ASH'S THEOREM FOR ABSTRACT STRUCTURES

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Abstract. We introduce and study the class of relatively α-intrinsic sets on partial abstract structures. The main results are the Abstract jump inversion theorem and the Normal form theorem for the relatively α-intrinsic sets.

1. Introduction

In this paper we are going to prove an analog of Ash's Theorem [1] for abstract structures. We shall consider *partial* structures $\mathfrak{A} = (\mathbb{N}; R_1, \dots, R_k)$, where each R_i is a subset of $\mathbb{N}^{r_i}$ and the equality "=" and unequality "$\neq$" are among the predicates $R_1, \dots, R_k$.

Evidently every total structure $\mathfrak{A} = (\mathbb{N}; R_1, \dots, R_k)$ can be considered as a partial structure, since we can replace a total predicate R_i on $\mathbb{N}$ by two partial predicates R_i^+ and R_i^-, where $R_i^+ = \{\overline{x} : R_i(\overline{x}) = true\}$ and $R_i^- = \{\overline{x} : R_i(\overline{x}) = false\}$.

A total mapping from $\mathbb{N}$ onto $\mathbb{N}$ is called *enumeration of* $\mathfrak{A}$.

Given an enumeration f of $\mathfrak{A}$ and a subset of A of $\mathbb{N}^a$, let

$$f^{-1}(A) = \{\langle x_1, \dots, x_a\rangle : (f(x_1), \dots, f(x_a)) \in A\}.$$

By $f^{-1}(\mathfrak{A})$ we shall denote the set $f^{-1}(R_1)\oplus\cdots\oplus f^{-1}(R_k)$. In particular, if $f = \lambda x.x$, then $f^{-1}(\mathfrak{A})$ will be denoted by $D(\mathfrak{A})$.

Next we define for every recursive ordinal α the relatively α-intrinsic sets. This notion is a generalization of the respective notion of relatively intrinsically Σ^0_α sets, introduced in [2] and independently in [3] for total structures. Given a set D of natural numbers and a recursive ordinal α, by $D_e^{(\alpha)}$ we shall denote the α-th enumeration jump of D. The exact definition will be given in the next section.

1.1. Definition. Let α be a constructive ordinal and let $A \subseteq \mathbb{N}^a$. The set A is relatively α-intrinsic on the partial structure $\mathfrak{A}$ if for every enumeration f of $\mathfrak{A}$ the set $f^{-1}(A)$ is enumeration reducible to $(f^{-1}(\mathfrak{A}))_e^{(\alpha)}$.

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From the properties of the enumeration jump it follows that for total structures, i.e. partial structures obtained from total ones, for every recursive α the relatively α-intrinsic sets coincide with the relatively intrinsic $\Sigma^0_{\alpha+1}$ sets.

In [2] and in [3] an internal characterization of the relatively intrinsic Σ^0_α sets is obtained. Namely it is shown that these sets coincide with the sets definable on $\mathfrak{A}$ by means of the so called recursive Σ^0_α formulae. Because of this result we may think that on total structures the relatively intrinsically Σ^0_α sets are the right counterpart of the classical Σ^0_α sets.

Here we shall obtain a similar characterization of the relatively α-intrinsic sets on partial structures generalizing the respective results for total structures. On the other hand the results in [2] and [3] admit another kind of generalization which is in the spirit of the Ash's Theorem [1].

Consider a set $B \subseteq \mathbb{N}^a$. Suppose that you want to add this set to the structure $\mathfrak{A}$ as a partial predicate which is relatively β-intrinsic on $\mathfrak{A}$. It is not clear how to give an explicit definition of this kind of expansion of $\mathfrak{A}$ for recursive ordinals $\beta > 0$. Nevertheless we can obtain a new class of relatively α-intrinsic sets by restricting the class of all enumerations of $\mathfrak{A}$ to the class of those enumerations f of $\mathfrak{A}$ for which $f^{-1}(B)$ is enumeration reducible to $(f^{-1}(\mathfrak{A}))^{(\beta)}_e$. In other words, we consider only those enumerations of $\mathfrak{A}$ which "know" that B is relatively β-intrinsic on $\mathfrak{A}$. More generally, consider a sequence $\{B_\gamma\}_{\gamma\leq\zeta}$, where each B_γ is a subset of $\mathbb{N}^{a_\gamma}$, ζ is a constructive ordinal and there exists a recursive function ρ such that $\rho(\gamma) = a_\gamma$ for all $\gamma \leq \zeta$.

1.2. Definition. Let $\alpha < \omega_1^{CK}$. A subset A of $\mathbb{N}^a$ is *relatively α-intrinsic on $\mathfrak{A}$ with respect to the sequence $\{B_\gamma\}_{\gamma\leq\zeta}$* if for every enumeration f of $\mathfrak{A}$ such that $(\forall\gamma \leq \zeta)(f^{-1}(B_\gamma) \leq_e (f^{-1}(\mathfrak{A}))^{(\gamma)})$ uniformly in γ, the set $f^{-1}(A)$ is enumeration reducible to $(f^{-1}(\mathfrak{A}))^{(\alpha)}$.

In what follows we are going to present an explicit internal characterization of the relatively α-intrinsic with respect to the sequence $\{B_\gamma\}$ sets. For the sake of simplicity we shall assume that all sets B_γ are subsets of $\mathbb{N}$, i.e. $(\forall\gamma \leq \zeta)(a_\gamma = 1)$. The proofs in the general case are similar.

2. Preliminaries

2.1. Ordinal notations. We shall consider only recursive ordinals α which are below a fixed recursive ordinal η. We shall suppose that a notation $e \in \mathcal{O}$ for η is fixed and the notations for the ordinals $\alpha < \eta$ are elements a of $\mathcal{O}$ such that $a <_o e$. For the definitions of the set $\mathcal{O}$ and the relation "$<_o$" the reader may consult [7] or [8]. We shall identify every ordinal with its notation and denote the ordinals by the letters α, β, γ and δ. In particular we shall write $\alpha < \beta$ instead of $\alpha <_o \beta$. If α is a limit ordinal then by $\{\alpha(p)\}_{p\in\mathbb{N}}$ we shall denote the unique strongly increasing sequence of ordinals with limit α, determined by the notation of α, and write $\alpha = \lim \alpha(p)$.

2.2. The enumeration jump. Given two sets of natural numbers A and B, we say that A is enumeration reducible to B ($A \leq_e B$) if $A = \Gamma_z(B)$ for some enumeration operator Γ_z. In other words, using the notation D_v for the finite set having canonical code v and $W_0, \dots, W_z, \dots$ for the Gödel enumeration of the r.e. sets, we have

$$A \leq_e B \iff \exists z \forall x (x \in A \iff \exists v (\langle v, x\rangle \in W_z \;\&\; D_v \subseteq B)).$$

The relation $\leq_e$ is reflexive and transitive and induces an equivalence relation $\equiv_e$ on all subsets of $\mathbb{N}$. The respective equivalence classes are called enumeration degrees. For an introduction to the enumeration degrees the reader might consult Cooper [5].

Given a set A denote by A^+ the set $A \oplus (\mathbb{N} \setminus A)$. The set A is called *total* iff $A \equiv_e A^+$. Clearly A is recursively enumerable in B iff $A \leq_e B^+$ and A is recursive in B iff $A^+ \leq_e B^+$. Notice that the graph of every total function is a total set.

Evidently if $\mathfrak{A}$ is a partial structure, obtained from a total one, then for every enumeration f of $\mathfrak{A}$ the set $f^{-1}(\mathfrak{A})$ is total. So we may give the following definition:

2.1. Definition. An abstract structure $\mathfrak{A}$ is *total* if for every enumeration f of $\mathfrak{A}$ the set $f^{-1}(\mathfrak{A})$ is total.

The enumeration jump operator is defined in Cooper [4] and further studied by McEvoy [6]. Here we shall use the following definition of the e-jump which is m-equivalent to the original one, see [6]:

2.2. Definition. Given a set A, let $K_A^0 = \{\langle x, z\rangle : x \in \Gamma_z(A)\}$. Define the *e-jump* A'_e of A to be the set $(K_A^0)^+$.

The following properties of the enumeration jump are proved in [6]:

Let A and B be sets of natural numbers. Set $B_e^{(0)} = B$ and $B_e^{(n+1)} = (B_e^{(n)})'_e$.

(J1) If $A \leq_e B$, then $A'_e \leq_e B'_e$.

(J2) A is Σ^0_{n+1} relatively to B iff $A \leq_e (B^+)_e^{(n)}$.

Let α be a recursive ordinal. To define the α-th enumeration jump of a set A we are going to use a construction very similar to that used in the definition of the α-th Turing jump. For every recursive ordinal α we define the set E_α^A by means of transfinite recursion on α:

2.3. Definition.

(i) $E_0^A = A$.

(ii) $E_{\beta+1}^A = (E_\beta^A)'_e$.

(iii) If $\alpha = \lim \alpha(p)$, then $E_\alpha^A = \{\langle p, x\rangle : x \in E^A_{\alpha(p)}\}$.

From now on $A_e^{(\alpha)}$ will stand for E_α^A.

Of course the definition of the set $A_e^{(\alpha)}$ depends on the fixed notation of the ordinal α. On the other hand, it is easy to see by a minor modification of the proof of the respective Theorem of Spector for the H_α^A sets, see [7] or [8], that if α_1 and α_2 are two notations of the same recursive ordinal, then $A_e^{(\alpha_1)} \equiv_e A_e^{(\alpha_2)}$.

The following properties of the transfinite iterations of the enumeration jump follow easily from the definition:

(E1) If $\beta \leq \alpha$ are recursive ordinals, then $A_e^{(\beta)} \leq_e A_e^{(\alpha)}$ uniformly in β and α.
(E2) If $A \leq_e B$, then for every recursive ordinal α, $A_e^{(\alpha)} \leq_e B_e^{(\alpha)}$.
(E3) If $\alpha > 0$, then $A_e^{(\alpha)}$ is a total set.

Finally, we have that for total sets the α-th enumeration jump and the α-th Turing jump are equivalent. Namely the following is true:

2.4. Proposition. *Let A be a total set of natural numbers. Then for every recursive ordinal α, $E_\alpha^A \equiv_e (H_\alpha^A)^+$ uniformly in α.*

Since we are going to consider only e-jumps here, from now on we shall omit the subscript e in the notation of the enumeration jump. So for every recursive ordinal α by $A^{(\alpha)}$ we shall denote the α-th enumeration jump of A.

2.3. The jump set of a sequence of sets. Let ζ be a recursive ordinal and let $\{B_\gamma\}_{\gamma\leq\zeta}$ be a sequence of sets of natural numbers. For every recursive ordinal α we define the *jump set $\mathcal{P}_\alpha$ of the sequence* $\{B_\gamma\}$ by means of transfinite recursion on α:

2.5. Definition.

(i) $\mathcal{P}_0 = B_0$.
(ii) Let $\alpha = \beta + 1$. Then let

$$\mathcal{P}_\alpha = \begin{cases} \mathcal{P}'_\beta \oplus B_\alpha & \text{if } \alpha \leq \zeta, \\ \mathcal{P}'_\beta & \text{otherwise.} \end{cases}$$

(iii) Let $\alpha = \lim \alpha(p)$. Then set $\mathcal{P}_{<\alpha} = \{\langle p, x\rangle : x \in \mathcal{P}_{\alpha(p)}\}$ and let

$$\mathcal{P}_\alpha = \begin{cases} \mathcal{P}_{<\alpha} \oplus B_\alpha & \text{if } \alpha \leq \zeta, \\ \mathcal{P}_{<\alpha} & \text{otherwise.} \end{cases}$$

Notice that if the sequence $\{B_\gamma\}$ contains only one member, i.e $\zeta = 0$, then for every recursive α, $\mathcal{P}_\alpha = B_0^{(\alpha)}$.

The properties of the jump sets $\mathcal{P}_\alpha$ are similar to the properties of the enumeration jumps. Again we have that if α_1 and α_2 are two notations of the same recursive ordinal, then $\mathcal{P}_{\alpha_1} \equiv_e \mathcal{P}_{\alpha_2}$. We shall omit the proof since it is very close to the proof of the respective result for the H_α^A sets mentioned above.

We shall use the following properties of the jump sets which follow easily from the definition:

($\mathcal{P}$1) If $\beta \leq \alpha$, then $\mathcal{P}_\beta \leq_e \mathcal{P}_\alpha$ uniformly in β and α.
($\mathcal{P}$2) If $\gamma \leq \min(\alpha, \zeta)$, then $B_\gamma \leq_e \mathcal{P}_\alpha$ uniformly in γ and α.
($\mathcal{P}$3) Let $(\forall\gamma \leq \min(\alpha,\zeta))(B_\gamma \leq_e A^{(\gamma)}$ uniformly in $\gamma)$. Then $\mathcal{P}_\alpha \leq_e A^{(\alpha)}$.
($\mathcal{P}$4) If α is a limit ordinal, then the set $\mathcal{P}_{<\alpha}$ is total.
($\mathcal{P}$5) If $\zeta < \alpha$, then the set $\mathcal{P}_\alpha$ is total.

We conclude the preliminaries by a jump inversion theorem proved in [9]:

2.6. Theorem. *Let $A \subseteq \mathbb{N}$ and $\{B_\gamma\}_{\gamma\leq\zeta}$ be a sequence of sets of natural numbers. Suppose that $\alpha < \zeta$ is a recursive ordinal such $A \not\leq_e \mathcal{P}_\alpha$. Let Q be a total subset of $\mathbb{N}$ such that $\mathcal{P}_\zeta \leq_e Q$ and $A^+ \leq_e Q$. Then there exists a total set F having the following properties:*

1. *For all $\gamma \leq \zeta$, $B_\gamma \leq_e F^{(\gamma)}$ uniformly in γ;*
2. *For all $\gamma \leq \zeta$ if $\gamma = \beta + 1$, then $F^{(\gamma)} \equiv_e F \oplus \mathcal{P}'_\beta$ uniformly in γ;*
3. *For all limit $\gamma \leq \zeta$, $F^{(\gamma)} \equiv_e F \oplus \mathcal{P}_{<\gamma}$ uniformly in γ;*
4. *$F^{(\zeta)} \equiv_e Q$.*
5. *$A \not\leq_e F^{(\alpha)}$.*

Here we shall use the following obvious corollary of the above Theorem.

2.7. Theorem. *Let $\{B_\gamma\}_{\gamma\leq\zeta}$ be a sequence of sets of natural numbers. Let α be a recursive ordinal and $\xi = \max(\alpha+1,\zeta)$. Suppose that $A \subseteq \mathbb{N}$ and $A \not\leq_e \mathcal{P}_\alpha$, Q is a total set and $\mathcal{P}_\xi \oplus A^+ \leq_e Q$. Then there exists a total set F with the following properties:*

1. *For all $\gamma \leq \zeta$, $B_\gamma \leq_e F^{(\gamma)}$ uniformly in γ;*
2. *$F^{(\xi)} \equiv_e Q$.*
3. *$A \not\leq_e F^{(\alpha)}$.*

Proof. For every γ such that $\zeta < \gamma \leq \xi$ set $B_\gamma = \emptyset$. Apply the previous Theorem for the sequence $\{B_\gamma\}_{\gamma\leq\xi}$. □

Throughout the rest of the paper we shall suppose fixed a partial structure $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_k)$ and a sequence $\{B_\gamma\}_{\gamma\leq\zeta}$ of sets of natural numbers. Without loss of generality we shall assume that all sets B_γ are not empty.

3. Forcing fundamentals

Let f be an enumeration of $\mathfrak{A}$. For every recursive ordinal α by $\mathcal{P}^f_\alpha$ we shall denote the α-th jump set of the sequence $f^{-1}(\mathfrak{A}) \oplus f^{-1}(B_0), f^{-1}(B_1), \ldots, f^{-1}(B_\zeta)$.

For every α, e and x in $\mathbb{N}$ we define the relations $f \models_\alpha F_e(x)$ and $f \models_\alpha \neg F_e(x)$ as follows:

(i) $f \models_0 F_e(x)$ iff there exists a v such that $\langle v, x\rangle \in W_e$ and for all $u \in D_v$ either
  a) $u = \langle 0, \langle i, x^u_1, \ldots, x^u_{r_i}\rangle\rangle$ & $(f(x^u_1), \ldots, f(x^u_{r_i})) \in R_i$ or
  b) $u = \langle 2, x_u\rangle$ & $f(x_u) \in B_0$.

(ii) Let $\alpha = \beta + 1$. Then
  a) if $\alpha \leq \zeta$, then

$$\begin{aligned} f \models_\alpha F_e(x) \iff & (\exists v)(\langle v, x\rangle \in W_e \;\&\; (\forall u \in D_v)(\\ & (u = \langle 0, e_u, x_u\rangle \;\&\; f \models_\beta F_{e_u}(x_u)) \vee \\ & (u = \langle 1, e_u, x_u\rangle \;\&\; f \models_\beta \neg F_{e_u}(x_u)) \vee \\ & (u = \langle 2, x_u\rangle \;\&\; f(x_u) \in B_\alpha))); \end{aligned}$$

b) if $\zeta < \alpha$, then

$$f \models_\alpha F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)((u = \langle 0, e_u, x_u\rangle \ \&\ f \models_\beta F_{e_u}(x_u)) \vee (u = \langle 1, e_u, x_u\rangle \ \&\ f \models_\beta \neg F_{e_u}(x_u)))).$$

(iii) Let $\alpha = \lim \alpha(p)$. Then

a) if $\alpha \le \zeta$, then

$$f \models_\alpha F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)((u = \langle 0, p_u, e_u, x_u\rangle \ \&\ f \models_{\alpha(p_u)} F_{e_u}(x_u)) \vee (u = \langle 2, x_u\rangle \ \&\ f(x_u) \in B_\alpha)));$$

b) if $\zeta < \alpha$, then

$$f \models_\alpha F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)((u = \langle 0, p_u, e_u, x_u\rangle \ \&\ f \models_{\alpha(p_u)} F_{e_u}(x_u)))).$$

(iv) $f \models_\alpha \neg F_e(x) \iff f \not\models_\alpha F_e(x)$.

An immediate corollary of the definitions above is the following:

3.1. Lemma. *Let $A \subseteq \mathbb{N}$ and let $\alpha \le \zeta$. Then $A \le_e \mathcal{P}^f_\alpha$ iff there exists an e such that $A = \{x : f \models_\alpha F_e(x)\}$.*

The forcing conditions, which we shall call *finite parts*, are arbitrary finite mappings of $\mathbb{N}$ into $\mathbb{N}$. We shall denote the finite parts by the greek letters τ, ρ and δ.

For every $\alpha \le \zeta$, e and x in $\mathbb{N}$ and every finite part τ we define the forcing relations $\tau \Vdash_\alpha F_e(x)$ and $\tau \Vdash_\alpha \neg F_e(x)$ following the definition of "$\models$":

(i) $\tau \Vdash_0 F_e(x)$ iff there exists a v such that $\langle v, x\rangle \in W_e$ and for all $u \in D_v$ either

a) $u = \langle 0, \langle i, x^u_1, \dots, x^u_{r_i}\rangle\rangle$, $x^u_1, \dots, x^u_{r_i} \in \mathrm{dom}(\tau)$ and $(\tau(x^u_1), \dots, \tau(x^u_{r_i})) \in R_i$ or

b) $u = \langle 2, x_u\rangle$, $x_u \in \mathrm{dom}(\tau)$ and $\tau(x_u) \in B_0$.

(ii) Let $\alpha = \beta + 1$. Then

a) if $\alpha \le \zeta$, then

$$\tau \Vdash_\alpha F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)((u = \langle 0, e_u, x_u\rangle \ \&\ \tau \Vdash_\beta F_{e_u}(x_u)) \vee (u = \langle 1, e_u, x_u\rangle \ \&\ \tau \Vdash_\beta \neg F_{e_u}(x_u)) \vee (u = \langle 2, x_u\rangle \ \&\ !\tau(x_u) \in B_\alpha)));$$

b) if $\zeta < \alpha$, then

$$\tau \Vdash_\alpha F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)((u = \langle 0, e_u, x_u\rangle \ \&\ \tau \Vdash_\beta F_{e_u}(x_u)) \vee (u = \langle 1, e_u, x_u\rangle \ \&\ \tau \Vdash_\beta \neg F_{e_u}(x_u)))).$$

(iii) Let $\alpha = \lim \alpha(p)$. Then

a) if $\alpha \leq \zeta$, then

$$\begin{aligned} \tau \Vdash_\alpha F_e(x) \iff & (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(\\ & (u = \langle 0, p_u, e_u, x_u\rangle \ \&\ \tau \Vdash_{\alpha(p_u)} F_{e_u}(x_u)) \vee \\ & (u = \langle 2, x_u\rangle \ \&\ !\tau(x_u) \in B_\alpha))); \end{aligned}$$

b) if $\zeta < \alpha$, then

$$\begin{aligned} \tau \Vdash_\alpha F_e(x) \iff & (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(\\ & (u = \langle 0, p_u, e_u, x_u\rangle \ \&\ \tau \Vdash_{\alpha(p_u)} F_{e_u}(x_u)))). \end{aligned}$$

(iv) $\tau \Vdash_\alpha \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \nVdash_\alpha F_e(x))$.

For every recursive ordinal α, $e, x \in \mathbb{N}$ set $X^\alpha_{\langle e,x\rangle} = \{\rho : \rho \Vdash_\alpha F_e(x)\}$.

3.2. Definition. An enumeration f of $\mathfrak{A}$ is *α-generic* if for every $\beta < \alpha$, $e, x \in \mathbb{N}$ the following condition holds:

$$(\forall \tau \subseteq f)(\exists \rho \in X^\beta_{\langle e,x\rangle})(\tau \subseteq \rho) \Rightarrow (\exists \tau \subseteq f)(\tau \in X^\beta_{\langle e,x\rangle}). \tag{3.1}$$

The following standard properties of the forcing relation follow immediately from the definitions:

3.3. Lemma. *(TLA)*

(1) *Let α be a recursive ordinal, $e, x \in \mathbb{N}$ and let $\tau \subseteq \rho$ be finite parts. Then*

$$\tau \Vdash_\alpha (\neg)F_e(x) \Rightarrow \rho \Vdash_\alpha (\neg)F_e(x).$$

(2) *Let f be an α-generic enumeration. Then*

$$f \models_\alpha F_e(x) \iff (\exists \tau \subseteq f)(\tau \Vdash_\alpha F_e(x)).$$

(3) *Let f be an $\alpha+1$-generic enumeration. Then*

$$f \models_\alpha \neg F_e(x) \iff (\exists \tau \subseteq f)(\tau \Vdash_\alpha \neg F_e(x)).$$

Finally we would like to estimate an upper bound of the complexity of the forcing relation.

Given a sequence $\{X_n\}$ of sets of natural numbers, say that $\{X_n\}$ is e-reducible to the set P if there exists a recursive function g such that for all n we have that $X_n = \Gamma_{g(n)}(P)$. The sequence $\{X_n\}$ is T-reducible to P, if the function $\lambda n, x.\chi_{X_n}(x)$ is recursive in P.

From the definition of the enumeration jump it follows immediately that if $\{X_n\}$ is e-reducible to P, then $\{X_n\}$ is T-reducible to P'.

For every recursive ordinal α let $\mathcal{P}_\alpha$ be the α-th jump set of the sequence $B_0 \oplus D(\mathfrak{A}), B_1, \ldots, B_\zeta$.

3.4. Lemma. *For every α the sequence $\{X^\alpha_n\}$ is uniformly in α e-reducible to $\mathcal{P}_\alpha$ and hence it is uniformly in α T-reducible to $\mathcal{P}'_\alpha$.*

Proof. Using effective transfinite recursion and following the definition of the forcing, one can define a recursive function $g(\alpha, n)$ such that for every α, $X_n^\alpha = \Gamma_{g(\alpha,n)}(\mathcal{P}_\alpha)$. □

4. An abstract jump inversion theorem

4.1. Definition. Let $A \subseteq \mathbb{N}$ and let α be a recursive ordinal. The set A is *forcing α-definable* on $\mathfrak{A}$ if there exist a finite part δ and $e, x \in \mathbb{N}$ such that

$$A = \{s : (\exists \tau \supseteq \delta)(\tau(x) \simeq s \ \&\ \tau \Vdash_\alpha F_e(x))\}.$$

Clearly if A is forcing α-definable on $\mathfrak{A}$, then $A \leq_e \mathcal{P}_\alpha$. The vice versa is not always true. As we shall see later the forcing α-definable sets coincide with the sets which are relatively α-intrinsic with respect to sequence $\{B_\gamma\}_{\gamma \leq \zeta}$.

4.2. Proposition. *(Poli) Let α be a recursive ordinal and let $A \subseteq \mathbb{N}$ be not forcing α-definable on $\mathfrak{A}$. Set $\xi = \max(\alpha + 1, \zeta)$. There exists an enumeration f of $\mathfrak{A}$ satisfying the following conditions:*

(1) $f \leq_e A^+ \oplus \mathcal{P}_\xi$.
(2) *If* $\gamma \leq \xi$, *then* $\mathcal{P}_\gamma^f \leq_e f \oplus \mathcal{P}_\gamma$.
(3) $f^{-1}(A) \not\leq_e \mathcal{P}_\alpha^f$.

Proof. We shall construct the enumeration f by steps. At each step q we shall define a finite part δ_q so that $\delta_q \subseteq \delta_{q+1}$ and take $f = \bigcup_q \delta_q$. We shall consider three kinds of steps. On steps $q = 3r$ we shall ensure that the mapping f is total and surjective. On steps $q = 3r + 1$ we shall ensure that f is ξ-generic and on steps $q = 3r + 2$ we shall ensure that f satisfies (3).

Let $\gamma_0, \gamma_1, \ldots$ be a recursive enumeration of all ordinals less than ξ. For every natural number n set $Y_n = X_{(n)_1}^{\gamma_{(n)_0}}$. Notice that the sequence $\{Y_n\}$ is T-reducible to $\mathcal{P}_\xi$.

Let δ_0 be the empty finite part and suppose that δ_q is defined.

a) Case $q = 3r$. Let x_0 be the least natural number which does not belong to $\mathrm{dom}(\delta_q)$ and let s_0 be the least natural number which does not belong to the range of δ_q. Set $\delta_{q+1}(x_0) \simeq s_0$ and $\delta_{q+1}(x) \simeq \delta_q(x)$ for $x \neq x_0$.

b) Case $q = 3r+1$. Consider the set Y_r. Check whether there exists an element ρ of Y_r such that $\delta_q \subseteq \rho$. If the answer is positive, then let δ_{q+1} be the least extension of δ_q belonging to Y_r. If the answer is negative then let $\delta_{q+1} = \delta_q$.

c) Case $q = 3r + 2$. Let x_q be the least natural number which does not belong to $\mathrm{dom}(\delta_q)$. Consider the set

$$C_r = \{s : (\exists \tau \supseteq \delta_q)(\tau(x_q) \simeq s \ \&\ \tau \Vdash_\alpha F_r(x_q))\}$$

Clearly C_r is forcing α-definable on $\mathfrak{A}$ and hence $C_r \neq A$. Notice that $C_r \leq_e \mathcal{P}_\alpha$ uniformly in r and δ_q. Therefore, since $\alpha < \xi$, the set C_r is recursive in $\mathcal{P}_\xi$ uniformly in r and δ_q. Let s_0 be the least natural number such that

$$s_0 \in C_r \ \&\ s_0 \notin A \vee s_0 \notin C_r \ \&\ s_0 \in A.$$

Suppose that $s_0 \in C_r$. Then there exists a τ such that

$$\delta_q \subseteq \tau \ \&\ \tau(x_q) \simeq s_0 \ \&\ \tau \Vdash_\alpha F_r(x_q). \tag{4.1}$$

Let δ_{q+1} be the least τ satisfying (4.1).

If $s_0 \notin C_r$, then set $\delta_{q+1}(x_q) \simeq s_0$ and $\delta_{q+1}(x) \simeq \delta_q(x)$ for $x \neq x_q$. Notice that in this case we have that $\delta_{q+1} \Vdash_\alpha \neg F_r(x_q)$.

From the construction above it follows immediately that $f = \bigcup_q \delta_q$ is e-reducible to $A^+ \oplus \mathcal{P}_\xi$ and hence it satisfies (1).

Let $\gamma \leq \xi$. Then there exists an e such that $\mathcal{P}^f_\gamma = \{x : f \models_\gamma F_e(x)\}$. Since f is ξ-generic, we can rewrite the last equality as $\mathcal{P}^f_\gamma = \{x : (\exists \tau \subseteq f)(\tau \Vdash_\gamma F_e(x))\}$ Therefore $\mathcal{P}^f_\gamma \leq_e f \oplus \mathcal{P}_\gamma$.

It remains to show that $f^{-1}(A) \not\leq_e \mathcal{P}^f_\alpha$. Towards a contradiction assume that $f^{-1}(A) \leq_e \mathcal{P}^f_\alpha$. Then there exists an r such that

$$A = \{f(x) : f \models_\alpha F_r(x)\}.$$

Consider the step $q = 3r + 2$. By the construction we have that

$$\delta_{q+1}(x_q) \notin A \ \&\ \delta_{q+1} \Vdash_\alpha F_r(x_q) \vee \delta_{q+1}(x_q) \in A \ \&\ \delta_{q+1} \Vdash_\alpha \neg F_r(x_q).$$

Hence by the genericity of f

$$f(x_q) \notin A \ \&\ f \models_\alpha F_r(x_q) \vee f(x_q) \in A \ \&\ f \models_\alpha \neg F_r(x_q).$$

A contradiction. □

The following theorem is an abstract version of Theorem 2.7.

4.3. Theorem. *(AJIT) Let α be a recursive ordinal and let $A \subseteq \mathbb{N}$ be not forcing α-definable on $\mathfrak{A}$. Set $\xi = \max(\alpha{+}1, \zeta)$ and let Q be a total set such that $A^+ \oplus \mathcal{P}_\xi \leq_e Q$. Then there exists an enumeration f of $\mathfrak{A}$ satisfying the following conditions:*

(1) *$f \leq_e Q$.*
(2) *The enumeration degree of $f^{-1}(\mathfrak{A})$ is total, i.e. it contains a total set.*
(3) *For all $\gamma \leq \zeta$, $f^{-1}(B_\gamma) \leq_e (f^{-1}(\mathfrak{A}))^{(\gamma)}$ uniformly in γ.*
(4) *$f^{-1}(A) \not\leq_e (f^{-1}(\mathfrak{A}))^{(\alpha)}$.*
(5) *$(f^{-1}(\mathfrak{A}))^{(\xi)} \equiv_e Q$.*

Proof. According Proposition 4.2 there exists an enumeration g of $\mathfrak{A}$ such that $g \leq_e Q$, $\mathcal{P}^g_\xi \leq_e Q$ and $g^{-1}(A) \not\leq_e \mathcal{P}^g_\alpha$. Since $A^+ \leq_e Q$, we have also that $(g^{-1}(A))^+ \leq_e Q$.

From Theorem 2.7 it follows that there exists a total set F such that the following assertions are true:

(i) $g^{-1}(\mathfrak{A}) \leq_e F$.
(ii) For all $\gamma \leq \zeta$, $g^{-1}(B_\gamma) \leq_e F^{(\gamma)}$ uniformly in γ.
(iii) $g^{-1}(A) \not\leq_e F^{(\alpha)}$.
(iv) $F^{(\xi)} \equiv_e Q$.

We shall construct the enumeration f so that $f^{-1}(\mathfrak{A}) \equiv_e F$. Let s and t be two distinct elements of $\mathbb{N}$. Fix also two numbers x_s and x_t such that $g(x_s) \simeq s$ and $g(x_t) \simeq t$.

For $x \in \mathbb{N}$ set

$$f(x) \simeq \begin{cases} g(x/2) & \text{if } x \text{ is even,} \\ s & \text{if } x = 2z+1 \text{ and } z \in F, \\ t & \text{if } x = 2z+1 \text{ and } z \notin F. \end{cases}$$

Since "=" and "$\neq$" are among the underlined predicates of $\mathfrak{A}$, we have that $F \leq_e f^{-1}(\mathfrak{A})$. To prove that $f^{-1}(\mathfrak{A}) \leq_e F$ consider the partial predicate R_i of $\mathfrak{A}$. Let $x_1, \ldots, x_{r_i}$ be arbitrary natural numbers. Define the natural numbers $y_1, \ldots, y_{r_i}$ by means of the following recursive in F procedure. Let $1 \leq j \leq r_i$. If x_j is even then let $y_j = x_j/2$. If $x_j = 2z+1$ and $z \in F$, then let $y_j = x_s$. If $x_j = 2z+1$ and $z \notin F$, then let $y_j = x_t$. Clearly

$$\langle x_1, \ldots, x_{r_i} \rangle \in f^{-1}(R_i) \iff \langle y_1, \ldots, y_{r_i} \rangle \in g^{-1}(R_i).$$

Since $g^{-1}(\mathfrak{A}) \leq F$, from the last equivalence it follows that $f^{-1}(R_i) \leq_e F$. So we obtain that $f^{-1}(\mathfrak{A}) \leq_e F$.

To prove (2) it is sufficient to show that if $\gamma \leq \zeta$, then $f^{-1}(B_\gamma) \leq_e F^{(\gamma)}$ uniformly in γ. Denote by E_f the set $f^{-1}(=)$. Clearly for all $\gamma \leq \zeta$ we have that E_f and $g^{-1}(B_\gamma)$ are e-reducible to $F^{(\gamma)}$ uniformly in γ. Let us fix a $\gamma \leq \zeta$. From the definition of f it follows that

$$f^{-1}(B_\gamma) = \{x : (\exists y \in g^{-1}(B_\gamma))(\langle x, 2y \rangle \in E_f)\}.$$

Therefore $f^{-1}(B_\gamma) \leq_e F^{(\gamma)}$ uniformly in γ.

It remains to see that $f^{-1}(A) \not\leq_e F^{(\alpha)}$. Assume that $f^{-1}(A) \leq_e F^{(\alpha)}$. Clearly

$$g^{-1}(A) = \{x : 2x \in f^{-1}(A)\}.$$

Then $g^{-1}(A) \leq_e f^{-1}(A) \leq_e F^{(\alpha)}$. A contradiction. □

4.4. Definition. Let Q be a total subset of $\mathbb{N}$ and $\xi < \omega_1^{CK}$. An enumeration f of $\mathfrak{A}$ is *ξ, Q-acceptable* (with respect to the sequence $\{B_\gamma\}_{\gamma \leq \zeta}$) if f satisfies the following conditions:

(i) The enumeration degree of $f^{-1}(\mathfrak{A})$ is total.
(ii) $(\forall \gamma \leq \zeta)(f^{-1}(B_\gamma) \leq_e (f^{-1}(\mathfrak{A}))^{(\gamma)})$ uniformly in γ.
(iii) $(f^{-1}(\mathfrak{A}))^{(\xi)} \equiv_e Q$.

4.5. Theorem. *(AJIT1) Given a total set Q such that $\mathcal{P}_\xi \leq_e Q$, one can construct a ξ, Q-acceptable enumeration $f \leq_e Q$.*

Proof. Repeat the proof of the previous Theorem without bothering about the set A. □

4.6. Theorem. *(AJIT2) Let $\alpha < \omega_1^{CK}$ and let $A \subseteq \mathbb{N}$. Let $\xi = \max(\alpha+1, \zeta)$. Suppose that $Q \geq_e \mathcal{P}_\xi$, Q is a total set and for all ξ, Q-acceptable enumerations f of $\mathfrak{A}$ we have that $f^{-1}(A) \leq_e (f^{-1}(\mathfrak{A}))^{(\alpha)}$. Then A is forcing α-definable on $\mathfrak{A}$.*

Proof. First we shall show that $A^+ \leq_e Q$. By the previous Theorem there exists an enumeration g of $\mathfrak{A}$ such that $g \leq_e Q$ and g is ξ, Q-acceptable. Then $g^{-1}(A) \leq_e (g^{-1}(\mathfrak{A}))^{(\alpha)}$. By the monotonicity of the enumeration jump we can conclude that

$$(g^{-1}(A))' \leq_e (g^{-1}(\mathfrak{A}))^{(\alpha+1)} \leq_e (g^{-1}(\mathfrak{A}))^{(\xi)} \leq_e Q.$$

Since $(g^{-1}(A))^+ \leq_e (g^{-1}(A))'$, we get that $(g^{-1}(A))^+ \leq_e Q$. Therefore both A and $\mathbb{N} \setminus A$ are enumeration reducible to Q and hence $A^+ \leq_e Q$.

Assume that A is not forcing α-definable on $\mathfrak{A}$. Applying Theorem 4.3 we obtain an ξ, Q-acceptable enumeration f such that $f^{-1}(A) \not\leq_e (f^{-1}(\mathfrak{A}))^{(\alpha)}$. A contradiction. □

5. Normal form of the forcing definable sets

In this section we shall show that the forcing definable sets on the partial structure $\mathfrak{A}$ coincide with the sets which are definable on $\mathfrak{A}$ by means of a certain kind of *positive* recursive Σ^0_α formulae. This formulae can be considered as a modification of the formulae introduced in [1], which is appropriate for their use on abstract structures.

Let $\mathfrak{L} = \{T_1, \ldots, T_k\}$ be the first order language corresponding to the structure $\mathfrak{A}$. So every T_i is an r_i-ary predicate symbol. Let $\{P_\gamma\}_{\gamma \leq \zeta}$ be a recursive sequence of unary predicate symbols intended to represent the sets B_γ. We shall suppose also fixed a sequence $\mathbb{X}_0, \ldots, \mathbb{X}_n, \ldots$ of variables. The variables will be denoted by the letters X, Y, W possibly indexed.

Next we define for $\alpha < \omega_1^{CK}$ the Σ^+_α formulae. The definition is by transfinite recursion on α and goes along with the definition of indices (codes) for every formula. We shall leave to the reader the explicit definition of the indices of our formulae which can be done in a natural way.

5.1. Definition.

(i) Let $\alpha = 0$. The elementary Σ^+_α formulae are formulae in prenex normal form with a finite number of existential quantifiers and a matrix which is a finite conjunction of atomic predicates built up from the variables and the predicate symbols $T_1, \ldots, T_k$ and P_0.

(ii) Let $\alpha = \beta + 1$ and $\alpha \leq \zeta$. An elementary Σ^+_α formula is in the form

$$\exists Y_1 \ldots \exists Y_l M(X_1, \ldots, X_l, Y_1, \ldots, Y_m),$$

where M is a finite conjunction of atoms of the form $P_\alpha(X_i)$ or $P_\alpha(Y_j)$, Σ^+_β formulae and negations of Σ^+_β formulae with free variables among $X_1, \ldots, X_l$, $Y_1, \ldots, Y_m$.

(iii) Let $\alpha = \beta + 1$ and $\alpha > \zeta$. An elementary Σ^+_α formula is in the form

$$\exists Y_1 \ldots \exists Y_l M(X_1, \ldots, X_l, Y_1, \ldots, Y_m),$$

where M is a finite conjunction of atoms of Σ^+_β formulae and negations of Σ^+_β formulae with free variables among $X_1, \ldots, X_l$, $Y_1, \ldots, Y_m$.

(iv) Let $\alpha = \lim \alpha(p)$ be a limit ordinal and $\alpha \leq \zeta$. The elementary Σ^+_α formula are in the form

$$\exists Y_1 \ldots \exists Y_l M(X_1, \ldots, X_l, Y_1, \ldots, Y_m),$$

where M is a finite conjunction of atoms of the form $P_\alpha(X_i)$ or $P_\alpha(Y_j)$ and $\Sigma^+_{\alpha(p)}$ formulae with free variables among $X_1, \ldots, X_l, Y_1, \ldots, Y_m$.

(v) Let $\alpha = \lim \alpha(p)$ be a limit ordinal and $\alpha > \zeta$. The elementary Σ^+_α formula are in the form

$$\exists Y_1 \ldots \exists Y_l M(X_1, \ldots, X_l, Y_1, \ldots, Y_m),$$

where M is a finite conjunction of $\Sigma^+_{\alpha(p)}$ formulae with free variables among $X_1, \ldots, X_l, Y_1, \ldots, Y_m$.

(vi) A Σ^+_α formula with free variables among $X_1, \ldots, X_l$ is an r.e. infinitary disjunction of elementary Σ^+_α formulae with free variables among $X_1, \ldots, X_l$.

Notice that the Σ^+_α formulae are effectively closed under existential quantification and infinitary r.e. disjunctions.

Let Φ be a Σ^+_α formula with free variables among $W_1, \ldots, W_n$ and let $t_1, \ldots, t_n$ be elements of $\mathbb{N}$. Then by $\mathfrak{A} \models \Phi(W_1/t_1, \ldots, W_n/t_n)$ we shall denote that Φ is true on $\mathfrak{A}$ under the variable assignment v such that $v(W_1) = t_1, \ldots, v(W_n) = t_n$.

5.2. Definition. Let $A \subseteq \mathbb{N}$ and let $\alpha < \zeta$. The set A is *formally α-definable* on $\mathfrak{A}$ with respect to the sequence $\{B_\gamma\}_{\gamma<\zeta}$ if there exists a Σ^+_α formula Φ with free variables among $W_1, \ldots, W_r, X$ and elements $t_1, \ldots, t_r$ of $\mathbb{N}$ such that for every element s of $\mathbb{N}$ the following equivalence holds:

$$s \in A \iff \mathfrak{A} \models \Phi(W_1/t_1, \ldots, W_r/t_r, X/s).$$

We shall show that every forcing α-definable set is formally α-definable.

Let *var* be an effective mapping of the natural numbers onto the variables. Given a natural number x, by X we shall denote the variable $var(x)$.

Let $y_1 < y_2 < \ldots < y_k$ be the elements of a finite set D, let Q be one of the quantifiers $\exists$ or $\forall$ an let Φ be an arbitrary formula. Then by $Q(y : y \in D)\Phi$ we shall denote the formula $QY_1 \ldots QY_k\Phi$.

5.3. Lemma. *Let $D = \{w_1, \ldots, w_r\}$ be a finite and not empty set of natural numbers and x, e be elements of $\mathbb{N}$. Let $\alpha < \omega_1^{CK}$. There exists an uniform recursive way to construct a Σ^+_α formula $\Phi^\alpha_{D,e,x}$ with free variables among $W_1, \ldots, W_r$ such that for every finite part δ such that $dom(\delta) = D$ the following equivalence is true:*

$$\mathfrak{A} \models \Phi^\alpha_{D,e,x}(W_1/\delta(w_1), \ldots, W_r/\delta(w_r)) \iff \delta \Vdash_\alpha F_e(x).$$

Proof. We shall construct the formula $\Phi^\alpha_{D,e,x}$ by means of effective transfinite recursion on α following the definition of the forcing.

1) Let $\alpha = 0$. Let $V = \{v : \langle v, x\rangle \in W_e\}$. Consider an element v of V. For every $u \in D_v$ define the atom Π_u as follows

a) If $u = \langle 0, \langle i, x^u_1, \dots, x^u_{r_i}\rangle\rangle$, where $1 \le i \le k$ and all $x^u_1, \dots, x^u_{r_i}$ are elements of D, then let $\Pi_u = T_i(X^u_1, \dots, X^u_{r_i})$.

b) If $u = \langle 2, x_u\rangle$ and $x_u \in D$, then let $\Pi_u = P_0(X_u)$.

c) Let $\Pi_u = W_1 \ne W_1$ in the other cases.

Set $\Pi_v = \bigwedge_{u\in D_v} \Pi_u$ and $\Phi^\alpha_{D,e,x} = \bigvee_{v\in V} \Pi_v$.

2) Let $\alpha = \beta + 1$ Let again $V = \{v : \langle v, x\rangle \in W_e\}$ and $v \in V$.

For every $u \in D_v$ define the formula Π_u as follows:

a) If $u = \langle 0, e_u, x_u\rangle$, then let $\Pi_u = \Phi^\beta_{D,e_u,x_u}$.

b) If $u = \langle 1, e_u, x_u\rangle$, then let

$$\Pi_u = \neg[\bigvee_{D^*\supseteq D} (\exists y \in D^* \setminus D)\Phi^\beta_{D^*,e_u,x_u}].$$

c) If $\alpha \le \zeta$, $u = \langle 2, x_u\rangle$ and $x_u \in D$ then let $\Pi_u = P_\alpha(X_u)$.

d) Let $\Pi_u = \Phi^\beta_{\{0\},0,0} \wedge \neg\Phi^\beta_{\{0\},0,0}$ in the other cases.

Now let $\Pi_v = \bigwedge_{u\in D_v} \Pi_u$ and set $\Phi^\alpha_{D,e,x} = \bigvee_{v\in V} \Pi_v$.

3) Let $\alpha = \lim \alpha(p)$ be a limit ordinal. Let $V = \{v : \langle v, x\rangle \in W_e\}$. Consider a $v \in V$. For every element u of D_v we define the formula Π_u as follows:

a) If $u = \langle 0, p_u, e_u, x_u\rangle$, then let $\Pi_u = \Phi^{\alpha(p_u)}_{D,e_u,x_u}$.

b) If $\alpha \le \zeta$, $u = \langle 2, x_u\rangle$ and $x_u \in D$, then let $\Pi_u = P_\alpha(X_u)$.

c) Let $\Pi_u = \Phi^{\alpha(0)}_{\{0\},0,0} \wedge \neg\Phi^{\alpha(0)}_{\{0\},0,0}$ in the other cases.

Set $\Pi_v = \bigwedge_{u\in D_v} \Pi_u$ and $\Phi^\alpha_{D,e,x} = \bigvee_{v\in V} \Pi_v$

An easy transfinite induction on α shows that for every α the Σ^+_α formula $\Phi^\alpha_{D,e,x}$ satisfies the requirements of the Lemma. □

5.4. Theorem. *Let $\alpha < \omega_1^{CK}$ and let $A \subseteq \mathbb{N}$ be forcing α-definable on $\mathfrak{A}$. Then A is formally α-definable on $\mathfrak{A}$.*

Proof. Suppose that for all $s \in \mathbb{N}$ we have that

$$s \in A \iff (\exists \tau \supseteq \delta)(\tau(x) \simeq s \;\&\; \tau \Vdash_\alpha F_e(x)),$$

where δ is a finite part, e, x are fixed elements of $\mathbb{N}$. Let $D = \mathrm{dom}(\delta) = \{w_1, \dots, w_r\}$ and let $\delta(w_i) = t_i$, $i = 1, \dots, r$. Consider a finite set $D^* \supseteq D\cup\{x\})$. By the previous Lemma

$$\mathfrak{A} \models \exists(y \in D^* \setminus (D \cup \{x\}))\Phi^\alpha_{D^*,e,x}(W_1/t_1, \dots, W_r/t_r, X/s)$$

if and only if there exists a finite part τ such that $\mathrm{dom}(\tau) = D^*$, $\tau \supseteq \delta$,$\tau(x) \simeq s$ and $\tau \Vdash_\alpha F_e(x)$. Hence we have that for all $s \in \mathbb{N}$ the following equivalence is true:

$$s \in A \iff \mathfrak{A} \models \bigvee_{D^*\supseteq D\cup\{x\}} \exists(y \in D^* \setminus (D \cup \{x\}))\Phi^\alpha_{D^*,e,x}(W_1/t_1, \dots, W_r/t_r).$$

From here we can conclude that A is formally α-definable on $\mathfrak{A}$. □

5.5. Theorem. *Let $A \subseteq \mathbb{N}$. Suppose that $\alpha < \omega_1^{CK}$ and $\xi = \max(\alpha + 1, \zeta)$. Let Q be a total set such that $\mathcal{P}_\xi \le_e Q$. Then the following are equivalent:*

(1) *A is relatively α-intrinsic with respect to the sequence $\{B_\gamma\}_{\gamma<\zeta}$.*

(2) *For every* ξ, Q*-acceptable enumeration* f *of* $\mathfrak{A}$, $f^{-1}(A) \leq_e (f^{-1}(\mathfrak{A}))^{(\alpha)}$.
(3) A *is forcing* α*-definable on* $\mathfrak{A}$.
(4) A *is formally* α*-definable on* $\mathfrak{A}$.

Proof. The implication (1) $\Rightarrow$ (2) is obvious.
The implication (2) $\Rightarrow$ (3) follows from Theorem 4.6.
The implication (3) $\Rightarrow$ (4) follows from the previous Theorem.
The last implication (4) $\Rightarrow$ (1) can be proved by transfinite induction on α. □

The characterization of the relatively α-intrinsic sets can be obtained from the Theorem above by taking $\zeta = 0$ and $B_0 = \mathbb{N}$. In particular if the structure $\mathfrak{A}$ is total, one can easyly derive from here the normal form of the relatively intrinsically Σ^0_α sets, obtained in [2] and [3]. Moreover we can get a slight improvement of the upper bound of the level of genericity compared to that obtained in [3]. Namely the following is true:

5.6. Corollary. *Suppose that* $\mathfrak{A}$ *is a partial structure with recursively enumerable underlined predicates and* $\alpha < \omega_1^{CK}$. *Let* $A \subseteq \mathbb{N}$ *and let for all enumerations* f *of* $\mathfrak{A}$ *such that* $(f^{-1}(\mathfrak{A}))^{(\alpha+1)} \equiv_e \emptyset^{(\alpha+1)}$ *we have that* $f^{-1}(A)$ *is enumeration reducible to* $(f^{-1}(\mathfrak{A}))^{(\alpha)}$. *Then* A *is relatively* α*-intrinsic on* $\mathfrak{A}$.

The last Corollary generalizes the respective result (Corollary V.18, [3]), where the same upper bound is obtained for recursive structures under the additional condition that A is a $\Delta^0_{\alpha+1}$ set.

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