A computability notion for locally finite lattices

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Abstract. A rather restricted kind of computability called calculability is considered on locally finite lattices. Besides arbitrary given functions, its definition uses as an initial function also a four-argument one whose value equals the third or the fourth argument depending on whether the first argument is dominated by the second one. New functions are constructed by means of substitution and of supremum and infimum attained when one of the arguments ranges between varying limits. In one of the examples the calculable functions have a certain relation to the Bounded Arithmetic of S. R. Buss, and in another one they coincide with the so-called elementary definable functions considered by the present author in 1967. Under some assumptions that are satisfied in these two examples, the calculable functions on the natural numbers are characterized through definability by formulas with bounded quantifiers and domination by termal functions.

1 Introduction

The study of restricted kinds of computability is often used for achieving a better applicability of the theory of computability to problems about complexity of computations. In the present paper one such restricted kind called calculability will be examined. It is a sort of relative computability that makes sense not only on the natural numbers, but on any non-degenerate locally finite lattice. An example to the definition of calculability will be shown to have a certain relation to the Bounded Arithmetic introduced in [1].

2 The main definition and some examples to it

Let L be a non-degenerate lattice that is locally finite, i.e. $\{y \in L \,|\, x \leq y \leq z\}$ is a finite set for any x, z in L. For instance, L could consist of all finite subsets of some infinite set and be partially ordered by the inclusion relation, or could be the set of the positive integers partially ordered by the divisibility relation. In fact we are interested mainly in the case when L is some subset of the set $\mathbb{Z}$ of the integers and L is linearly ordered in the usual way (the most important case will be the one when L is the set $\mathbb{N}$ of the natural numbers, but there are also other cases that deserve some interest – for example the case of $L = \mathbb{Z}$ and the case of $L = \{0, 1\}$).

Each function considered in this paper will be, except explicitly said something else, a total function of some non-zero number of arguments in the set L, i.e. a mapping of L^n into L for some positive integer n. We shall denote by $\mathcal{P}$ the set of the projection functions, i.e. of the functions $\lambda x_1 \ldots x_n.\, x_i$ with $1 \leq i \leq n$. The function $\lambda xyst.\,(\text{if } x \leq y \text{ then } s \text{ else } t)$ will be denoted by δ_L. Whenever f is an m-ary function, and $g_1, \ldots, g_m$ are n-ary functions, the function

$$\lambda x_1 \ldots x_n.\, f(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$$

will be called their *composition*. For any non-negative integer n and any $(n+1)$-ary function f one can consider the two $(n+2)$-ary functions

$$\lambda x_1 \ldots x_n uv.\, \bigvee_{y \in [u..v]_L} f(x_1, \ldots, x_n, y)\,, \quad \lambda x_1 \ldots x_n uv.\, \bigwedge_{y \in [u..v]_L} f(x_1, \ldots, x_n, y)\,,$$

where $[u..v]_L = \{y \in L \,|\, u \wedge v \leq y \leq u \vee v\}$ (the subscript L will be usually omitted). These two functions will be denoted by f^+ and f^-, respectively.

Let $\mathcal{A}$ be a set of functions. The functions *termal in* $\mathcal{A}$ form the smallest class of functions containing $\mathcal{A} \cup \mathcal{P}$ and closed under compositions. The functions that are termal in $\mathcal{A} \cup \{\delta_L\}$ will be called *piecewise termal in* $\mathcal{A}$. Our main subject will be the smallest class of functions containing $\mathcal{A} \cup \{\delta_L\} \cup \mathcal{P}$ that is closed under compositions and contains f^+ and f^- for any f in it. The functions belonging to this class will be called *calculable in* $\mathcal{A}$ or $\mathcal{A}$-*calculable*, for short. Obviously, all functions termal in $\mathcal{A}$ are piecewise termal in $\mathcal{A}$, and all functions piecewise termal in $\mathcal{A}$ are $\mathcal{A}$-calculable. (The converse statements are not true in the general case.)

Remark 1. Each $\mathcal{A}$-calculable function is absolutely search computable (in the sense of [2]) in $\lambda xy.\, x \vee y$, $\lambda xy.\, x \wedge y$, $\lambda xz.\, \mathrm{card}\{y \in L \,|\, x \leq y \leq z\}$ and the functions from some finite subset of $\mathcal{A}$.

Remark 2. For any positive integer n the functions $\lambda x_1 \ldots x_n.\, x_1 \vee \ldots \vee x_n$ and $\lambda x_1 \ldots x_n.\, x_1 \wedge \ldots \wedge x_n$ are $\mathcal{A}$-calculable, and they are even piecewise termal in $\mathcal{A}$ if L is linearly ordered. This statement is trivial for $n = 1$. For $n = 2$ one can use the equalities

$$u \vee v = \bigvee_{y \in [u..v]} y\,, \quad u \wedge v = \bigwedge_{y \in [u..v]} y\,,$$

in the general case and the equalities

$$u \vee v = \delta_L(u, v, v, u)\,, \quad u \wedge v = \delta_L(u, v, u, v)$$

in the case of linearly ordered L. The general statement can be proved by induction.

The following denotations will be used sometimes in the sequel:

$$f_a^+ = \lambda x_1 \ldots x_n v.\, f^+(x_1, \ldots, x_n, a, v)\,,$$
$$f_{a,b}^+ = \lambda x_1 \ldots x_n.\, f^+(x_1, \ldots, x_n, a, b)$$

for any natural number n, any $(n+1)$-ary function f and any a, b in L.

Remark 3. Suppose L is linearly ordered, there is a least element a in L, and the constant a regarded as a one-argument function is piecewise termal in $\mathcal{A}$. Then the class of the $\mathcal{A}$-calculable functions is the smallest class of functions containing $\mathcal{A} \cup \{\delta_L\} \cup \mathcal{P}$ that is closed under compositions and contains f_a^+ for any f in it. To prove this, it is sufficient to prove the following two statements:

(i) For any $\mathcal{A}$-calculable function f the function f_a^+ is also $\mathcal{A}$-calculable.
(ii) Whenever $\mathcal{C}$ is a class of functions with the properties formulated above, and f is a function belonging to $\mathcal{C}$, the functions f^+ and f^- also belong to $\mathcal{C}$.

The statement (i) is obvious, since f_a^+ is obtained from f^+ by means of a substitution of a into it. As to the statement (ii), it follows (assuming f is $(n+1)$-ary) from the equalities

$$\begin{aligned} f^+(\bar{x},u,v) &= \delta_L(u,v,g(\bar{x},u,v),g(\bar{x},v,u))\,, \\ f^-(\bar{x},u,v) &= h_a^+(\bar{x},u,v,f(\bar{x},u))\,, \end{aligned}$$

where $\bar{x}$ stands for $x_1,\ldots,x_n$,

$$g(\bar{x},s,t) = \bigvee_{y\in[a..t]} \delta_L(y,s,f(\bar{x},s),f(\bar{x},y))\,,$$

and h is defined by consecutively setting

$$\begin{aligned} l(\bar{x},u,v,z) &= \bigvee_{y\in[u..v]} \delta_L(z,f(\bar{x},y),a,z)\,, \\ h(\bar{x},u,v,z) &= \delta_L(l(\bar{x},u,v,z),a,z,a) \end{aligned}$$

(*comment*: for any given $\bar{x},u,v$ the inequality $l(\bar{x},u,v,z) \le a$ is equivalent to the equality $l(\bar{x},u,v,z)=a$, and it is satisfied by an element z of L if and only if $z \le f(\bar{x},y)$ for all y in $[u..v]$, therefore $\{\,h(\bar{x},u,v,z)\,|\,z \in [a..f(\bar{x},u)]\}$ is the set of all such z).

Remark 4. Suppose L is linearly ordered, there are a least element a and a greatest element b in L, and the constants a and b regarded as one-argument functions are piecewise termal in $\mathcal{A}$. Then the class of the $\mathcal{A}$-calculable functions is the smallest class of functions containing $\mathcal{A} \cup \{\delta_L\} \cup \mathcal{P}$ that is closed under compositions and contains $f_{a,b}^+$ for any f in it. Indeed, $f_{a,b}^+$ is $\mathcal{A}$-calculable for any $\mathcal{A}$-calculable function f, and, on the other hand, if f is an $(n+1)$-ary function for some non-negative integer n, then $f_a^+ = g_{a,b}^+$, where g is defined by means of the equality

$$g(\bar{x},v,y) = \delta_L(y,v,f(\bar{x},y),f(\bar{x},v))$$

($\bar{x}$ stands again for $x_1,\ldots,x_n$), hence any class of functions with the properties formulated in the present remark has also the properties formulated in Remark 3.

Remark 5. If L is linearly ordered, there are a least and a greatest element in L, and all constants regarded as one-argument functions are piecewise termal in $\mathcal{A}$, then all $\mathcal{A}$-calculable functions are piecewise termal in $\mathcal{A}$. This can be seen by using Remarks 2 and 4.

Remark 6. All $\mathcal{A}$-calculable functions are piecewise termal in $\mathcal{A}$ also in the case when L has exactly two elements (hence L is linearly ordered). Indeed, then $[u..v] = \{u, v\}$, therefore

$$f^+(\bar{x}, u, v) \; = \; f(\bar{x}, u) \vee f(\bar{x}, v) \, , \quad f^-(\bar{x}, u, v) \; = \; f(\bar{x}, u) \wedge f(\bar{x}, v)$$

for all $\bar{x}, u, v$, and we may use Remark 2.

Now we shall consider some examples to the definition of calculability. In each of them except for the last one, L will be a subset of $\mathbb{Z}$, and the usual linear ordering will be tacitly assumed to be the partial order on L (according to Remark 6 the examples with $L = \{0, 1\}$ concern in fact the ordinary theory of Boolean functions).

Example 1. Let $L = \mathbb{N}$, $\mathcal{A} = \{\lambda x.\, 0,\ \lambda x.\, 1,\ \lambda xy.\, x + y,\ \lambda xy.\, x \cdot y\}$. Then the $\mathcal{A}$-calculable functions will be shown to coincide with the class of functions considered in [4] and called elementary definable there.[1] By its definition (slightly paraphrased), the class of the elementary definable functions is the smallest class of functions in $\mathbb{N}$ containing $\lambda xy.\,(\text{if } x = y \text{ then } 1 \text{ else } 0)$, $\lambda xy.\, x + y$, $\lambda xy.\, x \cdot y$ and the projection functions, that is closed under compositions and contains f_0^+ for any f in it. Therefore (in view of Remark 3) it is sufficient to prove the calculability of the function $\lambda xy.\,(\text{if } x = y \text{ then } 1 \text{ else } 0)$ in the concrete $\mathcal{A}$ considered now and the elementary definability of the functions $\lambda x.\, 0$, $\lambda x.\, 1$ and δ_L. This can be done by means of the equalities

$$\begin{aligned}
&(\text{if } x = y \text{ then } 1 \text{ else } 0) \; = \; \delta_L(x, y, \delta_L(y, x, 1, 0), 0)\, , \\
&1 \; = \; (\text{if } x = x \text{ then } 1 \text{ else } 0)\, , \quad 0 \; = \; (\text{if } 1 = 1 + 1 \text{ then } 1 \text{ else } 0), \\
&\delta_L(x, y, s, t) \; = \; g_0^+(x, y, y) \cdot s \; + \; g_0^+(y + 1, x, x) \cdot t\, ,
\end{aligned}$$

where $g = \lambda t_1 t_2 z.(\text{if } t_1 + z = t_2 \text{ then } 1 \text{ else } 0)$.

Example 2. Let $L = \mathbb{N}$, $\mathcal{A} = \{\lambda x.\, 0\, ,\ \lambda xy.\, x{+}y\, ,\ \lambda xy.\, x{\cdot}y\, ,\ \lambda x.\, |x|\, ,\ \lambda xy.\, x\#y\}$, where $|x|$ is the length of the binary representation of x if $x > 0$ and 0 otherwise, $x\#y$ is $2^{|x|\cdot|y|}$. In [1] some theories of arithmetic are studied that have the functions from $\mathcal{A}$ and the functions $\lambda x.\, x + 1$, $\lambda x.\, \lfloor x/2 \rfloor$ as their primitive functions and $\leq$ as their primitive relation (the study of these theories, and especially of formulas in them with bounded quantifiers, is related to problems of feasible computability). The functions $\lambda x.\, x + 1$, $\lambda x.\, \lfloor x/2 \rfloor$ are $\mathcal{A}$-calculable thanks to the equalities

$$1 \; = \; 0\#0\, , \quad \lfloor x/2 \rfloor \; = \; \bigvee_{y \in [0..x]} \delta_L(y + y, x, y, 0)\, ,$$

hence adding them to $\mathcal{A}$ would not enlarge the class of the $\mathcal{A}$-calculable functions. A comparison with Example 1 shows that all elementary definable functions belong to this class. The converse is not true, because the elementary definable

[1] The class in question is a subclass (most likely a proper one) of the class of the lower elementary functions introduced in [3].

functions are dominated by polynomials, whereas the function $\lambda xy.\, x\#y$ is not. By a certain fairly general statement concerning calculability (to be proved in the last section), the class considered now can be characterized in a simple way through definability by formulas with bounded quantifiers in the language of the above-mentioned theories (the elementary definable functions can be characterized in a similar way by formulas with bounded quantifiers in a more restricted language).

Example 3. Let $L = \mathbb{N}$ and the ternary functions f_1 and f_2 be defined as follows:

$$f_1(x, y, z) = (\text{if } x + y = z \text{ then } 0 \text{ else } 1)\,,$$
$$f_2(x, y, z) = (\text{if } x \cdot y = z \text{ then } 0 \text{ else } 1)\,.$$

Since f_1 and f_2 are piecewise termal in the set $\mathcal{A}$ from Example 1, all functions calculable in $\{f_1, f_2\}$ are calculable in that $\mathcal{A}$, hence they are elementary definable. However, the converse is not true. For example, the constant 1 (regarded say as a one-argument function) is not calculable in the set $\{f_1, f_2\}$, since, whenever an n-ary function f is calculable in this set, the inequality $f(x_1, \ldots, x_n) \leq x_1 \vee \ldots \vee x_n$ holds for all $x_1, \ldots, x_n$ in $\mathbb{N}$. We note that the constant 0 is still calculable (and even termal) in $\{f_1, f_2\}$ thanks to the equality $0 = f_2(f_2(x, x, x), f_2(x, x, x), f_2(x, x, x))$. By Remark 2, the functions of the form $\lambda x_1 \ldots x_n.\, x_1 \vee \ldots \vee x_n$ are also calculable in $\{f_1, f_2\}$. As an example of $\{f_1, f_2\}$-calculable function, that is not piecewise termal in $\{f_1, f_2\}$, we shall mention the predecessor function (defined as 0 at 0) – its value at x can be represented as

$$\bigvee_{y \in [0..x]} \delta_L(f_1(y, f_1(x, x, x), x), 0, y, 0)\,.$$

Example 4. Let $L = \mathbb{Z}$, $\mathcal{A} = \{\, \lambda x.\, 1\,,\ \lambda xy.\, x - y\,,\ \lambda xy.\, x \cdot y \,\}$. Then the functions $\lambda x.\, 0$, $\lambda x.\, -x$, $\lambda xy.\, x + y$ are termal in $\mathcal{A}$ thanks to the equalities $0 = x - x$, $-x = 0 - x$, $x + y = x - (-y)$. The class of the $\mathcal{A}$-calculable functions is the smallest class of functions containing $\mathcal{A} \cup \{\delta_L\} \cup \mathcal{P}$ that is closed under compositions and contains f_0^+ for any f in it – this can be shown by using the equalities

$$f^+(\bar{x}, u, v) = \bigvee_{z \in [0..v-u]} f(\bar{x}, v - z)\,, \quad f^-(\bar{x}, u, v) = -\bigvee_{z \in [0..v-u)]} -f(\bar{x}, v - z)\,.$$

The $\mathcal{A}$-calculable functions can be characterized also in the following more explicit way: an n-argument function f is $\mathcal{A}$-calculable if and only if there exist two elementary definable $2n$-argument functions ψ and θ in $\mathbb{N}$ such that

$$f(i_1 - j_1, \ldots, i_n - j_n) = \psi(i_1, j_1, \ldots, i_n, j_n) - \theta(i_1, j_1, \ldots, i_n, j_n)$$

for all $i_1, j_1, \ldots, i_n, j_n$ in $\mathbb{N}$.

Example 5. Let $L = \{0, 1\}$, and let $\mathcal{A}$ consist of the constants 0 and 1 regarded as one-argument functions. Since $\lambda xy.\, \delta_L(x, y, 1, 0)$ is the Boolean implication, the Post Theorem shows that all Boolean functions are $\mathcal{A}$-calculable. Another proof of this will be given in the next section.

Example 6. Let $L = \{0,1\}$, $\mathcal{A} = \emptyset$. Then any $\mathcal{A}$-calculable function f belongs to both Post classes T_0 and T_1, i.e. $f(0,\ldots,0) = 0$ and $f(1,\ldots,1) = 1$. A proof of the converse statement will be given in the next section.

Example 7. Let $L = \{0,1\}$, $\mathcal{A} = \{\lambda x.\, 0\}$. Then all $\mathcal{A}$-calculable functions belong to T_0, and the converse statement will be proved in the next section.

Example 8. Let L be an infinite subset of $\mathbb{N}$, f be a unary function that transforms any element of L into some greater one, and g be the unary function that transforms any x from L into the least element of L greater than x. Then g is $\{f\}$-calculable thanks to the equality

$$g(x) = \bigwedge_{y \in [x..f(x)]_L} \delta_L(y, x, f(x), y)\,.$$

(However, a choice of L and f is possible such that g cannot be obtained by means of a recursive operator from f and the functions $\lambda xy.\, x \vee y$, $\lambda xy.\, x \wedge y$, $\lambda xz.\, \mathrm{card}\{y \in L \mid x \le y \le z\}$ mentioned in Remark 1.)

Example 9. Let L consist of all finite subsets of some infinite set, and $\mathcal{A}$ be any set of functions in L. The equalities

$$x_1 \setminus x_2 = \bigwedge_{y \in [\emptyset..x_1]_L} (\text{if } x_1 \le x_2 \vee y \text{ then } y \text{ else } x_1)\,, \quad \emptyset = x \setminus x$$

show that $\lambda x_1 x_2.\, x_1 \setminus x_2$ is $\mathcal{A}$-calculable if and only if $\lambda x.\, \emptyset$ is $\mathcal{A}$-calculable.

3 $\mathcal{A}$-calculable predicates

We continue adhering to the assumptions and conventions from the beginning of the previous section. Thus a non-degenerate locally finite lattice L and a set $\mathcal{A}$ of functions in it will be again supposed to be given. For any positive integer n and any n-argument predicate p on L the $(n+2)$-argument function

$$\lambda x_1 \ldots x_n st.\, (\text{if } p(x_1,\ldots,x_n) \text{ then } s \text{ else } t)$$

will be denoted by p^*, and it will be called *the representing function of* p (obviously there is a one-to-one correspondence between the predicates and their representing functions). A predicate will be called *calculable in* $\mathcal{A}$ or $\mathcal{A}$*-calculable*, for short, if the representing function of this predicate is $\mathcal{A}$-calculable.

Remark 7. If there are two distinct elements a and b of L such that the corresponding one-argument constant functions are $\mathcal{A}$-calculable then one can characterize the $\mathcal{A}$-calculability of predicates in a more usual way. In such a situation, an n-argument predicate p on L is $\mathcal{A}$-calculable if and only if the n-argument function

$$p^*_{a,b} = \lambda x_1 \ldots x_n.\, (\text{if } p(x_1,\ldots,x_n) \text{ then } a \text{ else } b)$$

is $\mathcal{A}$-calculable. This is clear from the equalities

$$\begin{aligned} p^*_{a,b}(\bar{x}) &= p^*(\bar{x}, a, b)\,, \\ p^*(\bar{x}, s, t) &= \delta_L(p^*_{a,b}(\bar{x}), a, \delta_L(a, p^*_{a,b}(\bar{x}), s, t), t)\,. \end{aligned}$$

Next example can be used together with Example 7 to show that the assumption about $\mathcal{A}$-calculability of a and b in the previous remark cannot be omitted (because the function $\lambda x.$ (if $x = 1$ then 0 else 1) is not $\mathcal{A}$-calculable in the situation from Example 7).

Example 10. Let $L = \{a, b\}$, where $a < b$. Then the predicates $p = \lambda x.\,(x = a)$ and $q = \lambda x.\,(x = b)$ are $\mathcal{A}$-calculable thanks to the equalities

$$\begin{aligned} p^*(x, s, t) = \delta_L(s, t, \delta_L(x, s, s, t), \delta_L(x, t, s, t))\,, \\ q^*(x, s, t) = \delta_L(s, t, \delta_L(t, x, s, t), \delta_L(s, x, s, t))\,. \end{aligned}$$

Immediate examples of $\mathcal{A}$-calculable predicates (for any $\mathcal{A}$) are the identically true and the identically false ones, as well as the predicate $\lambda x_1 x_2.\,(x_1 \leq x_2)$. The class of the $\mathcal{A}$-calculable n-argument predicates is closed under negation, conjunction and disjunction, as seen from the equalities

$$\begin{aligned} (\text{not } p)^*(\bar{x}, s, t) &= p^*(\bar{x}, t, s), \\ (p \text{ and } q)^*(\bar{x}, s, t) &= p^*(\bar{x}, q^*(\bar{x}, s, t), t), \\ (p \text{ or } q)^*(\bar{x}, s, t) &= p^*(\bar{x}, s, q^*(\bar{x}, s, t)). \end{aligned}$$

The class of the $\mathcal{A}$-calculable predicates is closed also under substitution of $\mathcal{A}$-calculable funcrions, i.e. whenever p is an $\mathcal{A}$-calculable m-argument predicate, and $f_1, \ldots, f_n$ are $\mathcal{A}$-calculable n-argument functions, the predicate

$$q = \lambda x_1 \ldots x_n.\, p(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$$

is also $\mathcal{A}$-calculable. This is clear from the equality

$$q^*(\bar{x}, s, t) = p^*(f_1(\bar{x}), \ldots, f_m(\bar{x}), s, t).$$

By using the above properties one can easily see the $\mathcal{A}$-calculability of a lot of other predicates. For example $\lambda x_1 x_2.\,(x_1 = x_2)$ is $\mathcal{A}$-calculable as a conjunction of $\lambda x_1 x_2.\,(x_1 \leq x_2)$ and $\lambda x_1 x_2.\,(x_2 \leq x_1)$. In the conditions of Example 10 each n-argument predicate on L turns out to be $\mathcal{A}$-calculable, since it can be obtained from predicates of the form $\lambda x_1 \ldots x_n.\,(x_i = a)$ by means of negation, conjunction and disjunction.

The definition of calculability ensures one more kind of closedness of the class of the $\mathcal{A}$-calculable predicates. Namely, if p is an $\mathcal{A}$-calculable $(n+1)$-argument predicate on L for some non-negative integer n then the following two predicates are also $\mathcal{A}$-calculable:

$$\begin{aligned} q &= \lambda x_1 \ldots x_n uv.\,(\,p(x_1, \ldots, x_n, y) \text{ for all } y \in [u..v]\,)\,, \\ r &= \lambda x_1 \ldots x_n uv.\,(\,p(x_1, \ldots, x_n, y) \text{ for some } y \in [u..v]\,)\,. \end{aligned}$$

This follows from the equalities

$$q^*(\bar{x},u,v,s,t) = \delta_L\Big(\bigvee_{y\in[u..v]} p^*(\bar{x},y,s\wedge t,s\vee t), s\wedge t, s, t\Big)\,,$$
$$r^*(\bar{x},u,v,s,t) = \delta_L\Big(\bigwedge_{y\in[u..v]} p^*(\bar{x},y,s\wedge t,s\vee t), s\wedge t, s, t\Big)\,.$$

For any n-argument function f the predicate $\lambda x_1 \dots x_n y.\,(y = f(x_1, \dots x_n))$ will be said to *represent* f. If f is $\mathcal{A}$-calculable then this predicate is also $\mathcal{A}$-calculable, as it follows from some of the above-mentioned properties. Next lemma gives a way for reasoning in the opposite direction, namely from the $\mathcal{A}$-calculability of the predicate representing a function to the $\mathcal{A}$-calculability of the function itself.

Lemma 1. *Let f, g, h be n-argument functions such that $g(\bar{x}) \leq f(\bar{x}) \leq h(\bar{x})$ for any $\bar{x}$ in L^n, and the functions g and h, as well as the predicate representing f, are $\mathcal{A}$-calculable. Then the function f is also $\mathcal{A}$-calculable.*

Proof. If p is the predicate that represents f then

$$f(\bar{x}) = \bigvee_{y\in[g(\bar{x})..h(\bar{x})]} p^*(\bar{x},y,y,g(\bar{x}))$$

for any $\bar{x}$ in L^n. □

An n-argument function f will be called *$\mathcal{A}$-calculable on* a subset X of L^n if f coincides on X with some $\mathcal{A}$-calculable n-argument function. Clearly an n-argument function is $\mathcal{A}$-calculable if and only if it is $\mathcal{A}$-calculable on the whole L^n. A combined application of this fact and the following lemma can be useful for proving the $\mathcal{A}$-calculability of some functions.

Lemma 2. *Let $X_1, \dots, X_k$, where $k \geq 2$, be subsets of L^n, and the predicates $\lambda x_1 \dots x_n.\,(\,(x_1, \dots, x_n) \in X_i\,)$, $i = 2, \dots, k$, be $\mathcal{A}$-calculable. If an n-argument function is $\mathcal{A}$-calculable on each of the sets $X_1, \dots, X_k$, then it is $\mathcal{A}$-calculable also on their union.*

Proof. It is sufficient to prove the statement of the lemma for the case of $k = 2$, since the statement for the general case can be obtained from there by induction. The reasoning for the case of $k = 2$ looks as follows: if an n-argument function f coincides on X_i with the $\mathcal{A}$-calculable function g_i for $i = 1, 2$, then f coincides on $X_1 \cup X_2$ with the function $\lambda x_1 \dots x_n.\, p^*(\bar{x}, g_2(\bar{x}), g_1(\bar{x}))$, where $p = \lambda x_1 \dots x_n.\,(\,(x_1, \dots, x_n) \in X_2\,)$. □

By application of Lemma 2, we shall fulfil now the promises made in the three examples with $L = \{0, 1\}$ of the previous section. In the case of Example 5, given an arbitrary n-ary function f, we take $k = 2$, $X_1 = \{\bar{x} \,|\, f(\bar{x}) = 0\}$ and $X_2 = \{\bar{x} \,|\, f(\bar{x}) = 1\}$. In the case of Example 6, given an n-ary function f belonging to $T_0 \cap T_1$, we take $k = n$ and $X_i = \{\bar{x} \,|\, f(\bar{x}) = x_i\}$ for $i = 1, \dots, n$.

Finally, in the case of Example 7, given an n-ary function f belonging to T_0, we take $k = n + 1$ and $X_{n+1} = \{\bar{x} \mid f(\bar{x}) = 0\}$, the sets $X_1, \ldots, X_n$ being defined in the same way as in the case of Example 6. In each of these cases, the given function f is $\mathcal{A}$-calculable on any of the corresponging sets $X_1, \ldots, X_k$, hence it is $\mathcal{A}$-calculable on their union (because all predicates are $\mathcal{A}$-calculable in the case of a two-element L). On the other hand, it is easy to show that the union in question is equal to $\{0, 1\}^n$ in each of the three cases.

4 Definability by formulas with bounded quantifiers

Let a signature Σ be given with some function symbols and only one predicate symbol, and let this predicate symbol be a binary one. Let S be a structure for Σ such that the domain of S is $\mathbb{N}$, and let the predicate symbol of Σ be interpreted in S as the relation $\leq$ between natural numbers. In addition to the interpretation in S of any functional symbol of Σ we shall consider also its *adapted interpretation* in S - meaning the interpretation of the symbol in S if the symbol is not a constant of Σ and the one-argument constant function whose value is the interpretation of the symbol otherwise. Taking L to be the set $\mathbb{N}$ with the relation $\leq$, we shall consider any set $\mathcal{A}$ of functions such that each function in $\mathcal{A}$ is the adapted interpretation in S of some function symbol of Σ, and the adapted interpretation of each function symbol of Σ is $\mathcal{A}$-calculable.[2] Under some additional assumptions we shall study the connection between $\mathcal{A}$-calculability and definability by bounded formulas of the first-order language $\mathbf{L}_\Sigma$ corresponding to Σ.

Having in mind the above-mentioned interpretation of the predicate symbol of Σ, we shall write the atomic formulas of $\mathbf{L}_\Sigma$ as inequalities between terms. By definition, a formula of $\mathbf{L}_\Sigma$ is *bounded* if it is constructed from atomic formulas by means of negation, conjunction, disjunction and bounded quantifiers of the forms $(\forall\xi \leq \tau)$, $(\exists\xi \leq \tau)$ with term τ not containing the variable ξ (assuming that $(\forall\xi \leq \tau)F$ and $(\exists\xi \leq \tau)F$ are abbreviations for $\forall\xi((\xi \leq \tau) \rightarrow F)$ and $\exists\xi((\xi \leq \tau) \ \&\ F)$, respectively).

The additional assumptions we make are the following ones: (i) the constant 0 regarded as a one-argument function is piecewise termal in $\mathcal{A}$, (ii) a two-argument function exists that is termal in $\mathcal{A}$ and dominates its arguments, and (iii) each function belonging to $\mathcal{A}$ is dominated by some function termal in $\mathcal{A}$ and monotonically increasing with respect to any of its arguments.

The assumption (i) allows to apply Remark 3, and the other two assumptions imply that each $\mathcal{A}$-calculable function is dominated by some function termal in $\mathcal{A}$. All three of them are satisfied if $\mathcal{A}$ is as in Example 1 or Example 2 (even

[2] The simplest choice is to take as $\mathcal{A}$ the set consisting of the adapted interpretations in S of all function symbols of Σ, but sometimes certain of these interpretations can be omitted. For instance we must have in Σ function symbols for the functions $\lambda x.\, x + 1$ and $\lambda x.\, \lfloor x/2 \rfloor$ in order to get the language of the theories studied in [1], but it is not necessary to include the functions themselves in the set $\mathcal{A}$ – this set can be as in Example 2.

if one strengthens the assumptions by replacing "is piecewise termal in" and "is termal in" with "belongs to" and by skipping the phrase "dominated by some function termal in $\mathcal{A}$ and").

Lemma 3. *For any $\mathcal{A}$-calculable function the predicate representing it is definable by means of some bounded formula of $\mathbf{L}_{\Sigma}$.*

Proof. By using the characterization of $\mathcal{A}$-calculability indicated in Remark 3. The domination property discussed above is used in the reasoning about composition for turning some quantifiers into bounded ones. □

Lemma 4. *All predicates definable by means of bounded formulas of $\mathbf{L}_{\Sigma}$ are $\mathcal{A}$-calculable.*

Proof. By induction using properties indicated in the previous section. □

Theorem 1. *A function f is $\mathcal{A}$-calculable if and only if the predicate representing f is definable by means of some bounded formula of $\mathbf{L}_{\Sigma}$ and f is dominated by some function termal in $\mathcal{A}$.*

Proof. By Lemmas 1, 3 and 4. □

Theorem 2. *A predicate in L is $\mathcal{A}$-calculable if and only if it is definable by means of some bounded formula of $\mathbf{L}_{\Sigma}$.*

Proof. The "if"-direction is given by Lemma 4. Suppose now that p is an $\mathcal{A}$-calculable n-argument predicate in L. Let q be the n-argument predicate that is false at $(0,\ldots,0)$ and coincides with p at all other elements of L^n. The predicate q is definable by means of a bounded formula of $\mathbf{L}_{\Sigma}$, since $q(x_1,\ldots,x_n)$ is true if and only if $x_i \neq 0$ and $x_i = p^*(x_1,\ldots,x_n,x_i,0)$ for some i in $\{1,\ldots,n\}$. Then it remains to note that either $p = q$ or p is equal to the disjunction of q and the predicate $\lambda x_1 \ldots x_n.\,(\,(x_1,\ldots,x_n) = (0,\ldots,0)\,)$. □

Remark 8. The assumptions made in this section are still not sufficient for the existence of $\mathcal{A}$-calculable non-zero constant functions (consider for instance the case of $\mathcal{A} = \{\lambda x.\,0,\ \lambda xy.\,x+y\}$). However, if there exists an $\mathcal{A}$-calculable non-zero constant function (for instance if $\mathcal{A}$ is as in Example 1 or Example 2) then the above proof can be replaced by an essentially simpler one. Indeed, if b is the value of such a constant function then the definability of the $\mathcal{A}$-calculable n-argument predicate p can be seen from the equality

$$p \;=\; \lambda x_1 \ldots x_n.\,(0 = p^*(x_1,\ldots,x_n,0,b)\,)\,.$$

References

1. Buss, S. R.: Bounded Arithmetic. Bibliopolis, Naples, 1986
2. Moschovakis, Y. N.: Abstract first order computability. I. Trans. Amer. Math. Soc. **138** (1969) 427–464
3. Skolem, Th.: A theorem on recursively enumerable sets. Abstr. of Short Comm., Int. Congress Math., Stockholm, 1962, p. 11
4. Skordev, D.: On a class of primitive recursive functions. Ann. de l'Univ. de Sofia, Fac. de Math. **60** (1967) 105–111 (in Russian, with English summary)