes at this point looks as follows:



It is worth noticing that the outcomes on the tree, strictly speaking, are outcomes relating to strategies, rather than outcomes telling us exactly how the requirement is satisfied, and these subsume the "not Δ_2" case of the P-requirements. The properly Σ_2 case only needs to be specially factored in when one considers in the verification what the strategies deliver.

2.6 All requirements

When all requirements are involved the construction becomes more complicated. We will start by describing the tree of outcomes.

The requirements are ordered in the following way:

$$N_0 < P_0 < N_1 < P_1 \dots$$

Each P-requirement has at least one node along each path in the tree. Each N-requirement has a whole subtree of nodes along each path, the size of which depends on the number of P-requirements of higher priority.

Consider the requirement N_i. It has to clear the markers from A of i P-requirements $P_0, P_1, \dots P_{i-1}$. Each of them can follow one of the three strategies (N_Ψ, Γ_i), (N_Ψ, Λ_i) or (N_Ψ, FM_i). There will be nodes for each of the possible combinations in the subtree.

We distinguish between the following strategies:

1. For every P_i-requirement we have two different strategies: (P_i, Γ) with outcomes $e <_L l$ and (P_i, Λ) with one outcome s.
2. For every N_i-requirement, where $i > 0$, we have strategies of the form $(N_i, S_0, \dots S_{i-1})$, where $S_j \in \{\Gamma_j, \Lambda_j, FM_j\}$. They are all equipped with working boundaries (L, R). The requirement N_0 has one strategy (N_0, FM) with $(L = 0, R = \infty)$. The outcomes are f, w and for each $j < i$ if $S_j \in \{\Gamma_j, \Lambda_j\}$ there is an outcome h_j, if $S_j = \Gamma_j$, there is an outcome g_j. They are ordered according to the following rules:
   - For all j_1 and j_2, $g_{j_1} <_L h_{j_2} <_L f <_L w$
   - If $j_1 < j_2$ then $g_{j_2} <_L g_{j_1}$ and $h_{j_1} <_L h_{j_2}$.

   Let $\mathbb{O}$ be the set of all possible outcomes and $\mathbb{S}$ be the set of all possible strategies.

Definition 4. *The tree of outcomes is a computable function $T : D(T) \subset \mathbb{O}^* \rightarrow \mathbb{S}$ which has the following properties:*

1. $T(\emptyset) = (N_0, FM)$

2. $T(\alpha) = S$ and O_S is the set of outcomes for the strategy S, then for every $o \in O_S$, $\alpha\hat{\ }o \in D(T)$.

3.If $S = (N_i, S_0, S_1, \dots, S_{i-1})$, then

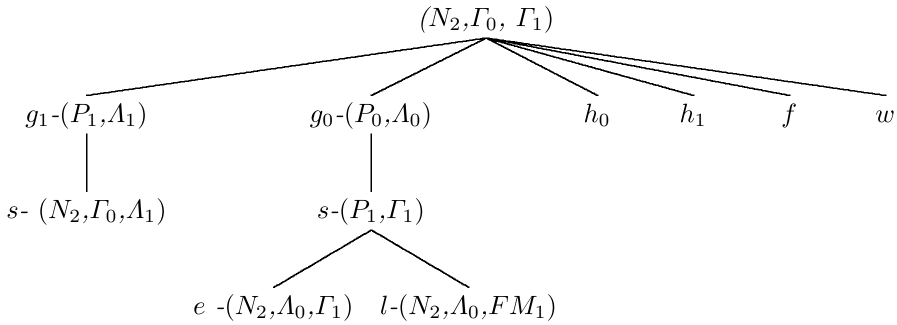
$T(\alpha\hat{\ }g_j) = (P_j, \Lambda_j)$ and $T(\alpha\hat{\ }g_j\hat{\ }s) = (P_{j+1}, \Gamma_{j+1}) \dots T(\alpha\hat{\ }g_j\hat{\ }s\hat{\ }o_{j+1}\hat{\ }o_{i-2}) = (P_{i-1}, \Gamma_{i-1})$, where $o_k \in \{e_k, l_k\}$ for $j+1 \le k \le i-2$.

All this means that, under an outcome g_j the strategy P_j starts its work on building the second possible functional Λ_j, and all strategies P_k for $k > j$ start their work from the beginning, i.e., start building the functional Γ_k again.

$T(\alpha\hat{\ }g_j\hat{\ }s\hat{\ }o_{j+1}\hat{\ }o_{i-1}) = (N_i, S_0, \dots, \Lambda_j, \dots, S_{i-1})$, where $S_k = \Gamma_k$ if $o_k = e_k$ and $S_k = FM_k$ if $o_k = l_k$ for every k such that $j < k < i$.

Then there is a copy of the strategy N_i which starts work with the old strategies S_l for $l < j$ and the new strategies S_k for $k \ge j$.

To illustrate this complicated definition here is a picture of this part of the tree in the simpler case of only two P- requirements.



The tree under outcome h_j is built in a similar fashion.

$T(\alpha\hat{\ } h_j) = (P_{j+1}, \Gamma_{j+1}) \ldots T(\alpha\hat{\ } h_j\hat{\ } o_{j+1}\hat{\ } o_{i-2}) = (P_{i-1}, \Gamma_{i-1})$, *where* $o_k \in \{e_k, l_k\}$ *for* $j+1 \leq k \leq i-2$.

Hence all strategies S_k for $k > j$ start their work from the beginning, building a new functional Γ_k.

$T(\alpha\hat{\ } h_j\hat{\ } o_{j+1}\hat{\ } o_{i-1}) = (N_i, S_0, \ldots, FM_j, \ldots, S_{i-1})$, *where* $S_k = \Gamma_k$ *if* $o_k = e_k$ *and* $S_k = FM_k$ *if* $o_k = l_k$ *for every k such that* $j < k < i$.

$T(\alpha\hat{\ } f) = (P_i, \Gamma_i)$

$T(\alpha\hat{\ } w) = (P_i, \Gamma_i)$

Say $S = (P_i, \Gamma)$, *and* $\alpha = \alpha'\hat{\ } f$ *or* $\alpha = \alpha'\hat{\ } w$, *and* $T(\alpha') = (N_i, S_0, S_1, \ldots, S_{i_1})$. *It follows that this is not the case described and* (P_i, Γ) *appears for the first time, and then*

$T(\alpha\hat{\ } e) = (N_{i+1}, S_0, \ldots, S_{i-1}, \Gamma_i)$

$T(\alpha\hat{\ } l) = (N_{i+1}, S_0, \ldots, S_{i-1}, FM_i)$.

Interaction between strategies: In order to prevent unwanted interaction between the different strategies on different nodes we will do the following:

Different P-strategies define and extract different A-markers at stages of the construction. Extraction of markers for one P-strategy may influence the validity of axioms for another P-strategy. Again we deal with this problem by separating the A-markers for the different possible strategies. We have countably many different nodes in the tree of outcomes, whose values are P-strategy. For each such node α we define an infinite computable set A_α, from which the strategy $T(\alpha)$ can choose A-markers. If $\alpha \neq \beta$ then $A_\alpha \cap A_\beta = \emptyset$.

Similarly we define separate nonintersecting sets D_α and X_α for the different nodes on the tree which are labelled with N- strategies, from which they choose their thresholds and witnesses.

As usual we give higher priority to nodes that are to the left or higher up in the tree of strategies. This is achieved via two forms of initialization.

1. On each stage initialization is performed on all nodes that are bigger than the last node visited on that stage.

2. The second case in which initialization is performed is when the thresholds are not correct:
   Every strategy α with $T(\alpha) = (N_i, S_1, \ldots, S_j, \ldots, S_{i-1})$ has a threshold d_j for each strategy S_j. If $\overline{K} \upharpoonright (d_j + 1)$ changes, then all successors of α that assume that d_j does not change infinitely many times are initialized. These are strategies γ such that $\gamma \supseteq \alpha\hat{\ }g_k$ for $k \leq j$ or $\gamma \supseteq \alpha\hat{\ }o$, where $o \in \{h_l, s, w | l < j\}$. And hence are all strategies below and to the right of outcome g_j.
   If α has not yet started an attack, then it continues from the Initialization step.
   If α has started an attack, and this change injures the equation that α is trying to preserve, then α will choose a new witness and start from Initialization. If the equation is not injured, then α will continue to restrain A in the hope that no later change in K will ever destroy its work.

The construction: At each stage s of the construction we build inductively a string $\delta_s \in D(T)$ of length s, by visiting nodes from the tree and acting according to their corresponding strategies.

$\delta_s(0) = \emptyset$.

Let $\delta_s \upharpoonright n = \alpha$.

1. $T(\alpha) = (P_i, \Gamma)$ – on active stages we perform the actions as stated in the main module. $\delta_s(n+1) = l$ at nonexpansionary stages. At expansionary stages $\delta_s(n+1) = e$ and $R_{\alpha\hat{\ }\delta_s(n+1)} = R_\alpha$.
   At nonactive stages no actions are performed. The strategy will have the same outcome as it did on the previous active stage.
2. $T(\alpha) = (P_i, \Lambda)$ – on active stages we perform the actions as stated in the main module. $\delta(n+1) = s$ and $R_{\alpha\hat{\ }\delta_s(n+1)} = R_\alpha$.
   At nonactive stages no actions are performed, $\delta(n+1) = s$.
3. $T(\alpha) = (N_i, S_0, \ldots, S_{i-1})$ –
   Let $Z_j = U$ if $S_j = \Gamma$ and $Z_j = \overline{W}$ if $S_j = \Lambda$.
   - **Initialization**

     On active stages:
     Each strategy $S_j \neq FM_j$ picks a threshold if it is not already defined. The different thresholds must be in the following order:

     $$L < d_{i-1} < d_{i-2} < \cdots < d_0 < R$$

     Strategy S_j picks its threshold so that it is bigger than any threshold it has picked before and such that its marker has not yet been defined.
     After all thresholds have been chosen, the strategy picks a witness x, bigger than any witness used until now and such that $d_0 < x$ and waits until $l(E, \Theta_j^{U_j, \overline{W}_j}) > x$ for all $j < i$. $\delta(n+1) = w$, working below $(R = R_\alpha)$.

On the first stage on which $l(E, \Theta_j^{U_j,\overline{W}_j}) > x$ for all $j < i$, extract all markers for all axioms old and new for all thresholds d_j, cancel all markers $z_j(n)$ for $n \geq d_j$ and let $z_j(d_j) > \theta_j(x)$.
For every element $y \leq x$, $y \in E$ enumerate in the list $Axioms_j$ the current valid axiom from Θ_j, that has been valid the longest.
Go to honestification at the next stage. Notice that this guarantees that any axiom $\langle n, Z_n, A_n\rangle$ enumerated in S_j for an element $n \geq d_j$, $n \in \overline{K}$ will have the property that for any $y \leq x$, $x \in E$ with axiom $\langle y, Z_y, V_y\rangle \in Axioms_j$, we will have that $Z_y \subset Z_n$.
$\delta(n+1) = w$, working below $(R = R_\alpha)$.

- **Honestification**

  On active stages:
  Scan all strategies from the list $S_0 \dots S_{i-1}$ in turn $(j = 0, 1, \dots i-1)$. Perform $Honestification_j$ from the main module for each $S_j \neq FM_j$. If the outcome of $Honestification_j$ is w go on to the next strategy. If it is h, then extract all markers old and new $s_k(d_k)$ for $k > j$ from A and empty their corresponding lists $Old_k(n)$ for elements $n \geq d_k$.
  The outcome is $\delta(n+1) = h_j$ working below $(R = \min(R_\alpha, s_j(d_j)))$. Start from $Honestification$ at the next stage.
- **Waiting**

  If all outcomes of all $Honestification_j$-modules are w, i.e all enumeration operators are honest, then wait for $x \in \Psi_i^A$ with $use(\Psi, A, x) < R_\alpha$. $f(n+1) = w$, working within below $R = R_\alpha)$. Return to $Honestification$ at the next stage.
- **Attack**

  (a) Let β be the biggest node such that $\alpha \supseteq \beta\hat{\ }g$. If there is such a node β, then wait for a β-nonactive stage.
  (b) If $x \in \Psi^A$, $use(\Psi, A, x) < R_\alpha$ and all operators are honest, then extract x from E and restrain A on $use(\Psi, A, x)$. This starts a nonactive stage for the strategies below the most recently visited outcome g_j (if none has been visited until now, then g_0).
  (c) Result –
  Wait until the length of agreement has returned for all strategies and they have been visited at an expansionary stage s with $ap_s^{U_j,W_j} > u(d_j)$.
  Scan all strategies $S_0, \dots S_{i-1}$ in turn, starting with S_0 and perform the corresponding $Result_j$ from the main module on each.
  - If the attack was successful for S_j, continue scan with S_{j+1}. If all the strategies are scanned, then $\delta(n+1) = f$, working below $(R = R_\alpha)$ go to Result at the next stage.
  - If the attack was unsuccessful for S_j, hence $S_j = \Gamma_j$, then $\gamma_j(d_j)$ has been extracted from A during $Result_j$ and the corresponding markers have been moved. In addition cancel the thresholds d_k, for $k < j$, cancel all markers $s_l(d_l)$ old and new for $l > j$

and extract them from A, emptying the corresponding lists Old_l. Cancel the witness. Start from Initialization at the next stage. $\delta(n+1) = g_j$, working below $(R = \min(x, R_\alpha))$.

Proof: The true path f is defined to be the leftmost path on the tree that is visited infinitely many times. Such a path exists, because the tree is finitely branching. We prove that the strategies along the true path satisfy their requirements.

Lemma 1. *For every n there is a stage s_n such that $f \upharpoonright n$ does not get initialized after stage s_n.*

Lemma 2.
1. *Let $\alpha \subset f$ be the biggest (P_j, Γ)-strategy and assume U_j is Δ_2. If $\gamma(n)$ moves off to infinity, then the following condition holds: If $\Theta_j^{U,\overline{W}} = E$ and then there is an outcome g_j along the true path.*
2. *Let $\alpha \subset f$ be the biggest P_j-strategy. It builds a function M. If $m(n)$ moves off to infinity then $\Theta_j^{U,\overline{W}} \neq E$.*

Corollary 1. *Every P_i-requirement is satisfied.*

Proof. If $\Theta_i^{U_i,\overline{W}_i} \neq E$ or U_i is properly Σ_2, then the requirement is trivially satisfied. Otherwise let $\alpha \subset f$ be the biggest (P_i, M) strategy along the true path. The properties of the approximation guarantee that there are infinitely many expansionary stages. According to the previous lemma all markers m used to build the operator M are bounded, hence for each element n, there are finitely many axioms in M.

Now it is easy to prove via induction on n that $\overline{K}(n) = M^{Z,A}(n)$.

Lemma 3. *Let $\alpha \subset f$ be an N_i requirement along the true path. And let s be a stage after which α is not initialized. Then*

1. *None of the nodes to the right or to the left of α extract elements from A that are less than R_α after stage s.*
2. *None of the N_j-nodes above α extract elements from A that are less than R_α after stage s.*
3. *Suppose $\beta \subset \alpha$ is a P_j node such that there is another P_j node β', with $\beta \subset \beta' \subset \alpha$. Then β does not extract elements from A that are less than R_α after stage s.*

Hence the strategies that can injure a restraint imposed by an N_i-strategy α along the true path are the active P_j-strategies at α and α itself.

Lemma 4. *Every N_i-requirement is satisfied.*

Proof. Let α be the last N_i requirement along the true path. We will prove that it satisfies N_i.

α has true outcome w or f, or else there will be a successive copy of N_i along the true path.

In the first case, α waits forever for $\Psi^A(x) = 0$. Hence $\Psi^A(x) \neq E(x)$.

Let the true outcome be f. And let $s > s_{\alpha\hat{}f}$. After stage s, $\overline{K} \upharpoonright d_1$ will not change anymore. Hence the active P_j-strategies can only extract markers of elements $n > d_j$.

If $S_j = \Gamma_j$ and the restraint is injured, then α would have outcome to the left, because this would mean that the attack for x is unsuccessful. Suppose $S_j = \Lambda_j$. Then α timed its attack, with some β, such that $\beta\hat{}g_j \subset \alpha$. Note that the entries in both lists $Axioms_j$ at α and β are the same. Hence an unsuccessful β-attack would mean that α gets the right $\overline{W}$-permission.

References

1. M. M. Arslanov, *Structural properties of the degrees below* $\mathbf{0}'$, Dokl. Akad. Nauk. SSSR **283** (1985), 270–273.
2. S. B. Cooper, *On a theorem of C. E. M. Yates*, (handwritten notes), 1974.
3. S. B. Cooper, *The strong anti-cupping property for recursively enumerable degrees*, J. Symbolic Logic **54** (1989), 527–539.
4. S. B. Cooper, *Enumeration reducibility, nondeterministic computations and relative computability of partial functions*, in *Recursion Theory Week, Proceedings Oberwolfach 1989* (K. Ambos-Spies, G. H. Müller, G. E. Sacks, eds.), Lecture Notes in Mathematics no. 1432, Springer-Verlag, Berlin, Heidelberg, New York, 1990, pp. 57–110.
5. S. B. Cooper, *Computability Theory*, Chapman & Hall/CRC Mathematics, Boca Raton, FL, New York, London, 2004.
6. S. B. Cooper, A. Sorbi, A. Li and Y. Yang, *Bounding and nonbounding minimal pairs in the enumeration degrees*, J. Symbolic Logic **70** (2005), 741–766.
7. R. G. Downey, Δ^0_2 *degrees and transfer theorems*, Illinois J. Math. **31** (1987), 419–427.
8. L. Harrington, *Understanding Lachlan's monster paper* (handwritten notes), 1980.
9. A. H. Lachlan, *A recursively enumerable degree which will not split over all lesser ones*, Ann. Math. Logic **9** (1975), 307–365.
10. A. H. Lachlan and R. A. Shore, *The n-rea enumeration degrees are dense*, Arch. Math. Logic **31** (1992), 277–285.
11. D. Miller, *High recursively enumerable degrees and the anti-cupping property*, in *Logic Year 1979–80: University of Connecticut* (M. Lerman, J. H. Schmerl, R. I. Soare, eds.), Lecture Notes in Mathematics No. 859, Springer-Verlag, Berlin, Heidelberg, Tokyo, New York, 1981.
12. P. G. Odifreddi, *Classical Recursion Theory, Volume II*, North-Holland/Elsevier, Amsterdam, Lausanne, New York, Oxford, Shannon, Singapore, Tokyo, 1999.
13. D. B. Posner and R. W. Robinson, *Degrees joining to* $\mathbf{0}'$, J. Symbolic Logic **46** (1981), 714–722.
14. G. E. Sacks, *On the degrees less than* $\mathbf{0}'$, Ann. of Math. **77** (2) (1963), 211–231.
15. T. A. Slaman and J. R. Steel, *Complementation in the Turing degrees*, J. Symbolic Logic **54** (1989), 160–176.
16. R. I. Soare, *Recursively enumerable sets and degrees*, Springer-Verlag, Berlin, Heidelberg, London, New York, Paris, Tokyo, 1987.