

EMBEDDING COUNTABLE PARTIAL ORDERINGS IN THE ENUMERATION DEGREES AND THE ω -ENUMERATION DEGREES

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1. INTRODUCTION

One of the most basic measures of the complexity of a given partially ordered structure is the quantity of partial orderings embeddable in this structure. In the structure of the Turing degrees, \mathcal{D}_T , this problem is investigated in a series of results: Mostowski [15] proves that there is a computable partial ordering in which every countable partial ordering can be embedded. Kleene and Post [10] introduce the notion of a computably independent sequence of sets and prove the existence of a countable computably independent sequence of sets $\{A_i\}_{i < \omega}$, so that the Turing degree of every member A_i of this class is uniformly below $0'$. Muchnik [16] proves the existence of a computably independent sequence of computably enumerable sets. Sacks [20] shows that one can embed any computable partial ordering using a computably independent sequence of sets, and as a corollary of the previously mentioned results obtains the embeddability of any countable partial ordering in the structure of the computably enumerable degrees, \mathcal{R} . Finally Robinson [19] generalizes Sacks' Density Theorem [22] by showing that one can embed any countable partial ordering in the computably enumerable degrees between any two given c.e. degrees $\mathbf{b} < \mathbf{a}$. (See Odifreddi [17, 18] for an extensive survey of these results.)

The structure of the enumeration degrees \mathcal{D}_e , which can be seen as an extension of the structure of the Turing degrees \mathcal{D}_T , naturally inherits this complexity. Further results on this topic are obtained by Case [2], who shows that any countable partial ordering can be embedded in the enumeration degrees below the e-degree of any given generic function, and Copestake [5], who shows that one can embed any countable partial ordering in the e-degrees below any given 1-generic enumeration degree. Lagemann [12] proves that the embedding of any countable partial ordering can be obtained below any nonzero Δ_2^0 enumeration degree. Finally the density of the structure of the Σ_2^0 enumeration degrees, \mathcal{G} , proved by Cooper [3] is strengthened by Bianchini [1] who shows that every countable partial ordering can be embedded in any non-empty interval of Σ_2^0 enumeration degrees; see also Sorbi [24] for a published proof of Bianchini's result.

In this article we study the embeddability problem further in the context of three different structures. We start with a slight improvement on the above mentioned embeddability results for the structure of the enumeration degrees. Then build onto our first result to solve the embeddability of countable partial orderings problem for the structure of the ω -enumeration degrees. Finally we apply our second result

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to prove the embeddability of countable partial orderings densely in the structure of the ω -enumeration degrees modulo iterated jump.

Lachlan and Shore introduce in [11] the notion of a good approximation and use it to extend Cooper's density result [3]. We shall call an enumeration degree *good* if it contains a member which has a good approximation.

Theorem 1.1 (Lachlan and Shore [11]). *If $\mathbf{b} < \mathbf{a}$ are enumeration degrees such that \mathbf{a} is good, then there is an enumeration degree \mathbf{c} such that $\mathbf{b} < \mathbf{c} < \mathbf{a}$.*

That every Σ_2^0 set has a good approximation is proved by Jockusch [8]. Lachlan and Shore [11] prove that furthermore all n -c.e.a. sets and all total sets have good approximations, but also provide an example of a Π_2^0 set which does not have a good approximation.

In Section 3.2 we combine the embeddability method via independent sequences together with the notion of a good approximation to prove our first result.

Theorem 1.2. *If $B <_e A$ are sets of natural numbers such that A has a good approximation, then there is an e -independent sequence of sets $\{C_i\}_{i < \omega}$ such that for every i $B <_e C_i <_e A$ uniformly in i .*

As an immediate corollary of this theorem we obtain a more general solution to the embeddability problem for the structure of the enumeration degrees.

Corollary 1.1. *If $\mathbf{b} < \mathbf{a}$ are enumeration degrees such that \mathbf{a} is good then there is an embedding of every countable partial ordering in the interval $[\mathbf{b}, \mathbf{a}]$.*

In particular one can embed any partial ordering in any nonempty interval of Σ_2^0 enumeration degrees, of n -c.e.a. enumeration degrees, or in any nonempty interval with endpoint any total enumeration degree.

In Section 4 we turn our attention to the structure of the ω -enumeration degrees, \mathcal{D}_ω . This structure is an upper semi-lattice with jump operation, where the building blocks of the degrees are of a higher type - sequences of sets of natural numbers. The structure is introduced by Soskov [25] and its properties are investigated in the works of Ganchev and Soskov [6, 7, 26]. We leave various formal definitions for Section 4.1. The main interest in this structure arises from the result that \mathcal{D}_ω is itself an extension of the structure of the enumeration degrees \mathcal{D}_e and furthermore the two structures have isomorphic automorphism groups. Soskov [25] proves a density results for the structure \mathcal{G}_ω of the Σ_2^0 ω -enumeration degrees, the the degrees bounded by the first jump $\mathbf{0}'_\omega$ of the least ω -enumeration degree. We extend the method used in the proof of Theorem 1.2 further to obtain a generalization of Soskov's density theorem:

Theorem 1.3. *Let $\mathbf{b} <_\omega \mathbf{a} <_\omega \mathbf{0}'_\omega$ be two Σ_2^0 ω -enumeration degrees. There is an embedding of every countable partial ordering in the interval $[\mathbf{b}, \mathbf{a}]$.*

In the last section of this article we consider the structure of the Σ_2^0 ω -enumeration degrees modulo iterated jump. The structure of the c.e. degrees modulo iterated jump is introduced and studied by Jockusch, Lerman, Soare and Solovay [9] and Lempp [13].

Definition 1.1 (Jockusch, Lerman, Soare and Solovay). ¹ *Let \mathbf{a} and \mathbf{b} be computably enumerable Turing degrees. $\mathbf{a} \sim_\infty \mathbf{b}$ iff there exists a natural number n such that $\mathbf{a}^n = \mathbf{b}^n$, where \mathbf{a}^n denotes the n -th Turing jump of the degree \mathbf{a} .*

¹The original definition is for a relation \sim_ω between c.e. sets.

This is obviously an equivalence relation on the c.e. Turing degrees and induces a degree structure \mathcal{R}/\sim_∞ with a reducibility relation defined by $[\mathbf{a}]_{\sim_\infty} \leq [\mathbf{b}]_{\sim_\infty}$ if and only if there exists a natural number n such that $\mathbf{a}^n \leq_T \mathbf{b}^n$. This structure has least element $L = \bigcup_{n < \omega} L_n$, the collection of all low $_n$ c.e. degrees, and greatest element $H = \bigcup_{n < \omega} H_n$, the collection of all high $_n$ c.e. degrees. Jockusch, Lerman, Soare and Solovay [9] prove that this is a dense structure. Lempp [13] proves furthermore that there is a splitting of the highest ∞ -degree and a minimal pair of ∞ -degrees.

The method for obtaining a degree structure modulo iterated jump can be applied to any degree structure with jump operation. We can consider for example the structure of all Δ_2^0 Turing degrees modulo iterated jump. However combining Shoenfield's Jump Inversion Theorem [23] with Sacks' Jump Inversion Theorem [21] yields that the range of the jump operator restricted to the c.e. Turing degrees coincides with the range of the jump operator restricted to the Δ_2^0 Turing degrees. It is namely the set of all Turing degrees c.e. in and above $\mathbf{0}'$. Thus the structure of the the Δ_2^0 Turing degrees modulo iterated jump is isomorphic to the structure \mathcal{R}/\sim_∞ .

Next consider the structure of the Σ_2^0 enumeration degrees modulo iterated enumeration jump, \mathcal{G}/\sim_∞ . As noted previously \mathcal{G} can be seen as an extension of the structure of the Δ_2^0 Turing degrees, as there is an embedding ι of the \mathcal{D}_T in \mathcal{D}_e which preserves the order, the least upper bound and the jump operation. The images of the c.e. Turing degrees under this embedding are exactly the Π_1^0 enumeration degrees. McEvoy [14] proves that the range of the enumeration jump operator restricted to the Σ_2^0 -enumeration degrees coincides with the range of the enumeration jump operator restricted to the Π_1^0 enumeration degrees. Thus the structure \mathcal{G}/\sim_∞ is as well isomorphic to the structure \mathcal{R}/\sim_∞ .

When we consider the structure of the Σ_2^0 ω -enumeration degrees modulo iterated jump, $\mathcal{G}_\omega/\sim_\infty$, however we obtain a proper extension of the structure \mathcal{R}/\sim_∞ . Soskov [25] proves that the structure of the enumeration degrees \mathcal{D}_e can be embedded in the structure of the ω -enumeration degrees \mathcal{D}_ω preserving the order, the least upper bound and the jump operation. From this we automatically get an embedding of \mathcal{R}/\sim_∞ as a partial ordering in the structure $\mathcal{G}_\omega/\sim_\infty$. That the image of this embedding is a proper substructure of $\mathcal{G}_\omega/\sim_\infty$ can also be seen easily. In Section 5 we define formally the structure $\mathcal{G}_\omega/\sim_\infty$ and study its properties. The main result is an application of Theorem 1.3.

Theorem 1.4. *If \mathbf{a} and \mathbf{b} are ω -enumeration degrees such that $[\mathbf{b}]_{\sim_\infty} <_\omega [\mathbf{a}]_{\sim_\infty}$, then there is an embedding of every countable partial ordering in the ∞ -degrees between $[\mathbf{b}]_{\sim_\infty}$ and $[\mathbf{a}]_{\sim_\infty}$*

2. THE EMBEDDABILITY METHOD VIA INDEPENDENT SEQUENCES

In this section we will review the embeddability method via independent sequences of sets. Let M be a nonempty set of objects and let \leq be a reflexive and transitive relation on M . Suppose also that for every computable index set C and every sequence $\{A_i\}_{i < \omega}$ of elements in M the operation $\bigoplus_{k \in C} A_k$ is defined and has the following properties:

- (1) If C is a computable index set and $i \in C$ then $A_i \leq \bigoplus_{k \in C} A_k$;
- (2) If $C_1 \subseteq C_2$ are two computable sets then $\bigoplus_{i \in C_1} A_i \leq \bigoplus_{i \in C_2} A_i$.

Definition 2.1. A countable sequence $\{A_i\}_{i < \omega}$ of elements in M is independent with respect to " \leq ", if for every natural number i

$$A_i \not\leq \bigoplus_{j \neq i} A_j.$$

Proposition 2.1 (Sacks). Let $\{A_i\}_{i < \omega}$ be an independent with respect to " \leq " sequence of elements in M and let $\mathcal{C} = \langle \mathbb{N}, \preceq \rangle$ be a computable partial ordering. There is an embedding of \mathcal{C} in $\langle M, \leq \rangle$. If furthermore A and B are elements of M such that B is a lower bound for the sequence $\{A_i\}_{i < \omega}$ and $\bigoplus_{i < \omega} A_i \leq A$ then the embedding is in the interval $[B, A]$.

Proof. The embedding $\kappa : \mathbb{N} \rightarrow M$ is defined as follows:

$$\kappa(i) = \bigoplus_{k \preceq i} A_k.$$

Suppose first that $i \preceq j$. Then sets $C_i = \{k \mid k \preceq i\}$ and $C_j = \{k \mid k \preceq j\}$ are computable and by transitivity of the relation " \preceq " we have that $C_i \subseteq C_j$. By property (2) of the operation \bigoplus it follows that $\kappa(i) \leq \kappa(j)$.

Suppose now that $i \not\preceq j$. Then $i \notin C_j = \{k \mid k \preceq j\}$ and hence $C_j \subseteq \{k \mid k \neq i\}$. On the other hand by reflexivity of " \preceq " we have that $i \in C_i = \{k \mid k \preceq i\}$. Hence assuming that $\kappa(i) \leq \kappa(j)$ leads to a contradiction with the independence of the sequence as follows:

$$A_i \leq \bigoplus_{k \in C_i} A_k = \kappa(i) \leq \kappa(j) = \bigoplus_{k \in C_j} A_k \leq \bigoplus_{k \neq i} A_k.$$

To prove the second part of the proposition we note that for every i we have that $B \leq A_i \leq \kappa(i) \leq \bigoplus_{i < \omega} A_i \leq A$. □

Combining Proposition 2.1 with Mostowski's result [15] we obtain a sufficient condition for the embeddability of any countable partial ordering in any pre-order $\langle M, \leq \rangle$.

We can transform the preorder $\langle M, \leq \rangle$ into a degree structure as follows:

- (1) First we define the equivalence relation \equiv by setting $A \equiv B$ if and only if $A \leq B$ and $B \leq A$ for any elements $A, B \in M$.
- (2) The equivalence class of A under the relation \equiv , denoted by $d(A)$, is called the degree of the element A . We define a reducibility relation " \leq " between degrees by setting $d(A) \leq d(B)$ if and only if $A \leq B$.
- (3) Let $\mathcal{D} = \langle M / \equiv, \leq \rangle$ be the set of all degrees of elements in M with the induced reducibility relation. This structure is a partial ordering.

Finally we note that the existence of an independent sequence in M is a sufficient condition for the embeddability of any countable partial ordering in the degree structure \mathcal{D} . We just modify the embedding κ to $\kappa' : \mathbb{N} \rightarrow \mathcal{D}$, by setting $\kappa'(i) = d(\kappa(i))$.

3. EMBEDDING RESULTS IN THE ENUMERATION DEGREES

3.1. Preliminaries. We assume that the reader is familiar with the notion of enumeration reducibility, and refer to Cooper [4] for a survey of basic results for the

structure of the enumeration degrees. For completeness we will nevertheless outline here basic definitions and properties of the enumeration degrees used in this article.

Intuitively a set of natural numbers B is enumeration reducible (\leq_e) to a set of natural numbers A if one can obtain an enumeration of the set B given any enumeration of the set A . More formally:

Definition 3.1. $B \leq_e A$ if there exists a c.e. set W such that $B = \{n \mid \exists u(\langle x, u \rangle \in W \wedge D_u \subset A)\}$, where D_u denotes the finite set with canonical index u .

The c.e. set W can be viewed as an operator on $\mathcal{P}(\mathbb{N})$ and will be referred to as an enumeration operator or e-operator. The elements of the set W will be called axioms. As each axiom consists of a natural number x and the code u of a finite set D_u , we will denote an axiom by $\langle x, D_u \rangle$.

The relation \leq_e defines a preorder on the powerset of \mathbb{N} . The degree structure \mathcal{D}_e is obtained from $\langle \mathcal{P}(\mathbb{N}), \leq_e \rangle$ using the method described in Section 2. Furthermore we have a definition of the join operation:

Definition 3.2. Let C be a computable set and $\{A_i\}_{i \in C}$ be a class of sets of natural numbers. Then $\bigoplus_{i \in C} A_i = \{\langle i, x \rangle \mid i \in C \wedge x \in A_i\}$.

This operation obviously has the two required properties, needed to apply the embeddability method via independent sequences. In order to construct an independent sequence we will need to use the notion of a good approximation.

Definition 3.3 (Lachlan, Shore [11]). Let $\{A^{\{s\}}\}_{s < \omega}$ be a uniformly computable sequence of finite sets. We say that $\{A^{\{s\}}\}_{s < \omega}$ is a good approximation to the set A if it has the following two properties:

G1: $(\forall n)(\exists s)[A \upharpoonright n \subseteq A^{\{s\}} \subseteq A]$ and

G2: $(\forall n)(\exists s)(\forall t > s)[A^{\{t\}} \subseteq A \Rightarrow A \upharpoonright n \subseteq A^{\{t\}}]$.

Stages s at which $A^{\{s\}} \subseteq A$ are called good stages. The set of good stages will be denoted by G_A .

It is convenient to use the following notion of a correct approximation to a set B with respect to a given good approximation to a set A . Intuitively a correct approximation behaves like a good approximation at good stages $s \in G_A$ of the given one, but might have more good stages $t \notin G_A$ at which we cannot guarantee the property G2.

Definition 3.4. Let $\mathbb{A} = \{A^{\{s\}}\}_{s < \omega}$ be a good approximation to A . A uniformly computable sequence of finite sets $\{B^{\{s\}}\}_{s < \omega}$ is a correct approximation to B with respect to \mathbb{A} if:

C1: $(\forall s)[A^{\{s\}} \subseteq A \Rightarrow B^{\{s\}} \subseteq B]$ and

C2: $(\forall n)(\exists s)(\forall t > s)[A^{\{t\}} \subseteq A \Rightarrow B \upharpoonright n \subseteq B^{\{t\}}]$.

This definition arises naturally when we approximate a set enumeration reducible to a given set with a good approximation. We prove a slightly more general statement:

Lemma 3.1. Let $\mathbb{A} = \{A^{\{s\}}\}_{s < \omega}$ be a good approximation to A . Let $\{\Gamma^{\{s\}}\}_{s < \omega}$ be a Σ_1^0 -approximation to the enumeration operator Γ and let $\mathbb{B} = \{B^{\{s\}}\}_{s < \omega}$ be a correct with respect to \mathbb{A} approximation to B . Then $\{\Gamma^{\{s\}}(B^{\{s\}})\}_{s < \omega}$ is a correct approximation to $\Gamma(B)$ with respect to \mathbb{A} .

Proof. C1: Let s be a stage such that $A^{\{s\}} \subseteq A$. Then by C1 for \mathbb{B} we have $B^{\{s\}} \subseteq B$ and by the properties of a Σ_1^0 approximation we have that $\Gamma^{\{s\}} \subseteq \Gamma$. Hence $\Gamma^{\{s\}}(B^{\{s\}}) \subseteq \Gamma(B)$.

C2: If $x \in \Gamma(B)$ then there is a valid axiom $\langle x, D \rangle \in \Gamma$. It follows from property C2 of the approximation \mathbb{B} for the element $\max(D)$ and the properties of a Σ_1^0 approximation that there is a stage s such that at all stages $t > s$ if $t \in G_A$ then $D \subset B^{\{t\}}$ and $\langle x, D \rangle \in \Gamma^{\{t\}}$ and hence $x \in \Gamma^{\{t\}}(B^{\{t\}})$. If on the other hand $x \notin \Gamma(B)$ then at all stages $t \in G_A$, $x \notin \Gamma^{\{t\}}(B^{\{t\}})$. \square

Lemma 3.2. *Let $\mathbb{A} = \{A^{\{s\}}\}_{s < \omega}$ be a good approximation to A . Let $\{B^{\{s\}}\}_{s < \omega}$ and $\{C^{\{s\}}\}_{s < \omega}$ be two correct with respect to \mathbb{A} approximations to B and C respectively. Then $\{B^{\{s\}} \oplus C^{\{s\}}\}_{s < \omega}$ is a correct approximation to $B \oplus C$ with respect to \mathbb{A} .*

Proof. C1: Let s be a stage such that $A^{\{s\}} \subseteq A$. Then by C1 we have $B^{\{s\}} \subseteq B$ and $C^{\{s\}} \subseteq C$, hence $B^{\{s\}} \oplus C^{\{s\}} \subseteq B \oplus C$.

C2: Fix n . Let s_B be a stage such that at all good stages $t > s_B$ we have $B \upharpoonright n \subseteq B^{\{s\}}$ and similarly let s_C be such that at all good stages $t > s_C$ we have $C \upharpoonright n \subseteq C^{\{s\}}$. Then at all good stages $t > \max\{s_B, s_C\}$ we have that $B \oplus C \upharpoonright n = (B \upharpoonright n \oplus C \upharpoonright n) \upharpoonright n = (B^{\{s\}} \upharpoonright n \oplus C^{\{s\}} \upharpoonright n) \upharpoonright n = B^{\{s\}} \oplus C^{\{s\}} \upharpoonright n$. \square

We will use the notion of a length of agreement function in the constructions and a basic property of this notion.

Definition 3.5. *The length of agreement between two sets A and B measured at stage s is $l(A, B, s) = \max\{u \leq s \mid \forall x < u [A(x) = B(x)]\}$.*

Lemma 3.3. *Let $\mathbb{A} = \{A^{\{s\}}\}_{s < \omega}$ be a good approximation to the set A and $\{B^{\{s\}}\}_{s < \omega}$ and $\{C^{\{s\}}\}_{s < \omega}$ be two correct with respect to \mathbb{A} approximations to sets B and C . Denote by $l_s = l(B^{\{s\}}, C^{\{s\}}, s)$.*

- (1) *If l_s grows unboundedly at good stages $s \in G_A$, i.e. for every n there is a stage $s \in G_A$ such that $l_s > n$, then $B = C$.*
- (2) *If $B = C$ then $\lim_{s \in G_A} l_s = \infty$, i.e. for every n there is a stage s such that at all good stages $t > s$, $t \in G_A$ we have $l_t > n$.*

Proof. Fix x . Let s_B be a stage such that $\forall t > s_B (t \in G_A \Rightarrow B \upharpoonright x+1 \subseteq B^{\{t\}} \subseteq B)$ and s_C be a stage such that $\forall t > s_C (t \in G_A \Rightarrow C \upharpoonright x+1 \subseteq C^{\{t\}} \subseteq C)$.

To prove (1) suppose that l_s grows unboundedly at good stages $s \in G_A$. Let $s > s_B, s_C$ be a stage such that $s \in G_A$ and $l_s > x$. Then $B(x) = B^{\{s\}}(x) = C^{\{s\}}(x) = C(x)$.

For (2) suppose that $B = C$. Then at all good stages $s > s_B, s_C, x$, such that $s \in G_A$ we have that $B^{\{s\}} \upharpoonright x+1 = B \upharpoonright x+1 = C \upharpoonright x+1 = C^{\{s\}} \upharpoonright x+1$ and hence $l_s > x$. \square

Finally we introduce one more notation:

Definition 3.6. *Let A be a set of natural numbers and i be a natural number:*

- (1) $A^{[i]} = \{\langle i, x \rangle \mid \langle i, x \rangle \in A\}$;
- (2) For $R \in \{\leq, <, \geq, >\}$ we set $A^{[Ri]} = \{\langle j, x \rangle \mid \langle j, x \rangle \in A \wedge (jRi)\}$.
- (3) $A[i] = \{x \mid \langle i, x \rangle \in A\}$.

3.2. Proof of theorem 1.2. Fix two sets of natural numbers $B <_e A$, such that A has a good approximation $\mathbb{A} = \{A^{\{s\}}\}_{s < \omega}$. As $B <_e A$ it follows that there is an operator Γ such that $B = \Gamma(A)$. Denote by $B^{\{s\}} = \Gamma^{\{s\}}(A^{\{s\}})$. Then by Lemma 3.1 $\mathbb{B} = \{B^{\{s\}}\}_{s < \omega}$ is a correct approximation to B with respect to \mathbb{A} .

We will construct an enumeration operator V , and define $A_i = V(A)[i]$. We will ensure that for every i we have that $B \oplus A_i \not\leq_e \bigoplus_{j \neq i} B \oplus A_j$. Then by setting $C_i = B \oplus A_i$ we will obtain the required e -independent sequence of sets. Indeed for every i we have that $A_i \leq_e A$ uniformly in i , hence $B \leq_e C_i \leq_e A$ uniformly in i . Furthermore it follows from the independence property that the sequence $\{C_i\}_{i < \omega}$ is an antichain with respect to e -reducibility, and hence we have strong inequalities: $B <_e C_i <_e A$.

To simplify the requirements one step further we note that $\bigoplus_{j \neq i} B \oplus A_j \equiv_e B \oplus \bigoplus_{j \neq i} A_j$ and that it is sufficient to prove that $A_i \not\leq_e B \oplus \bigoplus_{j \neq i} A_j$ as $A_i \leq_e B \oplus A_i$. Thus our requirements can be stated as follows:

$$\mathcal{P}_{e,i} : A_i \not\leq_e W_e(B \oplus \bigoplus_{j \neq n} A_j),$$

for every pair of natural numbers e, i .

Fix some computable linear ordering $\mathcal{R}_0 < \mathcal{R}_1 < \dots$ of the requirements $\mathcal{P}_{e,n}$. As usual we say that requirements which appear in earlier positions in this ordering have higher priority.

Approximations and conventions We will construct a Σ_1^0 approximation $\{V^{\{s\}}\}_{s < \omega}$ to the set V .

For every i and s let $A_i^{\{s\}} = V^{\{s\}}(A^{\{s\}})[i]$. Note that $A_i = V_i(A)$, where $V_i = \{\langle x, D \rangle \mid \langle \langle i, x \rangle, D \rangle \in V\}$ and $A_i^{\{s\}} = V_i^{\{s\}}(A^{\{s\}})$, where $V_i^{\{s\}} = \{\langle x, D \rangle \mid \langle \langle i, x \rangle, D \rangle \in V^{\{s\}}\}$. As $\{V_i^{\{s\}}\}_{s < \omega}$ is a Σ_1^0 approximation to V_i , it follows from Lemma 3.1 that $\{A_i^{\{s\}}\}_{s < \omega}$ is a correct with respect to \mathbb{A} approximation to A_i .

To keep notation simple we will introduce a convention, an abbreviation of a certain action that will be used in the construction.

Convention 1: *At stage s we will use the phrase “Enumerate z in A_i ” as an abbreviation of the action “Enumerate the axiom $\langle \langle i, z \rangle, A^{\{s\}} \rangle$ in the operator $V^{\{s+1\}}$ ”.*

The discussion above shows that the effect of the abbreviated action is exactly its abbreviation, provided that the stage s is good.

Furthermore for every i we have that $\bigoplus_{j \neq i} A_j = U_i(A)$, where $U_i = \{\langle \langle j, x \rangle, D \rangle \mid j \neq i \wedge \langle \langle j, x \rangle, D \rangle \in V\}$. It follows that for every s $\bigoplus_{j \neq i} A_j^{\{s\}} = U_i^{\{s\}}(A^{\{s\}})$, where $U_i^{\{s\}} = \{\langle \langle j, x \rangle, D \rangle \mid j \neq i \wedge \langle \langle j, x \rangle, D \rangle \in V^{\{s\}}\}$ and hence by Lemma 3.1 $\{\bigoplus_{j \neq i} A_j^{\{s\}}\}_{s < \omega}$ is as well a correct with respect to \mathbb{A} approximation to $\bigoplus_{j \neq i} A_j$.

Finally we note that for every i by Lemma 3.2 we get that $\{B^{\{s\}} \oplus \bigoplus_{j \neq i} A_j^{\{s\}}\}_{s < \omega}$ is a correct with respect to \mathbb{A} approximation to $B \oplus \bigoplus_{j \neq i} A_j$. For every e by Lemma 3.1 it follows that $\{W_e^{\{s\}}(B^{\{s\}} \oplus \bigoplus_{j \neq i} A_j^{\{s\}})\}_{s < \omega}$ is a correct with respect to \mathbb{A} approximation to $W_e(B \oplus \bigoplus_{j \neq i} A_j)$.

Construction. The construction is in stages. Let $V^{\{0\}} = \emptyset$. At stage $s \geq 0$ we construct $V^{\{s+1\}}$ from its value constructed at the previous stage, $V^{\{s\}}$, by allowing certain requirements to enumerate new axioms.

We consider all requirements \mathcal{R}_k , where $k < s$ and for each in order of priority we make the following actions:

Let $\mathcal{R}_k = \mathcal{P}_{e,i}$. Define $l_k^{\{s\}} = l(W_e^{\{s\}}(B^{\{s\}} \oplus \bigoplus_{j \neq i} A_j^{\{s\}}), A_i^{\{s\}}, s)$. For every $x < l_k^{\{s\}}$:

- If $x \in A^{\{s\}}$ but $\langle k, x \rangle \notin A_i^{\{s\}}$ then enumerate $\langle k, x \rangle$ in A_i .
- If $x \notin W_e^{\{s\}}(B^{\{s\}} \oplus \bigoplus_{j \neq i} A_j^{\{s\}})$ and there is a finite set $L = L_B \oplus \bigoplus_{j \neq i} L_j$ such that $\langle x, L \rangle \in W_e^{\{s\}}$, $L_B \subseteq B^{\{s\}}$ and for every $j \neq i$ we have $L_j^{\{\leq k\}} \subseteq A_j^{\{s\}}$ then for every $j \neq i$ and every $y \in L_j^{\{>k\}}$ enumerate y in A_j .

This completes the construction.

Lemma 3.4. *For every $k < \omega$:*

- (1) \mathcal{R}_k is satisfied.
- (2) The actions for \mathcal{R}_k enumerate finitely many axioms in V at good stages of the construction.

Proof. We will prove both statements of the lemma simultaneously by induction. Assume that both statements of the lemma are true for $j < k$ and consider $\mathcal{R}_k = \mathcal{P}_{e,i}$. Towards a contradiction assume that $W_e(B \oplus \bigoplus_{j \neq i} A_j) = A_i$. As established we are dealing with correct with respect to \mathbb{A} approximations to $W_e(B \oplus \bigoplus_{j \neq i} A_j)$ and A_i , so by Lemma 3.3 we have that $\lim_{s \in G_A} l_k^{\{s\}} = \infty$. We will prove that in this case $A \leq_e A_i$ and $W_e(B \oplus \bigoplus_{j \neq i} A_j) \leq_e B$. This would yield a contradiction as $A \leq_e A_i = W_e(B \oplus \bigoplus_{j \neq i} V_j(A)) \leq_e B$ and by assumption $A \not\leq_e B$.

Claim 3.4.1. $A \leq_e A_i$

Proof. We will prove that $A =^* A_i[k]$. The only requirements other than \mathcal{R}_k that can enumerate elements $\langle k, x \rangle$ in A_i , i.e. enumerate axioms of the form $\langle \langle i, \langle k, x \rangle \rangle, A^{\{s\}} \rangle$ in $V^{\{s+1\}}$ at good stages $s \in G_A$ are \mathcal{R}_j where $j < k$. By the induction hypothesis each such requirement enumerates only finitely many axioms in V at good stages $s \in G_A$. Hence the set F of elements $\langle k, x \rangle \in A_i$, which were not enumerated in A_i by \mathcal{R}_k is a finite.

Fix $x \in A$. There is a stage $s \in G_A$ such that $x < l_k^{\{s\}}$ and $x \in A^{\{s\}}$. If $\langle k, x \rangle \in A_i^{\{s\}}$ then $\langle k, x \rangle \in A_i$ by property C1 of the correct approximation $\{A_i^{\{s\}}\}_{s < \omega}$. If $\langle k, x \rangle \notin A_i^{\{s\}}$ then the actions of \mathcal{R}_k at stage s would enumerate $\langle k, x \rangle$ in A_i , i.e. would enumerate the axiom $\langle \langle i, \langle k, x \rangle \rangle, A^{\{s\}} \rangle$ in $V^{\{s+1\}}$ and as $A^{\{s\}} \subseteq A$ we will have that $\langle i, \langle k, x \rangle \rangle \in V(A)$, hence $\langle k, x \rangle \in V(A)[i] = A_i$. Thus $A \subseteq A_i[k]$.

For the converse side fix x such that $\langle k, x \rangle \in A_i \setminus F$. It follows that $\langle k, x \rangle$ is enumerated in A_i by \mathcal{R}_k at a good stage s . By construction this is possible only if $x \in A^{\{s\}} \subseteq A$. Thus $(A_i \setminus F)[k] \subseteq A$ which proves the claim. \square

Claim 3.4.2. $W_e(B \oplus \bigoplus_{j \neq i} A_j) \leq_e B$.

Proof. We will prove that there is a finite set F such that $W_e(B \oplus \bigoplus_{j \neq i} A_j) = W_e(B \oplus (F \cup \bigoplus_{j \neq i} \mathbb{N}^{[>k]}))$. As $F \cup \bigoplus_{j \neq i} \mathbb{N}^{[>k]}$ is a computable set it follows that $W_e(B \oplus (F \cup \bigoplus_{j \neq i} \mathbb{N}^{[>k]})) \leq_e B$.

Let $F = \bigoplus_{j \neq i} A_j^{[\leq k]}$. If $\langle j, x \rangle \in F$ then $j \neq i$, $x \in A_j^{[\leq k]}$ and x was enumerated in A_j at a good stage $t \in G_A$. The only requirements that may enumerate elements $y \in \mathbb{N}^{[\leq k]}$ in A_j are \mathcal{R}_l where $l < k$. Indeed \mathcal{R}_k enumerates in A_j only axioms for elements $y \in \mathbb{N}^{[> k]}$ and every lower priority requirement \mathcal{R}_m , $m > k$ enumerates in A_j only elements $y \in \mathbb{N}^{[\geq m]}$. By the induction hypothesis F is finite.

Note that $\bigoplus_{j \neq i} A_j = \bigoplus_{j \neq i} A_j^{[\leq k]} \cup \bigoplus_{j \neq i} A_j^{[> k]}$. Thus $B \oplus \bigoplus_{j \neq i} A_j \subseteq B \oplus (F \cup \bigoplus_{j \neq i} \mathbb{N}^{[> k]})$. Hence by the monotonicity of the enumeration operators we get automatically the first inclusion, namely $W_e(B \oplus \bigoplus_{j \neq i} A_j) \subseteq W_e(B \oplus (F \cup \bigoplus_{j \neq i} \mathbb{N}^{[> k]}))$.

Let $x \in W_e(B \oplus (F \cup \bigoplus_{j \neq i} \mathbb{N}^{[> k]}))$. Then there is an axiom $\langle x, M \rangle \in W_e$ such that $M = M_B \oplus \bigoplus_{j \neq n} M_j$, $M_B \subseteq B$ and for every j , $M_j^{[\leq k]} \subseteq F$. Consider a stage $s \in G_A$ such that:

- $s > k$;
- $\langle x, M \rangle \in W_e^{\{s\}}$;
- $B \upharpoonright \max M_B + 1 \subseteq B^{\{s\}}$;
- $\bigoplus_{j \neq i} A_j \upharpoonright \max F + 1 \subseteq \bigoplus_{j \neq i} A_j^{\{s\}}$ (the choice of s is possible by property C2 of the correct approximation $\{\bigoplus_{j \neq i} A_j^{\{s\}}\}_{s < \omega}$);
- $x < l_k^{\{s\}}$.

Now consider the actions of \mathcal{R}_k at stage s . If $x \in W_e^{\{s\}}(B^{\{s\}} \oplus \bigoplus_{j \neq i} A_j^{\{s\}})$ then $x \in W_e(B \oplus \bigoplus_{j \neq i} A_j)$ because $s \in G_A$ and by property C1 of the correct approximation $\{W_e^{\{s\}}(B^{\{s\}} \oplus \bigoplus_{j \neq i} A_j^{\{s\}})\}_{s < \omega}$.

If $x \notin W_e^{\{s\}}(B^{\{s\}} \oplus \bigoplus_{j \neq i} A_j^{\{s\}})$ then there is an axiom $\langle x, L \rangle$ in $W_e^{\{s\}}$ such that $L = L_B \oplus \bigoplus_{j \neq i} L_j$, $L_B \subseteq B^{\{s\}}$ and for every j we have $L_j^{[\leq k]} \subseteq A_j^{\{s\}}$, namely $\langle x, M \rangle$. The actions of \mathcal{R}_k will select such an axiom, say $\langle x, L \rangle$ and will enumerate in A_j every $y \in L_j^{[> k]}$, making the axiom $\langle x, L \rangle$ valid. Ultimately we get that $x \in W_e(B \oplus \bigoplus_{j \neq i} A_j)$ and establish the second inclusion. \square

The assumption that $W_e(B \oplus \bigoplus_{j \neq i} A_j) = A_i$ leads to a contradiction and hence is wrong. This establishes the first statement of the lemma. From this and by Lemma 3.3 it follows that there is a natural number l such that for all s , $s \in G_A$ we have $l_k^{\{s\}} \leq l$.

Suppose \mathcal{R}_k enumerates in $V^{\{s+1\}}$ an axiom for an element z at a good stage s .

If this action is performed under the first point in the construction, then $z = \langle i, k, x \rangle$, where $x < l$. Furthermore as the stage is good $\langle k, x \rangle \in A_i$. By property C2 of the correct approximation to A_i there will be a stage s_1 such that $\langle k, x \rangle \in A_i^{\{t\}}$ at all good stages $t > s_1$, and \mathcal{R}_k will not enumerate any more axioms for z in V . There are finitely many possible choices for z , hence finitely many axioms are enumerated in V at good stages $s \in G_A$ under the first point of the construction.

If the axiom is enumerated under the second point of the construction then this is on account of some $x < l$ such that $x \notin W_e^{\{s\}}(B^{\{s\}} \oplus \bigoplus_{j \neq i} A_j^{\{s\}})$, for which there is an appropriate axiom. The actions of \mathcal{R}_k then enumerate finitely many axioms in the operator $V^{\{s+1\}}$, enumerating the element x in $W_e^{\{s\}}(B^{\{s\}} \oplus \bigoplus_{j \neq i} A_j^{\{s+1\}}) \subseteq W_e(B \oplus \bigoplus_{j \neq i} A_j)$. By the properties of a correct approximation there is a stage s_1

such that at all good stages $t > s_1$ we have that $x \in W_e^{\{t\}}(B^{\{t\}} \oplus \bigoplus_{j \neq i} A_j^{\{t\}})$ and no further axioms will be enumerated in V an account of x .

Thus altogether \mathcal{R}_k enumerates finitely many axioms in V at good stages of the approximation \mathbb{A} . □

4. EMBEDDING RESULTS IN THE ω -ENUMERATION DEGREES

4.1. Preliminaries. Soskov [25] introduces a reducibility, \leq_ω , between sequences of sets of natural numbers. The original definition involves the so called *jump set* of a sequence and can be found in [25]. We use an equivalent definition in terms of operators which is more approachable, as it resembles the definition of e -reducibility. Before we define ω -reducibility we will need to introduce two more notations. Let \mathcal{S} denote the class of all sequences of sets of natural numbers of length ω . With every member $\mathcal{A} \in \mathcal{S}$ we connect a jump sequence $P(\mathcal{A})$.

Definition 4.1. *Let $\mathcal{A} = \{A_n\}_{n < \omega} \in \mathcal{S}$. The jump sequence of the sequence \mathcal{A} , denoted by $P(\mathcal{A})$ is the sequence $\{P_n(\mathcal{A})\}_{n < \omega}$ defined inductively as follows:*

- $P_0(\mathcal{A}) = A_0$.
- $P_{n+1}(\mathcal{A}) = A_{n+1} \oplus P'_n(\mathcal{A})$, where $P'_n(\mathcal{A})$ denotes the enumeration jump of the set $P_n(\mathcal{A})$.

The jump sequence $P(\mathcal{A})$ transforms a sequence \mathcal{A} into a monotone sequence of sets of natural numbers with respect to \leq_e . Every member of the jump sequence contains full information on previous members.

Next we extend the notion of an e -operator so that it can be applied to members of \mathcal{S} .

Definition 4.2. *Let $\mathcal{A} = \{A_n\}_{n < \omega}$ be a sequence of sets natural numbers and V be an e -operator. The result of applying the enumeration operator V to the sequence \mathcal{A} , denoted by $V(\mathcal{A})$, is the sequence $\{V[n](A_n)\}_{n < \omega}$. We say that $V(\mathcal{A})$ is enumeration reducible (\leq_e) to the sequence \mathcal{A} .*

Intuitively enumeration reducibility extended to \mathcal{S} combines two notions. The first one is enumeration reducibility between members of the sequence: the n -th member of the sequence $V(\mathcal{A})$ is enumeration reducible to the n -th member of \mathcal{A} via an enumeration operator V_n . The second notion is uniformity: the sequence of enumeration operators $\{V_n\}_{n < \omega}$ is uniform.

The motivation behind the definition of ω -reducibility is an attempt to capture the information content of a set of natural numbers together with all of its enumeration jumps. It turns out that e -reducibility between sequences of sets is too strong for this purpose.

Definition 4.3. *Let $\mathcal{A}, \mathcal{B} \in \mathcal{S}$. We shall say that \mathcal{B} is ω -enumeration reducible to \mathcal{A} , denoted by $\mathcal{B} \leq_\omega \mathcal{A}$, if $\mathcal{B} \leq_e P(\mathcal{A})$.*

Clearly “ \leq_ω ” is a reflexive and transitive relation and defines a preorder on \mathcal{S} . The degree structure obtained from \leq_ω by the standard method described in Section 2 is the structure of the ω -enumeration degrees, $\mathcal{D}_\omega = \langle \{d_\omega(\mathcal{A}) \mid \mathcal{A} \in \mathcal{S}\}, \leq_\omega \rangle$. This is a partial ordering with least element $\mathbf{0}_\omega$ the degree of the sequence \emptyset_ω , where all members of the sequence \emptyset_ω are equal to \emptyset or equivalently the degree of the sequence $\{\emptyset^n\}_{n < \omega}$.

Given two sequences $\mathcal{A} = \{A_n\}_{n < \omega}$ and $\mathcal{B} = \{B_n\}_{n < \omega}$ let $\mathcal{A} \oplus \mathcal{B} = \{A_n \oplus B_n\}_{n < \omega}$. Is it easy to see that $d_\omega(\mathcal{A} \oplus \mathcal{B})$ is the least upper bound of $d_\omega(\mathcal{A})$ and $d_\omega(\mathcal{B})$ and hence \mathcal{D}_ω is an upper semi-lattice. The operation \oplus can be extended as in Definition 3.2 to any computable class of sequences of sets as follows.

Definition 4.4. *Let C be a computable set and $\{\mathcal{A}_i\}_{i \in C}$ be a class of sequences of set, where for every $i \in C$, $\mathcal{A}_i = \{A_{i,n}\}_{n < \omega}$. Then*

$$\bigoplus_{i \in C} \mathcal{A}_i = \left\{ \bigoplus_{i \in C} A_{i,n} \right\}_{n < \omega}.$$

Finally we define a jump operation: for every sequence \mathcal{A} , let $d_\omega(\mathcal{A})' = d_\omega(\mathcal{A}')$, where $\mathcal{A}' = \{P_{n+1}(\mathcal{A})\}_{n < \omega}$.

The structure of the ω -enumeration degrees as an upper semilattice with jump operation, $\langle \mathcal{D}_\omega, \leq_\omega, \vee, ' \rangle$, can be seen as an extension of the structure of the enumeration degrees $\langle \mathcal{D}_e, \leq_e, \vee, ' \rangle$. Let A be any set of natural numbers and let $\mathcal{A} = \{A_n\}_{n < \omega}$ be the sequence defined by $A_0 = A$ and $A_{n+1} = \emptyset$. Then define

$$\kappa(d_e(A)) = \mathcal{A}.$$

The embedding κ preserves the order, the least upper bound and the jump operation. The images of the enumeration degrees under the embedding κ forms a substructure of the ω -enumeration degrees, which we will denote by \mathcal{D}_1 . In [26] Soskov and Ganchev prove that the structure \mathcal{D}_1 is first order definable in \mathcal{D}_ω .

The jump operation gives rise to the local structure of the ω -enumeration degrees \mathcal{G}_ω , consisting of all ω -enumeration degrees below the first jump of the least degree. We will call these degrees Σ_2^0 ω -enumeration degrees. It is not difficult to check that every degree $\mathbf{a} \leq \mathbf{0}'_\omega$ contains a member $\mathcal{A} = \{A_n\}_{n < \omega}$, such that for every n the set A_n is $\Sigma_2^0(\emptyset^n)$. This structure is itself an extension of the local structure of the Σ_2^0 enumeration degrees, \mathcal{G} .

In [25] Soskov proves that \mathcal{G}_ω is a dense structure and that its elements admit a generalized notion of a good approximation, which we will use in the proof of Theorem 3.2.

Definition 4.5. *Let $\{A_n^{\{s\}}\}_{n,s < \omega}$ be a uniformly computable matrix of finite sets. We say that $\{A_n^{\{s\}}\}_{s < \omega}$ is a good approximation to the sequence $\mathcal{A} = \{A_n\}_{n < \omega}$ if:*

$$GS0: (\forall s)(\forall k)[A_k^{\{s\}} \subseteq A_k \Rightarrow (\forall m \leq k)[A_m^{\{s\}} \subseteq A_m]];$$

$$GS1: (\forall n)(\forall k)(\exists s)(\forall m \leq k)[A_m \upharpoonright n \subseteq A_m^{\{s\}} \subseteq A_m] \text{ and}$$

$$GS2: (\forall n)(\forall k)(\exists s)(\forall t > s)[A_k^{\{t\}} \subseteq A_k \Rightarrow (\forall m \leq k)[A_m \upharpoonright n \subseteq A_m^{\{t\}}]].$$

Stages s at which $A_k^{\{s\}} \subseteq A_k$ are called k -good stages.

This definition essentially says that we have a good approximation to every set in the given sequences, which are coordinated in a certain way, namely it is straightforward to check that the following holds:

Proposition 4.1. *Let $\mathbb{A} = \{A_i^{\{s\}}\}_{i,s < \omega}$ be a uniformly computable matrix of finite sets and for every i denote by $\mathbb{A}_i = \{A_i^{\{s\}}\}_{s < \omega}$. Then \mathbb{A} is a good approximation to the sequence $\mathcal{A} = \{A_n\}_{n < \omega}$ if and only if:*

- (1) *For every i the sequence \mathbb{A}_i is a good approximation to the set A_i .*
- (2) *For every $i < j$ we have that \mathbb{A}_i is a correct approximation to A_i with respect to the approximation \mathbb{A}_j*

We have the corresponding notion of a correct approximation to a sequence with respect to a given good approximation:

Definition 4.6. Let $\mathcal{A} = \{A_i\}_{i < \omega}$ and $\mathcal{B} = \{B_i\}_{i < \omega}$ be sequences of sets of natural numbers and let $\mathbb{A} = \{A_i^{\{s\}}\}_{i, s < \omega}$ be a good approximation to \mathcal{A} . A uniform matrix $\{B_i^{\{s\}}\}_{i, s < \omega}$ of finite sets is a correct (with respect to \mathbb{A}) approximation to \mathcal{B} if the following two conditions hold:

CS1: $(\forall s, k)[A_k^{\{s\}} \subseteq A_k \Rightarrow (\forall m \leq k)B_m^{\{s\}} \subseteq B_m]$ and

CS2: $(\forall n, k)(\exists s)(\forall t > s)[A_k^{\{s\}} \subseteq A_k \Rightarrow (\forall m \leq k)[B_m \upharpoonright n \subseteq B_m^{\{t\}}]]$.

Here as well we can restate this definition in a more approachable form:

Proposition 4.2. Let $\mathcal{A} = \{A_n\}_{n < \omega}$ and $\mathcal{B} = \{B_n\}_{n < \omega}$ be two sequences of sets of natural numbers. Let $\mathbb{A} = \{A_n^{\{s\}}\}_{n, s < \omega}$ be a good approximation to \mathcal{A} and let $\mathbb{B} = \{B_{n,s}\}_{n, s < \omega}$ be a uniform matrix of finite sets. Then \mathbb{B} is correct with respect to \mathbb{A} if and only if for every n the sequence $\mathbb{B}_n = \{B_n^{\{s\}}\}_{s < \omega}$ is correct with respect to the good approximation $\mathbb{A}_n = \{A_n^{\{s\}}\}_{s < \omega}$.

The proof of this fact is also straightforward, but rather technical and we omit it. We can use it to transfer certain properties of the good approximations of sets to the setting of sequences of sets as follows.

Lemma 4.1. Let $\mathbb{A} = \{A_n^{\{s\}}\}_{n, s < \omega}$ be a good approximation to the sequence $\mathcal{A} = \{A_n\}_{n < \omega}$. Let $\{\Gamma^{\{s\}}\}_{s < \omega}$ be a Σ_1^0 -approximation to the enumeration operator Γ and let $\mathbb{B} = \{B_n^{\{s\}}\}_{n, s < \omega}$ be a correct with respect to \mathbb{A} approximation to the sequence $\mathcal{B} = \{B_n\}_{n < \omega}$. Then $\{\Gamma[n]^{\{s\}}(B_n^{\{s\}})\}_{n, s < \omega}$ is a correct approximation to $\Gamma(\mathcal{B})$ with respect to \mathbb{A} .

Proof. This follows immediately from Lemma 3.1 using the equivalent notion of a correct approximation given in Proposition 4.2. Namely by Lemma 3.1 have that that for every n $\{\Gamma[n]^{\{s\}}(B_n^{\{s\}})\}_{s < \omega}$ is a correct approximation with respect to $\{A_n^{\{s\}}\}_{s < \omega}$ to the set $\Gamma[n](B_n)$ which in turn gives us by Proposition 4.2 that $\{\Gamma[n]^{\{s\}}(B_n^{\{s\}})\}_{n, s < \omega}$ is a correct approximation to $\{\Gamma[n](B_n)\}_{n < \omega} = \Gamma(\mathcal{B})$ with respect to \mathbb{A} . \square

Lemma 4.2. Let $\mathbb{A} = \{A_n^{\{s\}}\}_{n, s < \omega}$ be a good approximation to the sequence $\mathcal{A} = \{A_n\}_{n < \omega}$. Let $\mathbb{B} = \{B_n^{\{s\}}\}_{n, s < \omega}$ and $\mathbb{C} = \{C_n^{\{s\}}\}_{n, s < \omega}$ be correct with respect to \mathbb{A} approximations to the sequences $\mathcal{B} = \{B_n\}_{n < \omega}$ and $\mathcal{C} = \{C_n\}_{n < \omega}$ respectively. Then $\{B_n^{\{s\}} \oplus C_n^{\{s\}}\}_{n, s < \omega}$ is a correct approximation to $\mathcal{B} \oplus \mathcal{C}$ with respect to \mathbb{A} .

Proof. Follows immediately from Lemma 3.2 using the equivalent notion of a correct approximation given in Proposition 4.2. \square

We can also transfer the notion of a length of agreement function to the setting of sequences of sets.

Definition 4.7. Let $\mathcal{A} = \{A_n\}_{n < \omega}$ and $\mathcal{B} = \{B_n\}_{n < \omega}$ be two sequences of sets of natural numbers. We define the length of agreement function at stage s as $l(\mathcal{A}, \mathcal{B}, s) = \max\{\langle n, x \rangle < s \mid (\forall m \leq n)(\forall y \leq x)(A_m(y) = B_m(y))\}$.

Lemma 4.3. *Let \mathbb{A} be a good approximation to the sequence \mathcal{A} and let $\mathbb{B} = \{B_n^{\{s\}}\}_{s,n < \omega}$ and $\mathbb{C} = \{C_n^{\{s\}}\}_{n,s < \omega}$ be two correct with respect to \mathbb{A} approximations the sequences \mathcal{B} and \mathcal{C} .*

Let G_A^k denote the set of k -good stages in the approximation \mathbb{A} and let $l^{\{s\}} = l(\mathcal{B}^{\{s\}}, \mathcal{C}^{\{s\}}, s)$, where $\mathcal{B}^{\{s\}} = \{B_n^{\{s\}}\}_{s < \omega}$ and $\mathcal{C}^{\{s\}} = \{C_n^{\{s\}}\}_{s < \omega}$. Then

- (1) *For every k if $l^{\{s\}}$ grows unboundedly at k -good stages $s \in G_A^k$ then $B_k = C_k$.*
- (2) *If $l^{\{s\}}$ grows unboundedly at k -good stages s of the approximation for every k then $\mathcal{B} = \mathcal{C}$.*
- (3) *If $\mathcal{B} = \mathcal{C}$ then $\lim_{s \in G_A^k} l^{\{s\}} = \infty$ for every $k < \omega$.*

Proof. Part (1) follows from Lemma 3.3, as for every k and every x if $\langle k, x \rangle \leq l(\mathcal{B}^{\{s\}}, \mathcal{C}^{\{s\}}, s)$ then $x \leq l(B_k^{\{s\}}, C_k^{\{s\}}, s)$ and on the other hand $\{B_k^{\{s\}}\}_{s < \omega}$ and $\{C_k^{\{s\}}\}_{s < \omega}$ are both correct with respect to the good approximation $\{A_k^{\{s\}}\}_{s < \omega}$. Hence if $l^{\{s\}}$ grows unboundedly at k -good stages $s \in G_A^k$ then $l(C_k^{\{s\}}, B_k^{\{s\}}, s)$ also grows unboundedly at good stages $s \in G_{A_k}$ of the good approximation $\{A_k^{\{s\}}\}_{s < \omega}$ and hence $C_k = B_k$.

If (1) is true for every $k < \omega$ then by the first part of the proposition we have that for every k , $C_k = B_k$ and hence $\mathcal{B} = \mathcal{C}$, proving part (2).

Now suppose that $\mathcal{B} = \mathcal{C}$. Fix k and a natural number $N = \langle n, x \rangle$. Let $M = \max(k, n)$. By properties *CS1* and *CS2* there exists a stage s_B such that at all M -good stages $t > s_B$ we have that for all $m \leq M$, $B_m \upharpoonright x+1 \subset B_m^{\{s\}} \subseteq B_m$. Similarly there is a stage s_C such that at all M -good stages $t > s_C$ we have that for all $m \leq M$, $C_m \upharpoonright x+1 \subset C_m^{\{s\}} \subseteq C_m$. Then at every M -good stage $t > \max(s_B, s_C)$ we will have that for every $m \leq M$ that $B_m^{\{t\}} \upharpoonright x+1 = B_m \upharpoonright x+1 = C_m \upharpoonright x+1 = C_m^{\{t\}} \upharpoonright x+1$ and hence $l^{\{t\}} \geq N$. Then as every M -good stage is a k -good stage we will have that $\lim_{s \in G_A^k} l^{\{s\}} > N$. The numbers N and k are arbitrary, we can conclude that $\lim_{s \in G_A^k} l^{\{s\}} = \infty$ for every k . \square

Note! If $\mathcal{B} \neq \mathcal{C}$ then there will be a number k such that $l^{\{s\}}$ is bounded at k -good stages $s \in G_A^k$. In fact the length of agreement will be bounded at all m -good stages for $m \geq k$ as every m -good stage is also k -good. But it is still possible that for some $n < k$ the length of agreement is not bounded as there are “more” n -good stages than k -good stages. However it will then follow that $C_n = B_n$.

4.2. Proof of Theorem 1.3. Let $\mathbf{b} <_\omega \mathbf{a} \leq_\omega \mathbf{0}'_\omega$ be two given ω -enumeration degrees. In [25] it is shown that every ω -enumeration degree $\mathbf{a} \leq_\omega \mathbf{0}'_\omega$ contains a member \mathcal{A} , such that $\mathcal{A} \equiv_e P(\mathcal{A})$ and such that \mathcal{A} has a good approximation. Let $\mathcal{A} = \{A_n\}_{n < \omega}$ be such a member of the given degree \mathbf{a} and let $\mathbb{A} = \{A_n^{\{s\}}\}_{s < \omega}$ be a good approximation to \mathcal{A} . Now we turn to \mathbf{b} . Select a member $\mathcal{B} = \{B_n\}_{n < \omega}$ of the ω -enumeration degree \mathbf{b} , such that $\mathcal{B} \equiv_e P(\mathcal{B})$. There is an e-operator Γ such that $\mathcal{B} = \Gamma(\mathcal{A})$ and hence $\{B_n^{\{s\}}\}_{s,n < \omega}$, where $B_n^{\{s\}} = \Gamma[n]^{\{s\}}(A_n^{\{s\}})$ is a correct approximation to \mathcal{B} with respect to \mathbb{A} by Proposition 4.1.

The method described in Section 2 can be applied in the context of ω -enumeration degrees as well. The operation \oplus has the required properties: let $\{\mathcal{A}_i\}_{i < \omega}$ be a sequence of sequences of sets of natural numbers. For every i and computable set C , with $i \in C$, we have even that $\mathcal{A}_i \leq_e \bigoplus_{j \in C} \mathcal{A}_j$ via the operator $V =$

$\{\langle n, x, \{\langle n, i, x \rangle\} \mid n, x < \omega\}\}$. If $C_1 \subseteq C_2$ are computable sets then $\bigoplus_{i \in C_1} \mathcal{A}_i \leq_e \bigoplus_{i \in C_2} \mathcal{A}_i$ via the operator $U = \{\langle n, i, x, \{\langle n, i, x \rangle\} \mid n, x < \omega \wedge i \in C_1\}$. Hence the embeddability of every countable partial ordering in the interval $[\mathbf{b}, \mathbf{a}]$ can be reduced to the existence of an ω -independent sequence $\{\mathcal{C}_i\}_{i < \omega}$, such that for every i $\mathcal{B} \leq_\omega \mathcal{C}_i \leq_\omega \mathcal{A}$ uniformly in i .

Definition 4.8. *A sequence of sequences of sets $\{\mathcal{C}_i\}_{i < \omega}$ is called ω -independent if for every i*

$$\mathcal{C}_i \not\leq_\omega \bigoplus_{j \neq i} \mathcal{C}_j.$$

4.2.1. *Easy case.* First we will examine the relationship between \mathcal{A} and \mathcal{B} . We know that $\mathcal{B} \leq_\omega \mathcal{A}$ and hence for every k , as $A_k \equiv_e P_k(\mathcal{A})$, we have that $B_k \leq_e A_k$. Suppose that there is some k such that $B_k <_e A_k$. By Proposition 4.1 we have that $\{A_k^{\{s\}}\}_{s < \omega}$ is a good approximation to A_k hence we may apply Theorem 1.2 to obtain an e -independent sequence $\{\mathcal{C}_i\}_{i < \omega}$ such that for every i we have that $B_k <_e \mathcal{C}_i <_e A_k$ uniformly in i . Now consider the sequences $\mathcal{C}_i = \{\mathcal{C}_{i,n}\}_{n < \omega}$, where $\mathcal{C}_{i,n} = B_n$ for $n \neq k$ and $\mathcal{C}_{i,k} = \mathcal{C}_i$.

It is easy to see that for every i we have $\mathcal{B} \leq_\omega \mathcal{C}_i \leq_\omega \mathcal{A}$. Indeed, $\mathcal{B} = V(\mathcal{C}_i)$, where V is the enumeration operator such that $V[n]$ for $n \neq k$ is the identity operator and $V[k]$ is the enumeration operator witnessing $B_k <_e \mathcal{C}_i$. For the second inequality we modify the operator Γ , for which $\mathcal{B} = \Gamma(\mathcal{A})$ by setting set $V[n] = \Gamma[n]$ for $n \neq k$ and $V[k]$ to be the operator witnessing $\mathcal{C}_i <_e A_k$. Then $\mathcal{C}_i = V(\mathcal{A})$.

We observe that the sequence $\{\mathcal{C}_i\}_{i < \omega}$ is ω -independent as follows: fix i . First note that $\bigoplus_{j \neq i} \mathcal{C}_j \equiv_e \mathcal{F}_i$, where $\mathcal{F}_i = B_n$ for $n \neq k$ and $\mathcal{F}_{i,k} = \bigoplus_{i \neq j} \mathcal{C}_j$. This follows from the fact that every set B is uniformly e -reducible to $\bigoplus_R B$, for any computable set R . If we assume that $\mathcal{C}_i \leq_\omega \bigoplus_{j \neq i} \mathcal{C}_j$ then $\mathcal{C}_i \leq_\omega \mathcal{F}_i$ and in particular $\mathcal{C}_i \leq_e P_k(\mathcal{F}_i)$. Now $P_k(\mathcal{F}_i) = P_{k-1}(\mathcal{F}_i)' \oplus \mathcal{F}_{i,k} = P_{k-1}(\mathcal{B})' \oplus \bigoplus_{i \neq j} \mathcal{C}_j$. By the choice of \mathcal{B} as e -equivalent to its jump sequence we have that $P_{k-1}(\mathcal{B})' \leq_e B_k \leq_e \bigoplus_{i \neq j} \mathcal{C}_j$ and hence $\mathcal{C}_i \leq_e P_k(\mathcal{F}_i) \leq_e \bigoplus_{i \neq j} \mathcal{C}_j$, contradicting the e -independence of the family $\{\mathcal{C}_i\}_{i < \omega}$.

Now we can easily deduce that for every i we have strong inequalities: $\mathcal{B} <_\omega \mathcal{C}_i <_\omega \mathcal{A}$. $\mathcal{C}_i \leq_\omega \mathcal{B}$ leads to $\mathcal{C}_i \leq_\omega \mathcal{C}_j$ for every j and hence $\mathcal{C}_i \leq_\omega \bigoplus_{i \neq j} \mathcal{C}_j$, contradicting the just proved ω -independence. On the other hand $\mathcal{A} \leq_\omega \mathcal{C}_i$ yields for any $j \neq i$ the inequality $\mathcal{C}_j \leq_\omega \mathcal{C}_i \leq_\omega \bigoplus_{l \neq j} \mathcal{C}_l$, again contradicting the ω -independence.

4.2.2. *Complicated case.* The more technically complex case is when for every k we have that $A_k \equiv_e B_k$. In this case we will not be able to deduce the statement of Theorem 1.3 from the structural properties of the enumeration degrees, as we did in the previous case. We will have to give a direct construction.

We will construct an operator V and define $\mathcal{A}_i = \{V[n](A_n)[i]\}_{n < \omega}$. The constructed operator will satisfy the following list of requirements:

$$\mathcal{P}_{i,e} : W_e(P(\mathcal{B} \oplus \bigoplus_{j \neq i} \mathcal{A}_j)) \neq \mathcal{A}_i,$$

for every pair of natural numbers i, e .

Then by setting $\mathcal{C}_i = \mathcal{A}_i \oplus \mathcal{B}$ we will obtain the required ω -independent sequence of sequences. Indeed for every i we have that $\mathcal{A}_i \leq_\omega \mathcal{A}$ and $\mathcal{B} \leq_\omega \mathcal{A}$, hence

$\mathcal{B} \leq_\omega \mathcal{A}_i \oplus \mathcal{B} = \mathcal{C}_i \leq_\omega \mathcal{A}$. On the other hand if we assume that $\mathcal{C}_i \leq_\omega \bigoplus_{i \neq j} \mathcal{C}_j$, then $\mathcal{A}_i \leq_\omega \mathcal{C}_i \leq_\omega \bigoplus_{i \neq j} (\mathcal{A}_j \oplus \mathcal{B}) \equiv_\omega \mathcal{B} \oplus \bigoplus_{i \neq j} \mathcal{A}_j$, hence there is some operator W_e such that $\mathcal{A}_i = W_e(P(\mathcal{B} \oplus \bigoplus_{j \neq i} \mathcal{A}_j))$, contradicting the requirement $\mathcal{P}_{i,e}$. This establishes the ω -independence of $\{\mathcal{C}_i\}_{i < \omega}$, which as in the previous section yields the strong inequalities $\mathcal{B} <_\omega \mathcal{C}_i <_\omega \mathcal{A}$.

Approximations and conventions

We start with a convention, which will simplify notation in the construction. At any given stage s , whenever we enumerate an axiom in the constructed operator $V = \bigoplus_{n < \omega} V[n]$, the axiom has a fixed structure. Thus we will use the following convention:

Convention 1: If at stage s we wish to enumerate an axiom in the operator $V[n]^{\{s+1\}}$ for some number z , then the axiom we enumerate in $V^{\{s+1\}}$ is $\langle n, z, A_n^{\{s\}} \rangle$.

Now we turn to the issue of finding correct approximations to the various sets involved in the construction.

To find a correct approximation to $\mathcal{A}_i = \{(V[n](A_n))[i]\}_{n < \omega}$ we observe the following. Consider the operator $V_i = \{\langle n, x, D \rangle \mid \langle n, i, x, D \rangle \in V\}$. It is straightforward to check that $\mathcal{A}_i = V_i(\mathcal{A})$. During the construction we define a Σ_1 -approximation $\{V^{\{s\}}\}_{s < \omega}$ to the c.e. set V . Denote by

$$V_i^{\{s\}} = \{\langle n, x, D \rangle \mid \langle n, i, x, D \rangle \in V^{\{s\}}\},$$

then $\{V_i^{\{s\}}\}$ is a Σ_1^0 approximation to V_i and by Lemma 4.1 we have that

$$\{A_{i,n}^{\{s\}} = V_i[n]^{\{s\}}(A_n^{\{s\}})\}_{s,n < \omega}$$

is a correct with respect to \mathbb{A} approximation to the sequence \mathcal{A}_i .

This allows us to introduce one further convention that will simplify the notation in the construction. Say at stage s we want to enumerate an axiom for x in $\mathcal{A}_{i,n}^{\{s+1\}}$. By the discussion above this can be achieved by enumerating an axiom for $\langle n, i, x \rangle$ in $V^{\{s+1\}}$. Thus we will have the following:

Convention 2: In the construction the action “Enumerate the element z in $A_{i,n}$ ” performed at stage s will be an abbreviation for the action “Enumerate an axiom for the element $\langle i, x \rangle$ in $V_n^{\{s+1\}}$ ” and by Convention 1 this translates to “Enumerate the axiom $\langle n, i, x, A_n^{\{s\}} \rangle$ in $V^{\{s+1\}}$ ”.

The second type of sequence that we will need to approximate correctly is $P(\mathcal{B} \oplus \bigoplus_{i \neq j} \mathcal{A}_j)$. We will do this in three steps. First for every i consider the operator $U_i = \{\langle n, j, x \rangle \mid i \neq j \wedge \langle n, j, x \rangle \in V\}$. Again it is straightforward to check that $\bigoplus_{j \neq i} \mathcal{A}_j = U_i(\mathcal{A})$. On the other hand note that there is a connection between the operator U_i and the operators V_j , for $j \neq i$, namely $U_i(\mathcal{A}) = \bigoplus_{j \neq i} V_j(\mathcal{A})$. This allows us to define a correct with respect to \mathbb{A} approximation

$$\{U_i[n]^{\{s\}}(A_n^{\{s\}}) = \bigoplus_{i \neq j} \mathcal{A}_{j,n}^{\{s\}}\}_{n,s < \omega}$$

to the sequence $\bigoplus_{i \neq j} \mathcal{A}_j$.

Secondly we note that $\{B_n^{\{s\}} \oplus \bigoplus_{i \neq j} \mathcal{A}_{j,n}^{\{s\}}\}_{s,n < \omega}$ is a correct approximation to $\mathcal{B} \oplus \bigoplus_{i \neq j} \mathcal{A}_j$ by Lemma 4.2.

The final step is a bit more difficult. The technique we use is introduced in the proof of the Density theorem for the local ω -enumeration degrees and can be found in [25]. By the Recursion theorem we may assume that we know in advance the index of the constructed c.e. set V . Recall that the monotonicity of the enumeration jump is effective, i.e. there is a computable function ρ such that if $A = W_e(B)$ then $A' = W_{\rho(e)}(B')$.

We obtain a correct with respect to \mathbb{A} approximation to $P(\mathcal{B} \oplus U_i(\mathcal{A}))$ as follows:

- $P_0(\mathcal{B} \oplus U_i(\mathcal{A})) = B_0 \oplus U_i[0](A_0)$ we approximate via the correct with respect to \mathbb{A}_0 approximation $\{B_0^{\{s\}} \oplus U_i[0]^{\{s\}}(A_0^{\{s\}})\}_{s < \omega}$. Using the index of the operator V , from which we immediately obtain an index of the operator $U_i[0]$ we can effectively obtain an index of an operator $W_{a_{i,0}}$ such that $P_0(\mathcal{B} \oplus U_i(\mathcal{A})) = W_{a_{i,0}}(A_0)$.
- Suppose we have constructed the correct with respect to \mathbb{A}_n approximation to $P_n(\mathcal{B} \oplus U_i(\mathcal{A}))$ and we have effectively obtained an index $a_{i,n}$ of an operator which reduces $P_n(\mathcal{B} \oplus U_i(\mathcal{A}))$ to A_n .

$P_{n+1}(\mathcal{B} \oplus U_i(\mathcal{A})) = P_n(\mathcal{B} \oplus U_i(\mathcal{A}))' \oplus B_{n+1} \oplus U_i[n+1](A_{n+1})$. We use the effective monotonicity of the jump to get $W_{\rho(a_{i,n})}(A_n') = P_n(\mathcal{B} \oplus U_i(\mathcal{A}))'$. Now using the fact that $\mathcal{A} \equiv_e P(\mathcal{A})$ we can effectively obtain an index $\mu(i, a_{i,n})$ such that $W_{\mu(i, a_{i,n})}(A_{n+1}) = P_n(\mathcal{B} \oplus U_i(\mathcal{A}))'$. From this we immediately get effectively an index $a_{i,n+1}$ such that $W_{a_{i,n+1}}(A_{n+1}) = P_{n+1}(\mathcal{B} \oplus U_i(\mathcal{A}))$ and a correct with respect to \mathbb{A}_{n+1} approximation to $P_{n+1}(\mathcal{B} \oplus U_i(\mathcal{A}))$ namely

$$\{W_{\mu(i, a_{i,n})}^{\{s\}}(A_{n+1}^{\{s\}}) \oplus B_{n+1}^{\{s\}} \oplus U_i[n+1]^{\{s\}}(A_{n+1}^{\{s\}})\}_{s < \omega}.$$

By Proposition 4.2 this inductive procedure defines a correct with respect to \mathbb{A} approximation to $P(\mathcal{B} \oplus U_i(\mathcal{A}))$.

Let \mathcal{Q}_i be the sequence defined by $Q_{i,0} = B_0$ and $Q_{i,n+1} = P_n(\mathcal{B} \oplus U_i(\mathcal{A}))' \oplus B_{n+1}$ for every $n > 0$. It follows that $P(\mathcal{B} \oplus U_i(\mathcal{A})) = \mathcal{Q}_i \oplus U_i(\mathcal{A})$ and that we have a correct with respect to \mathbb{A} approximation to this sequence as well, defined by $Q_{i,0}^{\{s\}} = B_0^{\{s\}}$ and for $n > 0$ let $Q_{i,n}^{\{s\}} = W_{\mu(i, a_{i,n-1})}^{\{s\}}(A_n^{\{s\}}) \oplus B_n^{\{s\}}$. For every i we have that $\{Q_{i,n}^{\{s\}} \oplus U_i[n]^{\{s\}}(A_n^{\{s\}})\}_{n, s < \omega}$ or equivalently

$$\{Q_{i,n}^{\{s\}} \oplus \bigoplus_{j \neq i} A_{j,n}^{\{s\}}\}_{n, s < \omega}$$

is a correct with respect to \mathbb{A} approximation to $P(\mathcal{B} \oplus \bigoplus_{i \neq j} \mathcal{A}_j)$.

Construction. Fix a computable priority ordering $\mathcal{R}_0 < \mathcal{R}_1 \dots$ of all requirements $\mathcal{P}_{e,i}$. The construction proceeds in stages. Let $V^{\{0\}} = \emptyset$. At stage $s \geq 0$ we construct $V^{\{s+1\}}$ from it value at the previous stage. We examine all requirements \mathcal{R}_k with $k < s$ and perform the following actions for each in order of priority.

Suppose $\mathcal{R}_k = \mathcal{P}_{i,e}$. We define $l_k^{\{s\}} = l(W_e^{\{s\}}(\{Q_{i,n}^{\{s\}} \oplus \bigoplus_{j \neq i} A_{j,n}^{\{s\}}\}_{n < \omega}, \{A_{i,n}^{\{s\}}\}_{n < \omega}))$.

For every $\langle n, x \rangle < l_k^{\{s\}}$:

- If $x \in A_n^{\{s\}}$ and $\langle k, x \rangle \notin A_{i,n}^{\{s\}}$ then enumerate $\langle k, x \rangle$ in $A_{i,n}$. (Recall Convention 2.)

- If $x \notin W_e[n]^{\{s\}}(Q_{i,n}^{\{s\}} \oplus \bigoplus_{j \neq i} A_{j,n}^{\{s\}})$ and there is a finite set $L = L_q \oplus \bigoplus_{j \neq i} L_j$ such that $\langle x, L \rangle \in W_e[n]^{\{s\}}$, $L_q \subseteq Q_{i,n}^{\{s\}}$ and for every j we have $L_j^{[\leq k]} \subseteq A_{j,n}^{\{s\}}$ then for every $j \neq i$ and every $y \in L_j^{[> k]}$ enumerate y in $A_{j,n}$.

This completes the construction.

Lemma 4.4. *Fore every $k < \omega$:*

- (1) *The requirement \mathcal{R}_k is satisfied.*
- (2) *There exists a number r_k such that for all $n \geq r_k$ the actions for \mathcal{R}_k do not enumerate any axioms in $V[n]$ at n -good stages of the approximation \mathbb{A} .*

Proof. We prove both statements of the lemma simultaneously by induction. Assume that they are true for every $m < k$ and let $\mathcal{R}_k = \mathcal{P}_{i,e}$ for some $e, i \in \mathbb{N}$. By the induction hypothesis for every $m < k$ there is a number r_m such that for all $n \geq r_m$ the strategy for \mathcal{R}_m does not enumerate any axioms in $V[n]$ at n -good stages of the approximation.

First we will prove that if there is a natural number $r \geq \max_{m < k} \{r_m\}$ such that the length of agreement:

$$l_k^{\{s\}} = l(W_e^{\{s\}}(\{Q_{i,n}^{\{s\}} \oplus \bigoplus_{j \neq i} A_{j,n}^{\{s\}}\}_{n < \omega}), \{A_{i,n}^{\{s\}}\})$$

is bounded by some number L at all r -good stages s then the two conditions of the lemma for \mathcal{R}_k are true. Indeed let r be such a number and suppose that the length of agreement $l_k^{\{s\}}$ is bounded by L . Using the fact that $\{Q_{i,n}^{\{s\}} \oplus \bigoplus_{j \neq i} A_{j,n}^{\{s\}}\}_{s,n < \omega}$ is a correct with respect to \mathbb{A} approximation to $P(\mathcal{B} \oplus \bigoplus_{j \neq i} \mathcal{A}_j)$ and Lemma 4.1 we get that $\{W_e[n]^{\{s\}}(Q_{i,n}^{\{s\}} \oplus \bigoplus_{j \neq i} A_{j,n}^{\{s\}})\}_{s,n < \omega}$ is a correct approximation with respect to \mathbb{A} to $W_e(P(\mathcal{B} \oplus \bigoplus_{j \neq i} \mathcal{A}_j))$. As $\{A_{i,n}^{\{s\}}\}_{s,n < \omega}$ is also a correct with respect to \mathbb{A} approximation to \mathcal{A}_i we can apply Lemma 4.3 and prove the first statement, namely that $\mathcal{A}_i \neq W_e(P(\mathcal{B} \oplus \bigoplus_{j \neq i} \mathcal{A}_j))$ and hence the requirement \mathcal{R}_k is satisfied.

The requirement \mathcal{R}_k enumerates axioms in $V[n]$ at stage s only if there is an element $\langle n, x \rangle < l_k^{\{s\}}$ with certain properties listed in the two cases of the construction. As the length of agreement is bounded by L at r -good stages and for every $n \geq r$ if a stage is n -good then it is r -good, there are only finitely many numbers n such that for some x $\langle n, x \rangle < L$. Let r_k be such that $L < \langle r_k, x \rangle$ for every $x < \omega$. Such a number r_k exists by the properties of the pairing function $\langle -, - \rangle$. Then \mathcal{R}_k will not enumerate any axioms in V_n for $n \geq r_k$ at any n -good stage.

Thus the only thing that remains to be shown is that the length of agreement is indeed bounded at all r -good stages for some natural number r . Towards a contradiction assume that this is not the case, i.e. for every r the length of agreement is unbounded at r -good stages. By Lemma 4.3 it follows that $\mathcal{A}_i = W_e(P(\mathcal{B} \oplus \bigoplus_{j \neq i} \mathcal{A}_j))$.

Fix $r = \max_{m < k} \{r_m\}$. First we observe the following:

Claim 4.4.1. *For every $n > r$, $A_n \leq_e A_{i,n}$ uniformly in n .*

Proof. Fix $n > r$. We will show that $A_n = \{x \mid \langle k, x \rangle \in A_{i,n}\}$.

Let $x \in A_n$. Let s be an n -good stage such that $\langle n, x \rangle < l_k^{\{s\}}$. Then by construction at stage s either $\langle k, n \rangle \in A_{i,n}^{\{s\}}$ and hence there is an axiom $\langle n, i, \langle k, x \rangle, D \rangle \in$

$V^{\{s\}}$ such that $D \subseteq A_n^{\{s\}} \subseteq A_n$ or the strategy enumerates $\langle k, x \rangle$ in $A_{i,n}$, i.e. enumerates the axiom $\langle n, i, \langle k, x \rangle, A_n^{\{s\}} \rangle$ in $V^{\{s+1\}}$. In both cases this axiom will turn out valid an $\langle k, x \rangle \in A_{i,n}$.

On the other hand suppose $\langle k, x \rangle \in A_{i,n}$. As $n > r$ this element was enumerated in $A_{i,n}$ through the actions of \mathcal{R}_k . Indeed by the choice of r higher priority strategies \mathcal{R}_m , $m < k$, do not enumerate any axioms in V_n at n -good stages and hence cannot enumerate an element in $A_{i,n}$. Lower priority strategies \mathcal{R}_l for $l > k$ only enumerate axioms for elements $z \in \mathbb{N}^{\geq l}$ in any set $A_{j,n}$, $j < \omega$. Hence \mathcal{R}_k enumerates an axiom for the element $\langle k, x \rangle$ at an n -good stage s . But by construction the strategy must have seen $x \in A_n^{\{s\}} \subseteq A_n$ and hence $x \in A_n$. \square

We will use this to prove that our assumption leads to a contradiction - namely that $\mathcal{A} \leq_\omega \mathcal{B}$.

Recall that we are proving the case of Theorem 1.3 where for every n we have that $A_n \leq_e B_n$, via some operator say W_{e_n} . The reason that $\mathcal{B} <_\omega \mathcal{A}$ is the lack of uniformity in the sequence $\{e_n\}_{n < \omega}$. We will obtain the desired contradiction by constructing an algorithm to obtain e_{n+1} from e_n, \dots, e_0 . More precisely we construct a computable function λ such that $A_n = W_{\lambda(n)}(B_n)$.

For $n \leq r$ set $\lambda(n) = e_n$. Suppose we have defined $\lambda(m)$ for all $m \leq n$, where $n \geq r$. Then $A_{n+1} = \{x \mid \langle k, x \rangle \in A_{i,n+1}\}$ by Claim 1. On the other hand $A_{i,n+1} = W_e[n+1](P_{n+1}(\mathcal{B} \oplus \bigoplus_{i \neq j} \mathcal{A}_j))$ by our assumption. Thus we will be able to effectively obtain the value of $\lambda(n+1)$ if we find an index of an operator which reduces $W_e[n+1](P_{n+1}(\mathcal{B} \oplus \bigoplus_{i \neq j} \mathcal{A}_j))$ to B_{n+1} .

Recall the algorithm which we used to obtain the correct approximation to $P(\mathcal{B} \oplus \bigoplus_{i \neq j} \mathcal{A}_j)$. We defined an effective sequence $\{a_{i,n}\}_{n < \omega}$ such that $P_n(\mathcal{B} \oplus \bigoplus_{i \neq j} \mathcal{A}_j) = W_{a_{i,n}}(A_n)$ and the sequence \mathcal{Q}_i so that $P(\mathcal{B} \oplus \bigoplus_{i \neq j} \mathcal{A}_j) = \mathcal{Q}_i \oplus \bigoplus_{i \neq j} \mathcal{A}_j$. Using $a_{i,n}$, the values of $\lambda(m)$ for $m \leq n$ and the effective monotonicity of the jump we can obtain effectively an index c such that $W_c(B_{n+1}) = P_n(\mathcal{B} \oplus \bigoplus_{j \neq i} \mathcal{A}_j)' \oplus B_{n+1} = \mathcal{Q}_{i,n+1}$.

Now $P_{n+1}(\mathcal{B} \oplus \bigoplus_{i \neq j} \mathcal{A}_j) = \mathcal{Q}_{i,n+1} \oplus \bigoplus_{j \neq i} \mathcal{A}_{j,n+1} = W_c(B_{n+1}) \oplus \bigoplus_{j \neq i} \mathcal{A}_{j,n+1}$.

We will show that $W_e[n+1](W_c(B_{n+1}) \oplus \bigoplus_{j \neq i} \mathcal{A}_{j,n+1}) = W_e[n+1](W_c(B_{n+1}) \oplus \bigoplus_{j \neq i} \mathbb{N}^{>k})$.

Having this equality and noting that $\bigoplus_{j \neq i} \mathbb{N}^{>k}$ is a computable set, it is now straightforward to obtain the value of $\lambda(n+1)$ so that $A_{n+1} = W_{\lambda(n+1)}(B_{n+1})$.

We turn to the proof of the last claim in this proof:

Claim 4.4.2. $W_e[n+1](W_c(B_{n+1}) \oplus \bigoplus_{j \neq i} \mathcal{A}_{j,n+1}) = W_e[n+1](W_c(B_{n+1}) \oplus \bigoplus_{j \neq i} \mathbb{N}^{>k})$.

Proof. As $n+1 > r$ for every j we have that $A_{j,n+1}^{[\leq k]} = \emptyset$. Hence

$$W_c(B_{n+1}) \oplus \bigoplus_{j \neq i} \mathcal{A}_{j,n+1} \subseteq W_c(B_{n+1}) \oplus \bigoplus_{j \neq i} \mathbb{N}^{>k}$$

and by the monotonicity of the e-operators we get the first inclusion

$$W_e[n+1](W_c(B_{n+1}) \oplus \bigoplus_{j \neq i} \mathcal{A}_{j,n+1}) \subseteq W_e[n+1](W_c(B_{n+1}) \oplus \bigoplus_{j \neq i} \mathbb{N}^{>k}).$$

Suppose that $x \in W_e[n+1](W_c(B_{n+1}) \oplus \bigoplus_{j \neq i} \mathbb{N}^{>k})$. Then there is a valid axiom $\langle x, L \rangle \in W_e[n+1]$ such that $L = L_q \oplus \bigoplus_{j \neq i} L_j$ and $L_q \subseteq W_c(B_{n+1})$ and

for every j , $L_j^{\leq k} = \emptyset$. As $\{Q_{i,n+1}^{\{s\}}\}$ is a correct approximation to the set $Q_{i,n+1}$ we know that there will be a stage s such that at all $n+1$ -good stages $t > s$, $L_q \subseteq Q_{i,n+1}^{\{t\}}$. Let $t > s$ be an $n+1$ -good stage such that $\langle n+1, x \rangle < l_k^{\{t\}}$ and $\langle x, L \rangle \in W_e[n+1]^{\{t\}}$. Then by construction the strategy \mathcal{R}_k will enumerate L_j in $\mathcal{A}_{j,n+1}$ for every $j \neq i$ at stage t and as the stage is $n+1$ -good we can conclude that $L \subseteq W_c(B_{n+1}) \oplus \bigoplus_{j \neq i} A_{j,n+1}$ and hence $x \in W_e[n+1](W_c(B_{n+1}) \oplus \bigoplus_{j \neq i} A_{j,n+1})$. This proves the second inclusion and concludes the proof of the theorem. \square

\square

5. EMBEDDING RESULTS IN THE ω -ENUMERATION DEGREES MODULO ITERATED JUMP

5.1. Preliminaries. In this section we will define the relation \leq_∞ for the ω -enumeration degrees, obtain from it the degree structure of Σ_2^0 ω -enumeration degrees modulo iterated jump, $\mathcal{G}_\omega / \sim_\infty$, and discuss certain basic properties of this structure.

Recall that the jump of an ω -enumeration degree \mathbf{a} is defined as the degree of the sequence $\mathcal{A}' = \{P_{n+1}(\mathcal{A})\}_{n < \omega}$, where \mathcal{A} is some representative of \mathbf{a} . We can iterate this definition to obtain the n -th jump of \mathbf{a} for every natural number n . Namely we set $\mathbf{a}^0 = \mathbf{a}$ and for every $n \geq 0$, $\mathbf{a}^{(n+1)} = (\mathbf{a}^n)'$. This definition gives rise to the following reducibility relation between ω -enumeration degrees.

Definition 5.1. *Let \mathbf{a} and \mathbf{b} be two Σ_2^0 ω -enumeration degrees. Then $\mathbf{a} \leq_\infty \mathbf{b}$ if and only if there is a natural number n such that $\mathbf{a}^n \leq_\omega \mathbf{b}^n$, where \mathbf{a}^n denotes the n -th ω -enumeration jump of the degree \mathbf{a} .*

The relation \leq_∞ is reflexive and transitive and induces an equivalence relation \sim_∞ , where $\mathbf{a} \sim_\infty \mathbf{b}$ if both $\mathbf{a} \leq_\infty \mathbf{b}$ and $\mathbf{b} \leq_\infty \mathbf{a}$. When we factorize \mathcal{G}_ω on the equivalence relation \sim_∞ we obtain the degree structure $\langle \mathcal{G}_\omega / \sim_\infty, \leq \rangle$ with domain, the set of all equivalence classes $\{[\mathbf{a}]_{\sim_\infty} \mid \mathbf{a} \in \mathcal{G}_\omega\}$ and the relation \leq defined by $[\mathbf{a}]_{\sim_\infty} \leq [\mathbf{b}]_{\sim_\infty}$ if and only if $\mathbf{a} \leq_\infty \mathbf{b}$. This is a partial ordering with least element $[\mathbf{0}_\omega]_{\sim_\infty}$ and greatest element $[\mathbf{0}'_\omega]_{\sim_\infty}$. The least element $[\mathbf{0}_\omega]_{\sim_\infty}$, denoted also as L , is exactly the union of the classes $L_n = \{\mathbf{a} \mid \mathbf{a}^n = \mathbf{0}_\omega^n\}$ of the low $_n$ Σ_2^0 ω -enumeration degrees. The greatest element $[\mathbf{0}'_\omega]_{\sim_\infty}$, denoted also as H , is exactly the union of the classes $H_n = \{\mathbf{a} \mid \mathbf{a}^n = \mathbf{0}_\omega^{(n+1)}\}$ of the high $_n$ Σ_2^0 ω -enumeration degrees. Every intermediate degree in $\mathcal{G}_\omega / \sim_\infty$ is therefore made up of members of the class $I = \mathcal{G}_\omega \setminus (L \cup H)$, the class of all intermediate Σ_2^0 ω -enumeration degrees.

Note that the embedding ι of the Turing degrees into the enumeration degrees, combined with the embedding κ of the enumeration degrees into the ω -enumeration degrees, both of which preserve the order and the jump operation, gives an embedding σ of the structure $\mathcal{R} / \sim_\infty$ into the structure $\mathcal{G}_\omega / \sim_\infty$. Hence the structure $\mathcal{G}_\omega / \sim_\infty$ is nontrivial as it contains elements different from the least degree L and the greatest degree H , namely $\sigma(I)$, where $I = [\mathbf{i}]_{\sim_\infty}$ is the iterated Turing jump degree of any intermediate c.e. Turing degree \mathbf{i} .

Some basic relationships between the relation \leq_ω and \leq_∞ are given in the following proposition.

Proposition 5.1. *Let \mathbf{a} and \mathbf{b} be two Σ_2^0 ω -enumeration degrees.*

- (1) *If $\mathbf{a} \leq_\omega \mathbf{b}$ then $[\mathbf{a}]_{\sim_\infty} \leq [\mathbf{b}]_{\sim_\infty}$.*

(2) If $[\mathbf{a}]_{\sim_\infty} \leq [\mathbf{b}]_{\sim_\infty}$ then there is a representative $\mathbf{c} \in [\mathbf{a}]_{\sim_\infty}$ such that $\mathbf{c} \leq_\omega \mathbf{b}$.

Proof. Part (1) follows directly from the definition of the relation \leq_∞ and the monotonicity of the ω -enumeration jump. Part (2) follows from the existence of a least n -th jump invert for every ω -enumeration degree above $\mathbf{0}_\omega^{(n)}$, proved in [26]. Let $\mathcal{A} = \{A_n\}_{n < \omega}$ be a representative of \mathbf{a} . Consider the sequence $\mathcal{I} = \{I_k\}_{k < \omega}$ defined by $I_k = \emptyset$ for $k < n$ and $I_k = P_k(\mathcal{A})$ for $k \geq n$. Then the degree of \mathcal{I} is the least ω -enumeration degree whose n -th jump is equal to \mathbf{a}^n and is denoted by $I^n(\mathbf{a})$.

Now if $[\mathbf{a}]_{\sim_\infty} \leq [\mathbf{b}]_{\sim_\infty}$ then there is some natural number n such that $\mathbf{a}^n \leq_\omega \mathbf{b}^n$. Let $\mathbf{c} = I^n(\mathbf{a})$. Then $\mathbf{c} \in [\mathbf{a}]_{\sim_\infty}$ and $\mathbf{c} \leq_\omega \mathbf{b}$. □

To understand the structure $\mathcal{G}_\omega / \sim_\infty$ we will need to introduce a special class of ω -enumeration degrees related to a given degree \mathbf{b} , the *almost- \mathbf{b}* enumeration degrees. The class of the almost zero ω -enumeration degrees was defined and studied by Soskov and Ganchev in [26].

Definition 5.2. Let \mathcal{B} be a sequence of sets of natural numbers. We shall say that the sequence \mathcal{A} is *almost- \mathcal{B}* if for every n we have that $P_n(\mathcal{B}) \equiv_e P_n(\mathcal{A})$.

If \mathcal{A} is almost- \mathcal{B} then we shall say that $d_\omega(\mathcal{A})$ is *almost- $d_\omega(\mathcal{B})$* .

The following proposition summarizes the properties of the almost- \mathbf{b} degrees and their relation to the structure $\mathcal{G}_\omega / \sim_\infty$.

Proposition 5.2. Let $\mathbf{b} \leq \mathbf{0}'_\omega$ be an ω -enumeration degree.

- (1) If \mathbf{a} is almost- \mathbf{b} and $\mathcal{B} \in \mathbf{b}$ then every $\mathcal{A} \in \mathbf{a}$ is almost- \mathcal{B} .
- (2) The class of almost- \mathbf{b} degrees is closed under least upper bound.
- (3) If $\mathbf{b} \leq_\omega \mathbf{c} \leq_\omega \mathbf{a}$ and \mathbf{a} is almost- \mathbf{b} then \mathbf{c} is almost- \mathbf{b} .
- (4) If $\mathbf{b} \in \mathcal{D}_1$ then \mathbf{b} is the least almost- \mathbf{b} Σ_2^0 ω -enumeration degree.
- (5) If \mathbf{a} and \mathbf{c} are almost- \mathbf{b} Σ_2^0 ω -enumeration degrees then $[\mathbf{a}]_{\sim_\infty} \leq [\mathbf{c}]_{\sim_\infty}$ if and only if $\mathbf{a} \leq_\omega \mathbf{c}$.

Proof. (1) Let \mathbf{a} be almost- \mathbf{b} . Then there are representatives $\mathcal{B} \in \mathbf{b}$ and $\mathcal{A} \in \mathbf{a}$ such that $P_n(\mathcal{B}) \equiv_e P_n(\mathcal{A})$ for every $n < \omega$. Let $\mathcal{A}_1 \in \mathbf{a}$ and $\mathcal{B}_1 \in \mathbf{b}$ be two other representatives. Then for every n we have that $P_n(\mathcal{A}_1) \equiv_e P_n(\mathcal{A}) \equiv_e P_n(\mathcal{B}) \equiv_e P_n(\mathcal{B}_1)$.

- (2) Let \mathbf{a}, \mathbf{c} be almost- \mathbf{b} . Fix $\mathcal{A} \in \mathbf{a}, \mathcal{C} \in \mathbf{c}$ and $\mathcal{B} \in \mathbf{b}$. Then for every n we have that $P_n(\mathcal{A}) \equiv_e P_n(\mathcal{B}) \equiv_e P_n(\mathcal{C})$.

It is straightforward to check that for every n we have that $P_n(\mathcal{B}) \leq_e P_n(\mathcal{C}) \leq_e P_n(\mathcal{A} \oplus \mathcal{C})$. We will prove by induction on n that for every n we have that $P_n(\mathcal{A} \oplus \mathcal{C}) \leq_e P_n(\mathcal{B})$. For $n = 0$ we have that $P_0(\mathcal{A} \oplus \mathcal{C}) = A_0 \oplus C_0 = P_0(\mathcal{A}) \oplus P_0(\mathcal{C}) \equiv_e P_0(\mathcal{B})$. Suppose we have proved that $P_n(\mathcal{A} \oplus \mathcal{C}) \leq_e P_n(\mathcal{B})$. Then by the monotonicity of the enumeration jump we have that $P_n(\mathcal{A} \oplus \mathcal{C})' \leq_e P_n'(\mathcal{B}) \leq_e P_{n+1}(\mathcal{B})$. On the other hand $A_{n+1} \oplus C_{n+1} \leq_e P_{n+1}(\mathcal{A}) \oplus P_{n+1}(\mathcal{C}) \leq_e P_{n+1}(\mathcal{B})$. Combining these together we get that $P_{n+1}(\mathcal{A} \oplus \mathcal{C}) = P_n(\mathcal{A} \oplus \mathcal{C})' \oplus (A_{n+1} \oplus C_{n+1}) \equiv_e P_{n+1}(\mathcal{B})$.

- (3) Let $\mathbf{b} \leq_\omega \mathbf{c} \leq_\omega \mathbf{a}$ be Σ_2^0 ω -enumeration degrees and let \mathbf{a} be almost- \mathbf{b} . Fix representatives $\mathcal{B} \in \mathbf{b}, \mathcal{C} \in \mathbf{c}$ and $\mathcal{A} \in \mathbf{a}$. Then for every n we have that $P_n(\mathcal{B}) \leq_e P_n(\mathcal{C}) \leq_e P_n(\mathcal{A}) \equiv_e P_n(\mathcal{B})$.
- (4) Let $\mathbf{b} \in \mathcal{D}_1$. There is a representative $\mathcal{B} = \{B_n\}_{n < \omega}$ of \mathbf{b} , such that for every $n > 0$ $B_n = \emptyset$. Suppose that \mathbf{a} is almost- \mathbf{b} and $\mathcal{A} \in \mathbf{a}$. Then

$B_0 = \Gamma(P_0(\mathcal{A}))$ for some enumeration operator Γ and hence $\mathcal{B} \leq_\omega \mathcal{A}$ via the operator $V = \{0\} \times \Gamma$. Thus $\mathbf{b} \leq_\omega \mathbf{a}$.

- (5) Let \mathbf{a} and \mathbf{c} be almost- \mathbf{b} degrees. Fix $\mathcal{A} \in \mathbf{a}$ and $\mathcal{C} \in \mathbf{c}$. Then for every k we have that $P_k(\mathcal{A}) = V_k(P_k(\mathcal{C}))$, for some enumeration operator V_k . Suppose $[\mathbf{a}]_{\sim_\infty} \leq [\mathbf{c}]_{\sim_\infty}$. Then $\mathbf{a} \leq_\infty \mathbf{c}$, i.e. there is a natural number n such that $\mathbf{a}^n \leq_\omega \mathbf{c}^n$ and hence $\mathcal{A}^n = \{P_{k+n}(\mathcal{A})\}$ is uniformly reducible to the sequence $\mathcal{C}^n = \{P_{k+n}(\mathcal{C})\}$. Let V be an operator such that $V[k](P_{n+k}(\mathcal{C})) = P_{n+k}(\mathcal{A})$. Now $\mathcal{A} \leq_\omega \mathcal{C}$ can be seen via the operator U , defined by $U[k] = V_k$ for $k < n$, and $U[n+k] = V[n+k]$ for $k \geq 0$. Hence $\mathbf{a} \leq_\omega \mathbf{c}$.

That $\mathbf{a} \leq_\omega \mathbf{c}$ yields $[\mathbf{a}]_{\sim_\infty} \leq [\mathbf{c}]_{\sim_\infty}$ follows from Proposition 5.1. \square

The almost zero Σ_2^0 ω -enumeration degrees turn out to have a very important relationship with the jump classes H and L . Soskov and Ganchev in [26] prove that the class L consists of exactly those elements of \mathcal{G}_ω , which do not bound any nonzero Σ_2^0 almost zero degree and that the class H consists of exactly those elements of \mathcal{G}_ω that bound every Σ_2^0 almost zero ω -enumeration degree. We prove a slight generalization of the above mentioned characterization of the class L :

Theorem 5.1. *Let $\mathbf{b} <_\omega \mathbf{a} \leq_\omega \mathbf{0}'_\omega$ be two ω -enumeration degrees. There exists an almost- \mathbf{b} degree \mathbf{z} such that $\mathbf{b} <_\omega \mathbf{z} \leq_\omega \mathbf{a}$ if and only if $\mathbf{b} <_\infty \mathbf{a}$.*

We will leave the rather technical proof of this property for Section 5.2. We end this section with two of its corollaries.

Firstly we can finally establish that the structure $\mathcal{G}_\omega / \sim_\infty$ is dense and properly extends the structure $\mathcal{R} / \sim_\infty$ as a partial ordering.

Corollary 5.1. *Let $[\mathbf{b}]_{\sim_\infty} < [\mathbf{a}]_{\sim_\infty}$ be two members of $\mathcal{G}_\omega / \sim_\infty$. There exists a Σ_2^0 ω -enumeration degree \mathbf{z} such that $[\mathbf{b}]_{\sim_\infty} < [\mathbf{z}]_{\sim_\infty} < [\mathbf{a}]_{\sim_\infty}$. Furthermore \mathbf{z} is an almost- \mathbf{b} degree and $[\mathbf{z}]_{\sim_\infty}$ is not the image of any element in $\mathcal{R} / \sim_\infty$ under the embedding σ .*

Proof. Fix $[\mathbf{b}]_{\sim_\infty} < [\mathbf{a}]_{\sim_\infty} \in \mathcal{G}_\omega / \sim_\infty$. By Proposition 5.1 without loss of generality we may assume that $\mathbf{b} <_\omega \mathbf{a}$.

We apply Theorem 5.1 to obtain an almost- \mathbf{b} degree \mathbf{x} such that $\mathbf{b} <_\omega \mathbf{x} \leq_\omega \mathbf{a}$ and then the density theorem for \mathcal{G}_ω to obtain \mathbf{z} such that $\mathbf{b} <_\omega \mathbf{z} <_\omega \mathbf{x} \leq_\omega \mathbf{a}$. By Part 3 of Proposition 5.2 the degree \mathbf{z} is almost- \mathbf{b} . Now, as \mathbf{b} , \mathbf{z} and \mathbf{x} are all almost- \mathbf{b} degrees it follows by Part 5 of Proposition 5.2 that $[\mathbf{b}]_{\sim_\infty} < [\mathbf{z}]_{\sim_\infty} < [\mathbf{x}]_{\sim_\infty}$. By Proposition 5.1 we have that $[\mathbf{x}]_{\sim_\infty} \leq [\mathbf{a}]_{\sim_\infty}$. Hence by transitivity of the relation “ \leq ” we have that $[\mathbf{b}]_{\sim_\infty} < [\mathbf{z}]_{\sim_\infty} < [\mathbf{x}]_{\sim_\infty} \leq [\mathbf{a}]_{\sim_\infty}$.

That $[\mathbf{z}]_{\sim_\infty}$ is not the image of any element in $\mathcal{R} / \sim_\infty$ under the embedding ι can be seen as follows. Assume towards a contradiction that $[\mathbf{z}]_{\sim_\infty}$ is the image of an element in $\mathcal{R} / \sim_\infty$. Then there is an element $\mathbf{c} \in D_1 \cap [\mathbf{z}]_{\sim_\infty}$. Fix representatives $\mathcal{Z} \in \mathbf{z}$, $\mathcal{B} \in \mathbf{b}$ and $\mathcal{C} \in \mathbf{c}$, such that $\mathcal{C} = \{C_k\}_{k < \omega}$ and $C_{k+1} = \emptyset$ for every $k < \omega$. It is easy to check that for every n $P_n(\mathcal{C}) \equiv_e C_0^n$ uniformly in n . Now from $\mathbf{c} \sim_\omega \mathbf{z}$ it follows that there is a natural number n such that $\mathbf{c}^n = \mathbf{z}^n$, hence $\mathcal{C}^n \equiv_\omega \mathcal{Z}^n$ and in particular $C_0^n \equiv_e P_n(\mathcal{C}) \equiv_e P_n(\mathcal{Z})$. On the other hand as \mathbf{z} is almost- \mathbf{b} it follows that $C_0^n \equiv_e P_n(\mathcal{Z}) \equiv_e P_n(\mathcal{B})$. Now we apply the effectiveness of the monotonicity of the enumeration jump to prove that for every $k < \omega$ we have that $C_0^{n+k} \leq_e P_{n+k}(\mathcal{B})$ uniformly in k and hence $\mathbf{c}^n \leq_\omega \mathbf{b}^n$. This yields that $[\mathbf{z}]_{\sim_\infty} = [\mathbf{c}]_{\sim_\infty} \leq [\mathbf{b}]_{\sim_\infty}$ contradicting the fact that $[\mathbf{b}]_{\sim_\infty} < [\mathbf{z}]_{\sim_\infty}$.

□

Secondly we give the proof of Theorem 1.4:

Proof of Theorem 1.4. Let \mathbf{b} and \mathbf{a} be Σ_2^0 ω -enumeration degrees such that $[\mathbf{b}]_{\sim_\infty} < [\mathbf{a}]_{\sim_\infty}$. Without loss of generality we may assume that $\mathbf{b} <_\omega \mathbf{a}$. By Theorem 5.1 there is an almost- \mathbf{b} ω -enumeration degree \mathbf{z} such that $\mathbf{b} <_\omega \mathbf{z} \leq_\omega \mathbf{a}$. By Proposition 5.2 the interval $[\mathbf{b}, \mathbf{z}]$ in \mathcal{G}_ω , which is entirely made up of almost- \mathbf{b} degrees, and the interval $[[\mathbf{b}]_{\sim_\infty}, [\mathbf{z}]_{\sim_\infty}]$ in $\mathcal{G}_\omega / \sim_\infty$ are isomorphic as partial orderings. By Theorem 1.3 every countable partial ordering can be embedded in the interval $[\mathbf{b}, \mathbf{z}]$, and hence every countable partial ordering can be embedded in the interval $[[\mathbf{b}]_{\sim_\infty}, [\mathbf{z}]_{\sim_\infty}] \subseteq [[\mathbf{b}]_{\sim_\infty}, [\mathbf{a}]_{\sim_\infty}]$

5.2. Proof of Theorem 5.1. Let $\mathbf{b} <_\omega \mathbf{a} <_\omega \mathbf{0}'_\omega$ be two Σ_2^0 enumeration degrees. It follows from Proposition 5.1 that $[\mathbf{b}]_{\sim_\infty} \leq [\mathbf{a}]_{\sim_\infty}$.

Suppose that $[\mathbf{b}]_{\sim_\infty} = [\mathbf{a}]_{\sim_\infty}$. If we assume that there is an almost- \mathbf{b} degree \mathbf{z} such that $\mathbf{b} <_\omega \mathbf{z} \leq_\omega \mathbf{a}$ then by Proposition 5.2 it follows that $[\mathbf{b}]_{\sim_\infty} < [\mathbf{z}]_{\sim_\infty} \leq [\mathbf{a}]_{\sim_\infty}$, contradicting $[\mathbf{b}]_{\sim_\infty} = [\mathbf{a}]_{\sim_\infty}$.

Now assume that $[\mathbf{b}]_{\sim_\infty} < [\mathbf{a}]_{\sim_\infty}$. Let $\mathcal{A} = \{A_n\}_{n < \omega}$ be a representative of \mathbf{a} , such that $P(\mathcal{A}) \equiv_e \mathcal{A}$ and such that \mathcal{A} has a good approximation, $\mathbb{A} = \{A_n^{\{s\}}\}_{n, s < \omega}$. Fix a representative $\mathcal{B} = \{B_n\}_{n < \omega}$ of \mathbf{b} such that $\mathcal{B} \equiv_e P(\mathcal{B})$ and a correct with respect to \mathbb{A} approximation to \mathcal{B} , $\{B_n^{\{s\}}\}_{n, s < \omega}$. We will construct an enumeration operator V such that for every n , $V[n](A_n) \leq_e B_n$ and $V(\mathcal{A}) \not\leq_\omega \mathcal{B}$. Then by setting $\mathbf{z} = d_\omega(\mathcal{B} \oplus V(\mathcal{A}))$ we obtain the required almost- \mathbf{b} ω -enumeration degree. It is straightforward to check that for every n : $P_n(\mathcal{B}) \equiv_e B_n \equiv_e P_n(\mathcal{B} \oplus V(\mathcal{A}))$. One side follows from $B_n \leq_e P_n(\mathcal{B} \oplus V(\mathcal{A})) = P_{n-1}(\mathcal{B} \oplus V(\mathcal{A}))' \oplus (B_n \oplus V[n](A_n))$. The other side is proved by induction: $P_0(\mathcal{B} \oplus V(\mathcal{A})) = B_0 \oplus V[0](A_0) \leq_e B_0$. Now assuming $P_n(\mathcal{B} \oplus V(\mathcal{A})) \leq_e B_n$ we get $P_n(\mathcal{B} \oplus V(\mathcal{A}))' \leq_e P_n(\mathcal{B})' \leq_e B_{n+1}$ and hence $P_{n+1}(\mathcal{B} \oplus V(\mathcal{A})) = P_n(\mathcal{B} \oplus V(\mathcal{A}))' \oplus (B_{n+1} \oplus V[n+1](A_{n+1})) \leq_e B_{n+1}$.

For every sequence $\mathcal{C} = \{C_k\}_{k < \omega}$ denote by $\mathcal{C}^{(i^*)}$ the sequence $\{C_{i+k}\}_{k < \omega}$. Note that if $\mathcal{C} \equiv_e P(\mathcal{C})$ then $\mathcal{C}^{(i^*)} \equiv_e \mathcal{C}^i$. If $\mathbb{C} = \{C_k^{\{s\}}\}_{s, k < \omega}$ is an approximation to the sequence \mathcal{C} , then by $\mathbb{C}^{(i^*)}$ we shall denote the approximation $\{C_{i+k}^{\{s\}}\}_{s, k < \omega}$ to $\mathcal{C}^{(i^*)}$.

To ensure that $V(\mathcal{A}) \not\leq_\omega \mathcal{B}$, the constructed set V will satisfy the following list of requirements, where i ranges over the natural numbers:

$$\mathcal{R}_i : W_i(\mathcal{B}^{(i^*)}) \neq V(\mathcal{A})^{(i^*)}$$

To see that the satisfaction of this list of requirements guarantees $V(\mathcal{A}) \not\leq_\omega \mathcal{B}$, assume the contrary: suppose that the constructed set satisfies all requirements, but we still have $V(\mathcal{A}) = W_e(\mathcal{B})$ for some e-operator W_e . There is a computable function g such that the sequence $V(\mathcal{A})^{(i^*)} = W_{g(i)}(\mathcal{B}^{(i^*)})$. By the recursion theorem there will be an index i such that $W_i = W_{g(i)}$, contradicting that the requirement \mathcal{R}_i is satisfied.

Construction. The construction is in stages. Set $V^{\{0\}} = \emptyset$. At stage $s \geq 0$ we construct $V^{\{s+1\}}$ from its value at the previous stage. We examine all requirements \mathcal{R}_i with $i < s$ and perform the following actions for each in order of priority.

Fix \mathcal{R}_i . We define $l_i^{\{s\}} = l(W_i^{\{s\}}(\{B_{i+k}^{\{s\}}\}_{k < \omega}, \{V[i+k]^{\{s\}}(A_{i+k}^{\{s\}})\}_{k < \omega, s}))$.

For every $\langle k, x \rangle < l_i^{\{s\}}$ if $x \in A_{i+k}^{\{s\}}$ and $\langle i, x \rangle \notin V[i+k]^{\{s\}}(A_{i+k}^{\{s\}})^{\{s\}}$ then enumerate the axiom $\langle\langle i, x \rangle, A_{i+k}^{\{s\}} \rangle$ in $V[i+k]^{\{s+1\}}$.

This completes the construction.

Lemma 5.1. *Fore every $i < \omega$ the requirement \mathcal{R}_i is satisfied.*

Proof. The proof is by induction. Fix i and assume that the statement is true for $j < i$. Towards a contradiction assume that $W_i(\mathcal{B}^{(i*)}) = V(\mathcal{A})^{(i*)}$. By Lemma 4.1 $W_i^{\{s\}}(\{B_{i+k}^{\{s\}}\}_{s,k < \omega})$ and $\{V[i+k]^{\{s\}}(A_{i+k}^{\{s\}})\}_{s,k < \omega}$ are correct with respect to $\mathbb{A}^{(i*)}$ approximations to $W_i(\mathcal{B}^{(i*)})$ and $V(\mathcal{A})^{(i*)}$ respectively. By Lemma 4.3 we have that $\lim_{s \in G_{i+k}} l_i^{\{s\}} = \infty$ for all $k \geq 0$. We will prove that in this case $\mathcal{A}^{(i*)} \leq_\omega V(\mathcal{A})^{(i*)}$. As $\mathcal{A}^{(i*)} \equiv_\omega \mathcal{A}^i$ and $\mathcal{B}^{(i*)} \equiv_\omega \mathcal{B}^i$ this would yield $\mathcal{A}^i \leq_\omega \mathcal{B}^i$ contradicting $[b]_{\sim_\infty} < [a]_{\sim_\infty}$.

Claim 5.1.1. $\mathcal{A}^{(i*)} \leq_\omega V(\mathcal{A})^{(i*)}$.

Proof. We show that for every $k \geq 0$ we have $A_{i+k} = \{x \mid \langle i, x \rangle \in V[i+k](A_{i+k})\}$.

Fix $k \geq 0$ and let $x \in A_{i+k}$. Let s be an $(i+k)$ -good stage such that $\langle k, x \rangle < l_i^{\{s\}}$. Then by construction at stage s either $\langle i, x \rangle \in V[i+k]^{\{s\}}(A_{i+k})^{\{s\}} \subseteq V[i+k](A_{i+k})$ or the strategy enumerates $\langle i, x \rangle$ in $V[i+k](A_{i+k})$, i.e. enumerates the axiom $\langle i+k, i, x, A_{i+k}^{\{s\}} \rangle$ in $V^{\{s+1\}}$. In both cases $\langle i, x \rangle \in V[i+k](A_{i+k})$.

On the other hand suppose $\langle i, x \rangle \in V[i+k](A_{i+k})$. The only requirement which enumerates axioms for elements of the form $\langle i, x \rangle$ in $V[i+k]$ is \mathcal{R}_i . Such an axiom must be enumerated at an $(i+k)$ -good stage s of the construction in order for it to be valid. But by construction the strategy must have seen $x \in A_{i+k}^{\{s\}} \subseteq A_{i+k}$ and hence $x \in A_{i+k}$. □

Lemma 5.2. *For every n , the set $V[n](A_n) \leq_e B_n$.*

Proof. Fix n . There are finitely many requirements, which enumerate axioms in the set $V[n]$, namely the requirements \mathcal{R}_i , where $i \leq n$.

For every $i \leq n$ consider the length of agreement measured at stage s :

$$l_{i,n}^{\{s\}} = l(W_i[n-i]^{\{s\}}(B_n^{\{s\}}), V[n]^{\{s\}}(A_n^{\{s\}}), s).$$

If for every $i \leq n$ the set $\{l_{i,n}^{\{s\}} \mid A_n^{\{s\}} \subseteq A_n\}$ is bounded, then for every i $V[n](A_n)^{[i]}$ is finite, hence $V[n](A_n)$ is finite and reducible to B_n .

Otherwise let i be such that the set $\{l_{i,n}^{\{s\}} \mid A_n^{\{s\}} \subseteq A_n\}$ is unbounded. By Lemma 3.1 $\{W_i[n-i]^{\{s\}}(B_n^{\{s\}})\}_{s < \omega}$ is a correct with respect to \mathbb{A}_n approximation to $W_i[n-i](B_n)$ and $\{V[n]^{\{s\}}(A_n^{\{s\}})\}_{s < \omega}$ is a correct with respect to \mathbb{A} approximation to $V[n](A_n)$. We can apply Lemma 3.3 and obtain $W_i[n-i](B_n) = V[n](A_n)$, hence $V[n](A_n)$ is in this case as well reducible to B_n .

This completes the proof of Theorem 5.1. □

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