INTRINSICALLY HYPERARITHMETICAL SETS

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ABSTRACT. The main result proved in the paper is that on every recursive structure the intrinsically hyperarithmetical sets coincide with the relatively intrinsically hyperarithmetical sets. As a side effect of the proof an effective version of the Kueker's theorem on definability by means of infinitary formulas is obtained.

1. INTRODUCTION

One of the main achievements of the classical recursion theory is the classification of certain sets based on the complexity of their definitions. So, we have complexity classes of sets organized in hierarchies, as the arithmetical hierarchy, the hyperarithmetical hierarchy, the analytical hierarchy, etc. All these hierarchies classify sets of natural numbers or sets of reals (usually considered as subsets of the Baire space). A natural problem is to obtain generalized versions of the classical hierarchies, which will work for subsets of the domains of arbitrary abstract structures. There are two approaches to this problem. The first one, called *internal*, is based on a direct development of recursion theory on abstract structures, as is done by Moschovakis [11, 12]. The second approach, called *external*, uses enumerations of the abstract structures. Let $\mathfrak{A}$ be a denumerable abstract structure. Assume that a subset A of the domain of $\mathfrak{A}$ is fixed and suppose that for every enumeration f of $\mathfrak{A}$ the set $f^{-1}(A)$ belongs to the same classical complexity class C relative to the atomic diagram of $f^{-1}(\mathfrak{A})$. Then we have evidence to think that A belongs to the complexity class C on $\mathfrak{A}$ and say that A is *relatively intrinsically* C on $\mathfrak{A}$. The external approach originates in [8] and is further extended in [6, 1, 5, 17, 18, 16]. All results in those papers confirm that both approaches are equivalent, i.e. the external and the respective internal complexity classes coincide.

Motivated by problems of recursive model theory Ash and Nerode initiated in [3] the study of an effective version of the external approach. They consider recursive (recursively presentable) abstract structures and instead of all enumerations of a

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structure $\mathfrak{A}$ they take into account only the effective enumerations of $\mathfrak{A}$, i.e. the enumerations f for which the structure $f^{-1}(\mathfrak{A})$ is recursive. The respective notions of external definability here are called *intrinsically* C instead of relatively intrinsically C. Obviously each relatively intrinsically C set on a recursive structure is also intrinsically C. The reverse inclusion is not always true. Examples of sets which are intrinsically r. e. but not relatively intrinsically r. e. can be found in [9] and [4]. It is quite probable that similar examples exist for all levels of the hyperarithmetical hierarchy.

In the present paper we prove that if we consider instead of fixed levels of the hyperarithmetical hierarchy all hyperarithmetical sets as a complexity class, then both versions of the external approach become equivalent. So, a set is intrinsically hyperarithmetical on a recursive structure $\mathfrak{A}$ iff it is relatively intrinsically hyperarithmetical on $\mathfrak{A}$. As a side effect of the proof we obtain an effective version of the Kueker's theorem [7] on definability by means of infinitary formulas.

The paper is organized as follows. After the preliminaries we introduce in section 3 the so called relatively intrinsically Σ sets. It is shown that those sets coincide with the sets which have inductive definitions with closure ordinals less than ω_1^{CK}. From here, we derive that on recursive structures the relatively intrinsically hyperarithmetical sets are relatively intrinsically Σ. In section 4, using a forcing argument, we obtain a normal form of the relatively intrinsically Σ sets and show that they are definable by means of recursive infinitary formulas. In the last section we combine the so far obtained results and prove the main theorem.

The present paper may be considered as a continuation of [16] and several results from [16] are used here. So, a preliminary knowledge of this paper would be very helpful for the understanding of the arguments.

2. Preliminaries

Let $\mathfrak{A} = (B; R_1, R_2, \dots, R_k)$ be a countable abstract structure, where each R_j is an a_j-ary predicate on B.

An one to one mapping f of the set of the natural numbers N onto B is called *enumeration* of $\mathfrak{A}$.

Every enumeration f of $\mathfrak{A}$ determines a unique structure

$$\mathfrak{B}_f = (N; R_1^{\mathfrak{B}_f}, R_2^{\mathfrak{B}_f}, \dots, R_k^{\mathfrak{B}_f}),$$

where for all $x_1, \dots, x_{a_j} \in N$, $R_j^{\mathfrak{B}_f}(x_1, \dots, x_{a_j}) = R_j(f(x_1), \dots, f(x_{a_j}))$.

By $D(\mathfrak{B}_f)$ we shall denote the set of all Gödel numbers of the elements of the diagram of $\mathfrak{B}_f$.

2.1. Definition. *Let* $A \subseteq B^n$. *Then* A *is* relatively intrinsically HYP (recursively enumerable) *on* $\mathfrak{A}$ *if for each enumeration* f *of* $\mathfrak{A}$, *there exists a hyperarithmetical (r. e.) relative to* $D(\mathfrak{B}_f)$ *subset* X *of* N^n, *such that for all* $x_1, \dots, x_n \in N$,

$$(x_1, \dots, x_n) \in X \iff (f(x_1), \dots, f(x_n)) \in A.$$

The complexity of an enumeration f of $\mathfrak{A}$ is mesured by the complexity of the respective set $D(\mathfrak{B}_f)$. So an enumeration is *effective (hyperarithmetical)* iff $D(\mathfrak{B}_f)$ is a recursive (hyperarithemtical) set. If $\mathfrak{A}$ admits an effective enumeration, then the structure $\mathfrak{A}$ is called *recursive (recursively presentable).*

2.2. Definition. *Let $\mathfrak{A}$ be a recursive structure with domain B. A subset A of B^n is* intrinsically HYP (r. e.) *on $\mathfrak{A}$ if for every effective enumeration f of $\mathfrak{A}$, $f^{-1}(A)$ is hyperarithmetical (r. e.).*

The least acceptable extension $\mathfrak{A}^*$ of $\mathfrak{A}$ is defined as follows.

Let 0 be an object which does not belong to B and $\langle\langle.,.\rangle\rangle$ be a pairing operation chosen so that neither 0 nor any element of B is an ordered pair. Let B^* be the least set containing all elements of $B_0 = B \cup \{0\}$ and closed under the operation $\langle\langle.,.\rangle\rangle$. We associate an element n^* of B^* with each integer n by the inductive definition:

$$\begin{aligned} 0^* &= 0 \\ (n+1)^* &= \langle\langle n^*, 0\rangle\rangle \end{aligned}$$

and put $N^* = \{0^*, 1^*, 2^*, \ldots\}$.

Let $\mathfrak{A}^*$ be the structure $(B^*; B_0, R_1, R_2, \ldots, R_k, N^*, G_{\langle\langle,\rangle\rangle})$, where $G_{\langle\langle,\rangle\rangle}$ is the graph of the pairing function and all predicates R_i are assumed false on $B^* \setminus B$.

The following proposition, proved in [16], shows that if $A \subseteq B^n$ is relatively intrinsically hyperarithmetical on $\mathfrak{A}^*$, then A is relatively hyperarithmetical on $\mathfrak{A}$.

2.3. Proposition. *Let f be an enumeration of $\mathfrak{A}$. There exists an enumeration f^* of $\mathfrak{A}^*$ such that $D(\mathfrak{B}_{f^*}) \leq_T D(\mathfrak{B}_f)$ and such that for every subset A of B^n, $f^{-1}(A) \leq_m f^{*-1}(A)$.*

We shall use also the following result from [16], which gives an internal characterization of the relatively intrisically hyperarithmetical sets.

2.4. Proposition. *Let $A \subseteq B^n$. Then A is relatively intrinsically hyperarithmetical on $\mathfrak{A}$ iff A is hyperelementary, i.e. inductive and coinductive in the sense of* [13], *on $\mathfrak{A}^*$.*

3. The relatively intrinsically Σ sets

Let $\mathfrak{A}$ be an abstract structure and f be a fixed enumeration of $\mathfrak{A}$. Set $D = D(\mathfrak{B}_f)$.

Roughly speaking, the Σ^D sets are the elements of the smallest effective $\sigma-$ring of sets containing all r. e. in D sets. The precise definition is a partial relativisation of the respective definition of the smallest effective $\sigma-$ring given in [15].

First we need an inductive definition of the set Ind of the indices of the hyperarithmetical sets.

Let $W_0, W_1, \ldots$ be a fixed Gödel enumeration of the r. e. sets.

3.1. Definition.

(1) *For all $e \in N$, $\langle 0, e\rangle \in Ind$;*
(2) *If $e \in Ind$, then $\langle 1, e\rangle \in Ind$;*

(3) *If* $W_e \subseteq Ind$, *then* $\langle 2, e\rangle \in Ind$.

Given an index $u \in Ind$, we define the norm $|u|$ of u to be equal to the least ordinal at which u appears in the definition of Ind:

3.2. Definition.

(1) $|\langle 0, e\rangle| = 0$;
(2) $|\langle 1, e\rangle| = |e| + 1$;
(3) $|\langle 2, e\rangle| = \sup(|z| + 1 : z \in W_e)$.

Since Ind is inductively defined on the structure of the Arithmetic, we have that for all $u \in Ind$, $|u| < \omega_1^{CK}$, where ω_1^{CK} is the least non constructive ordinal.

For each $e \in N$ denote by Γ_e the e-th enumeration operator [14]. Let $W_e^D = \Gamma_e(D)$. Since $N \setminus D$ is enumeration reducible to D, the sets W_e^D coincide with the r. e. in D sets.

Let $n \geq 1$. For each $u \in Ind$ the subset $J_u^{D,n}$ of N^n is defined by means of induction on $|u|$:

3.3. Definition.

(1) $J_{\langle 0,e\rangle}^{D,n} = \{(x_1, \dots, x_n) : \langle x_1, \dots, x_n\rangle \in W_e^D\}$;
(2) $J_{\langle 1,e\rangle}^{D,n} = N^n \setminus J_e^{D,n}$;
(3) $J_{\langle 2,e\rangle}^{D,n} = \bigcup_{z \in W_e} J_z^{D,n}$.

Let S be a subset of N^n. The set S is called Σ in D (Σ^D for short), if $S = J_u^{D,n}$ for some $u \in Ind$.

Although the Σ^D sets are in general a proper subclass of the hyperarithmetical in D sets, many of the properties of the hyperarithmetical sets remain true for the Σ^D sets. In particular, the assertions $H1$–$H4$ from chapter 7 of [15], which show that the hyperarithmetical sets are uniformly closed with respect to recursive substitutions, boolean operations and quantification over the integers, can be proved for the Σ^D sets with almost the same arguments. We shall refer to those properties as *standard properties* of the Σ^D sets.

Next we shall show that if a subset A of B has an inductive definition on the least acceptable extension $\mathfrak{A}^*$ of $\mathfrak{A}$ with closure ordinal less than ω_1^{CK}, then $f^{-1}(A)$ is Σ^D. We start by recalling some definitions from [13].

Let $\varphi(p_1, \dots p_r, p_{r+1}, S)$ be a first order formula in the language of $\mathfrak{A}^*$ with the new relational symbol S which is $r+1$-ary. We shall suppose that S occurs positively in φ and call φ S-positive.

Using transfinite recursion on ξ we define for each ordinal ξ the set $I_\varphi^\xi \subseteq (B^*)^{r+1}$ by

$$I_\varphi^\xi = \{(p_1, \dots, p_{r+1}) : \mathfrak{A}^* \models \varphi(p_1, \dots, p_{r+1}, \bigcup_{\eta<\xi} I_\varphi^\eta)\}.$$

Set $I_\varphi^{<\xi} = \bigcup_{\eta<\xi} I_\varphi^\eta$ and let ξ_0 be the least ordinal such that $I_\varphi^{\xi_0} = I_\varphi^{<\xi_0}$. We call ξ_0 *the closure ordinal* of the definition φ. Clearly $I_\varphi^{\xi_0}$ is equal to the least fixed point I_φ of φ.

We shall denote the closure ordinal of a S-positive formula φ by κ^φ. The closure ordinal $\kappa^{\mathfrak{A}^*}$ of the structure $\mathfrak{A}^*$ is defined by

$$\kappa^{\mathfrak{A}^*} = \sup(\kappa^\varphi : \varphi \text{ is a } S\text{-positive first order formula in the language of } \mathfrak{A}^*).$$

3.4. Definition. *Let $A \subseteq B^*$. Then A is inductive on $\mathfrak{A}^*$ if there exist a S-positive first order formula φ and finite list $t_1, \dots, t_r$ of elements of B^* such that for all $s \in B^*$,*

$$s \in A \iff (t_1, \dots, t_r, s) \in I_\varphi.$$

3.5. Definition. *A set $A \subseteq B^*$ is low inductive on $\mathfrak{A}^*$ if there exist a S-positive first order formula φ, finite list $t_1, \dots, t_r$ of elements of B^* and ordinal $\xi_0 < \omega_1^{CK}$ such that for all $s \in B^*$,*

$$s \in A \iff (t_1, \dots, t_r, s) \in I_\varphi^{\xi_0}.$$

3.6. Lemma. *Let $A \subseteq B^n$ be low inductive on $\mathfrak{A}^*$. Then $f^{-1}(A)$ is Σ^D.*

Proof. For the sake of simplicity assume that $n = 1$.

Let $A = \{s : (t_1, \dots, t_r, s) \in I_\varphi^{\xi_0}\}$, where $\varphi(p_1, \dots, p_r, p_{r+1}, S)$ is a S-positive formula on $\mathfrak{A}^*$ and $\xi_0 < \omega_1^{CK}$. According to Proposition 2.3, there exists an enumeration f^* of $\mathfrak{A}^*$ such that if $D^* = D(\mathfrak{B}_{f^*})$, then $D^* \leq_T D$ and $f^{-1}(A) \leq_m f^{*-1}(A)$. Since every Σ^{D^*} set is Σ^D and the Σ^D sets are closed with respect to recursive substitutions, it is sufficient to show tat $f^{*-1}(A)$ is Σ^{D^*}.

Replace every constant c in the formula φ by $f^{*-1}(c)$. Call the resulting formula φ^*. Clearly for every sequence $p_1, \dots, p_{r+1}$ in B^* and every subset P of $(B^*)^{r+1}$

$$\mathfrak{A}^* \models \varphi(p_1, \dots, p_{r+1}, P) \iff \mathfrak{B}_{f^*} \models \varphi^*(f^{*-1}(p_1), \dots, f^{*-1}(p_{r+1}), f^{*-1}(P))$$

and for every ordinal ξ

$$f^{*-1}(I_\varphi^\xi) = I_{\varphi^*}^\xi. \tag{3.1}$$

Let $\mathcal{O}$ be the set of the Curch-Kleene ordinal notations and $<_o$ be the respective wellfounded relation, see [14]. Using effective transfinite recursion on $<_o$, we shall construct a recursive function g such that if $a \in \mathcal{O}$ and a is a notation of the ordinal ξ, then $g(a) \in Ind$ and $J_{g(a)}^{D^*,r+1} = I_{\varphi^*}^\xi$.

By the standard properties of the Σ^{D^*} sets, there exists a recursive function m such that if $u \in Ind$, then $m(u) \in Ind$ and

$$J_{m(u)}^{D^*,r+1} = \{(z_1, \dots, z_{r+1}) : \mathfrak{B}_{f^*} \models \varphi^*(z_1, \dots, z_{r+1}, J_u^{D^*,r+1})\}.$$

Let $h(v, a)$ be a recursive function such that if $a \in \mathcal{O}$, then $h(v, a)$ equals to the r. e. index of the set $\{\{v\}(b) : b <_o a\}$ and let η be defined by the equality

$$\eta(v, a) = m(\langle 2, h(v, a)\rangle).$$

Finally, let g be a partial recursive function having index e such that for all z, $\eta(e, z) = \{e\}(z)$. Obviously g is total. A simple transfinite induction on $<_o$ shows that g has the needed properties.

Now let $a \in \mathcal{O}$ be a notation of ξ_0. Using (3.1) we get that

$$f^{*-1}(A) = \{z : (f^{*-1}(t_1), \dots, f^{*-1}(t_r), z) \in J^{D^*, r+1}_{g(a)}\}.$$

From the last equality, using once more the standard properties, we obtain that $f^{*-1}(A)$ is Σ^{D^*}. □

3.7. Definition. *A subset A of B^n is* relatively intrinsically Σ *on $\mathfrak{A}$ if $f^{-1}(A)$ is Σ in $D(\mathfrak{B}_f)$ for every enumeration f of $\mathfrak{A}$.*

From Lemma 3.6 we obtain directly the following:

3.8. Proposition. *Let $A \subseteq B^n$ be low inductive on $\mathfrak{A}^*$. Then A is relatively intrinsically Σ on $\mathfrak{A}$.*

3.9. Corollary. *Suppose that the closure ordinal $\kappa^{\mathfrak{A}^*}$ of the structure $\mathfrak{A}^*$ is equal to ω_1^{CK}. Then every hyperelementary on $\mathfrak{A}^*$ subset of B^n is relatively intrinsically Σ on $\mathfrak{A}$.*

Proof. Let $A \subseteq B^n$ be hyperelementary on $\mathfrak{A}^*$. Then A and the complement of A are inductive on $\mathfrak{A}^*$. Let φ be an S-positive formula and $t_1, \dots, t_r$ be a finite list of elements of B^* such that $\overline{s} \in A \iff (t_1, \dots, t_r, \overline{s}) \in I_\varphi$.

Let σ be an inductive norm on I_φ, defined by $\sigma(\overline{p}, \overline{s}) =$ least $\xi((\overline{p}, \overline{s}) \in I^\xi_\varphi)$.

Since A is coinductive, by the Covering theorem [13], there exists a $\xi_0 < \omega_1^{CK}$ such that $\overline{s} \in A \Rightarrow \sigma(t_1, \dots, t_r, \overline{s}) \leq \xi_0$. Hence, $\overline{s} \in A \iff (t_1, \dots, t_r, \overline{s}) \in I^{\xi_0}_\varphi$.

So, A is low inductive on $\mathfrak{A}^*$ and therefore A is relatively intrinsically Σ on $\mathfrak{A}$. □

Given a subset D of N, denote by ω_1^D the least ordinal which is not constructive relative to D. The following external characterization of $\kappa^{\mathfrak{A}^*}$ is proved in [16]:

3.10. Proposition. $\kappa^{\mathfrak{A}^*} = \min(\omega_1^{D(\mathfrak{B}_f)}$: *f is an enumeration of* $\mathfrak{A})$.

So, for structures $\mathfrak{A}$ which admit recursive and even hyperarithmetical enumerations $\kappa^{\mathfrak{A}^*} = \omega_1^{CK}$. Therefore on such structures all hyperelementary sets are relatively intrinsically Σ. Combining this observation and Proposition 2.4 we obtain the following:

3.11. Proposition. *Let $\mathfrak{A}$ be a structure which admits a hyperarithmetical enumeration. Then every relatively intrinsically HYP on $\mathfrak{A}$ set is relatively intrinsically Σ on $\mathfrak{A}$.*

4. Normal form of the relatively intrinsically Σ sets

Let us fix a countable abstract structure $\mathfrak{A} = (B; R_1, R_2, \dots, R_k)$. The normal form theorem for the ralatively intrinsically Σ sets on $\mathfrak{A}$ will be deduced as a consequence of the general normal form theorem for the relatively intrinsically definable sets proved in [16]. To apply this general theorem we need to define appropriate satisfaction and forcing relations.

To simplify the notation from now on we shall consider only subsets of the domain B of $\mathfrak{A}$. However all results can be easily generalized for subsets of B^n, $n \geq 1$.

Suppose that for each element u of Ind a unary predicate letter J_u is fixed. Given an enumeration f of $\mathfrak{A}$ and natural number x, let

$$f \models J_u(x) \iff x \in J_u^{D(\mathfrak{B}_f)}.$$

The conditions of the forcing are the finite injective mappings of N into B which we call *finite parts*. We shall use δ, τ, ρ to denote finite parts. The forcing relation $\delta \Vdash J_u(x)$ is defined as follows.

Assume fixed an effective coding of all finite sets of natural numbers. By E_v we shall denote the finite set having code v. Recall that, by definition, for every enumeration f of $\mathfrak{A}$ the set $D(\mathfrak{B}_f)$ consists of codes of literals which are true on $\mathfrak{B}_f$. Let δ be a finite part. Given a $c \in N$, let $\delta \Vdash c$ if c is a code of a literal $L(x_1, \dots, x_a)$, all $x_1, \dots, x_a \in dom(\delta)$ and $\mathfrak{A} \models L(\delta(x_1), \dots, \delta(x_a))$. Further, if $E = \{c_1, \dots, c_r\}$ is a finite set, then let

$$\delta \Vdash E \iff \delta \Vdash c_1 \& \dots \& \delta \Vdash c_r.$$

Finally, note that by the definition of the enumeration operators in [14], we have for every enumeration f of $\mathfrak{A}$:

$$x \in W_e^{D(\mathfrak{B}_f)} \iff x \in \Gamma_e(D(\mathfrak{B}_f)) \iff \exists v(\langle v, x\rangle \in W_e \& E_v \subseteq D(\mathfrak{B}_f)).$$

Now we are ready to define the forcing $\delta \Vdash J_u(x)$ for all $u \in Ind$ by induction on $|u|$:

4.1. Definition.

(1) *If* $\exists v(\langle v, x\rangle \in W_e \ \& \ \delta \Vdash E_v)$, *then* $\delta \Vdash J_{\langle 0,e\rangle}(x)$;
(2) *If* $\forall \rho(\rho \supseteq \delta \Longrightarrow \rho \nVdash J_e(x))$, *then* $\delta \Vdash J_{\langle 1,e\rangle}(x)$;
(3) *If* $\exists z(z \in W_e \ \& \ \delta \Vdash J_z(x))$, *then* $\delta \Vdash J_{\langle 2,e\rangle}(x)$.

From the definition above it follows immediately the monotonicity of the forcing, i.e. if $\delta \Vdash J_u(x)$ and $\delta \subseteq \tau$, then $\tau \Vdash J_u(x)$.

Denote by $\mathcal{F}$ the family of sets of finite parts containing for all $u \in Ind$ and $x \in N$ the set

$$X_{u,x} = \{\rho : \rho \Vdash J_u(x)\}.$$

An enumeration f of $\mathfrak{A}$ is $\mathcal{F}$*-generic* if whenever $X \in \mathcal{F}$ and X is *dense* in f, i.e. $(\forall \delta \subseteq f)(\exists \tau \in X)(\delta \subseteq \tau)$, then f *meets* X, i.e. $(\exists \delta \subseteq f)(\delta \in X)$.

Next follows the Truth Lemma:

4.2. Lemma. *Let f be an $\mathcal{F}$-generic enumeration, $u \in Ind$ and $x \in N$. Then*

$$f \models J_u(x) \iff (\exists \delta \subseteq f)(\delta \Vdash J_u(x)). \tag{4.1}$$

Proof. Induction on $|u|$. Let $u \in Ind$. We have three cases.

1) $u = \langle 0, e\rangle$. In this case (4.1) follows directly from the definitions of $\models$ and $\Vdash$.

2) $u = \langle 1, e\rangle$. Let $f \models J_u(x)$. Assume that for all $\delta \subseteq f, \delta \not\Vdash J_u(x)$. Then, by Definition 4.1, the set $X_{e,x} = \{\rho : \rho \Vdash J_e(x)\}$ is dense in f. By genericity, f meets $X_{e,x}$. Hence, by induction, $f \models J_e(x)$. A contradiction.

Suppose now that for some $\delta \subseteq f$, $\delta \Vdash J_u(x)$. Assume that $f \not\models J_u(x)$. Then $f \models J_e(x)$. By induction, there exists a $\rho \subseteq f$ such that $\rho \Vdash J_e(x)$. By the monotonicity of $\Vdash$, we may assume that $\delta \subseteq \rho$. A contradiction.

3) $u = \langle 2, e\rangle$. By induction,

$$\begin{aligned} f \models J_u(x) &\iff (\exists z \in W_e)(f \models J_z(x)) \iff (\exists z \in W_e)(\exists \delta \subseteq f)(\delta \Vdash J_z(x)) \\ &\iff (\exists \delta \subseteq f)(\delta \Vdash J_u(x)). \quad \square \end{aligned}$$

Now we are ready to apply the Normal form theorem from [16]. Given a finite part δ and $x \in N$, denote by $\mathcal{R}(\delta, x)$, the set $\{s : s \in B \ \&\ \exists \tau \supseteq \delta(\tau(x) = s)\}$.

4.3. Theorem. *Let $A \subseteq B$ be relatively intrinsically Σ on $\mathfrak{A}$. There exist finite part δ and $u \in Ind$ such that if $x \in N$, then for every $s \in \mathcal{R}(\delta, x)$*

$$s \in A \iff (\exists \tau \supseteq \delta)(\tau(x) = s \ \&\ \tau \Vdash J_u(x)).$$

4.4. Corollary. *Every relatively intrinsically Σ set A has an inductive definition on $\mathfrak{A}^*$ with closure ordinal less than ω_1^{CK}.*

Proof. Suppose that A is relatively intrinsically Σ. Then there exist finite part δ and $u \in Ind$ such that for all $s \in B$,

$$s \in A \iff (\exists x \in N)(\exists \tau \supseteq \delta)(\tau(x) = s \ \&\ \tau \Vdash J_u(x))$$

Using this equivalence we can get easily an inductive definition of A. For we represent each finite part δ mapping $w_1, \dots, w_r$ onto $t_1, \dots, t_r$, respectively, by the element $\langle\langle\langle\langle w_1, t_1\rangle\rangle, \dots, \langle\langle w_r, t_r\rangle\rangle, 0^*\rangle\rangle$ of B^* and translate the inductive definition of the forcing in terms of $\mathfrak{A}^*$. Clearly the obtained this way inductive definition of A has a closure ordinal which is less than ω_1^{CK}. $\square$

As a second application of Theorem 4.3 we shall get a formal representation of the relatively intrinsically Σ sets in the spirit of [1] and [5].

Let $\mathcal{L} = \{T_1, \dots, T_k\}$ be the language of the structure $\mathfrak{A}$. The *recursive* $\Sigma_\alpha(\Pi_\alpha)$, $\alpha < \omega_1^{CK}$, formulas in the language $\mathcal{L}_{\omega_1^{CK}\omega}$ are defined as in [2]. Roughly speaking, the Σ_0 and the Π_0 formulas are the quantifier free formulas in $\mathcal{L}$. The $\Sigma_{\alpha+1}$ formulas are of the form $\bigvee_i \exists \overline{Y_i}\varphi_{h(i)}$, where $\{\varphi_{h(i)}\}$ is a recursive sequence of Π_α formulas and $\overline{Y_i}$ are finite sequences of variables; the $\Pi_{\alpha+1}$ formulas are negations of $\Sigma_{\alpha+1}$ formulas. If λ is a limit ordinal, then the Σ_λ formulas are $\bigvee_i \varphi_{h(i)}$, where $\{\varphi_{h(i)}\}$ is a recursive sequence of Σ_α, $\alpha < \lambda$, formulas; the Π_λ formulas are again negations of Σ_λ formulas. The precise definition is by effective transfinite recursion on $<_o$, where for each $a \in \mathcal{O}$ the $\Sigma_{|a|}$ and $\Pi_{|a|}$ formulas are defined simultaneously with their Gödel numbers.

Note that the recursive Σ_α formulas are closed with respect to existential quantification, finite conjunctions and r. e. infinite disjunctions, while the recursive Π_α formulas are closed with respect to universal quantification, finite disjunctions and r. e. infinite conjunctions.

A formula F is called *recursive* Σ if it is a recursive Σ_α formula for some $\alpha < \omega_1^{CK}$.

Next we are going to show that the relatively intrinsically Σ sets on $\mathfrak{A}$ are definable on $\mathfrak{A}$ by means of recursive Σ formulas.

Let us fix a recursive bijective mapping var of the natural numbers onto the set of all variables of the language $\mathfrak{L}$. Let F be a formula and D be a finite set of natural numbers. Let $y_1 < y_2 < \ldots < y_k$ be the elements of D and Q be one of the quantifiers $\exists$ or $\forall$. Then by $Q(y : y \in D)F$ we shall denote the formula $Qvar(y_1)\ldots Qvar(y_k)F$. By $Neq(D)$ we shall denote the conjunction $\bigwedge_{i,j\in D \ \& \ i<j} var(y_i) \neq var(y_j)$.

4.5. Lemma. *There exists a uniform effective way given natural numbers x, v and finite set $\{z_1, \ldots, z_r\}$ to define an existential first order formula C with free variables among $var(z_1), \ldots, var(z_r), var(x)$ such that if $Z_i = var(Z_i)$, $X = var(x)$ and δ is a finite part with domain $\{z_1, \ldots, z_r\}$, then for all $s \in \mathcal{R}(\delta, x)$*

(4.2) $\mathfrak{A} \models C(Z_1/\delta(z_1), \ldots, Z_r/\delta(z_r), X/s) \iff (\exists\tau \supseteq \delta)(\tau(x) = s \ \& \ \tau \Vdash E_v).$

Proof. Set $C = X \neq X$ if some of the elements of E_v is not a code of a literal. Otherwise, let $E_v = \{c_1, \ldots, c_m\}$, where c_i is the code of the literal $L^i(x_1^i, \ldots, x_{m_i}^i)$. Denote by π the conjunction $\bigwedge_{i=1}^m L_i(var(x_1^i), \ldots, var(x_{m_i}^i))$, let D be the finite set $\{z_1, \ldots, z_r, x\} \cup \bigcup_{i=1}^m \{x_1^i, \ldots, x_{m_i}^i\}$ and $D' = D \setminus \{z_1, \ldots, z_r, x\}$. Set

$$C = \exists(y : y \in D')(Neq(D) \wedge \pi).$$

Now following the definition of the forcing $\tau \Vdash E_v$ one can easily check the validity of (4.2). □

4.6. Lemma. *There exists an uniform effective way given $x \in N, u \in Ind$ and finite set $D = \{z_1, \ldots, z_r\}$ to define a recursive Σ formula $F_{x,u}^D$ with free variables among $var(z_1), \ldots, var(z_r), var(x)$ such that if $Z_i = var(z_i)$, $X = var(x)$ and δ is a finite part with domain $\{z_1, \ldots, z_r\}$, then for all $s \in \mathcal{R}(\delta, x)$*

$$\mathfrak{A} \models F_{x,u}^D(Z_1/\delta(z_1), \ldots, Z_r/\delta(z_r), X/s) \iff (\exists\tau \supseteq \delta)(\tau(x) = s \ \& \ \tau \Vdash J_u(x)).$$

Proof. We shall define the formula $F_{x,u}^D$ by means of effective transfinite recursion on $|u|$. Let us fix $D = \{z_1, \ldots, z_r\}, x \in N$ and $u \in Ind$. Then we have three cases:

1) $u = \langle 0, e\rangle$. According Definition 4.1 for any finite part δ and $s \in \mathcal{R}(\delta, x)$,

$$(\exists\tau \supseteq \delta)(\tau(x) = s \ \& \ \tau \Vdash J_u(x)) \iff (\exists\langle v, x\rangle \in W_e)(\exists\tau \supseteq \delta)(\tau(x) = s \ \& \ \tau \Vdash E_v).$$

For each $v \in N$ denote by C_v the existential formula satisfying (4.2) with respect to x, v and D. Let h be a recursive function with range equal to the set $\{v : \langle v, x\rangle \in W_e\}$. Set $F_{x,u}^D = \bigvee_n C_{h(n)}$.

2) $u = \langle 1, e\rangle$. Let δ be a finite part and $s \in \mathcal{R}(\delta, x)$. By Definition 4.1

$$(\exists\tau \supseteq \delta)(\tau(x) = s \;\&\; \tau \Vdash J_u(x)) \iff (\exists\tau \supseteq \delta)(\tau(x) = s \;\&\; (\forall\rho \supseteq \tau)(\rho \nVdash J_e(x)))$$
$$\iff (\exists\tau \supseteq \delta)(\tau(x) = s \;\& \neg(\exists\rho \supseteq \tau)(\rho(x) = s \;\&\; \rho \Vdash J_e(x))).$$

Notice also that if $\tau(x) = s$, then $s \in \mathcal{R}(\rho, x)$ for all $\rho \supseteq \tau$.

So a possible definition of the formula $F^D_{x,u}$ in this case is the following, where $\overline{D}$ varies over all finite sets of natural numbers:

$$F^D_{x,u} = \bigvee_{\overline{D} \supseteq D \cup \{x\}} \exists(y : y \in \overline{D} \setminus (D \cup \{x\}))(Neq(\overline{D}) \wedge \neg F^{\overline{D}}_{x,e}).$$

3) $u = \langle 2, e\rangle$. Using again Definition 4.1, we get for any δ and $s \in \mathcal{R}(\delta, x)$

$$(\exists\tau \supseteq \delta)(\tau(x) = s \;\&\; \tau \Vdash J_u(x)) \iff (\exists z \in W_e)(\exists\tau \supseteq \delta)(\tau(x) = s \;\&\; \tau \Vdash J_z(x)).$$

So we may define $F^D_{x,u} = \bigvee_{z \in W_e} F^D_{x,z}$. □

The following theorem shows that every relatively intrinsically Σ set is definable by means of some recursive Σ formula on $\mathfrak{A}$.

4.7. Theorem. *Let $A \subseteq B$ be relatively intrinsically Σ on $\mathfrak{A}$. Then there exist a recursive Σ formula $F(W_1, \dots, W_r, X)$ and elements $t_1, \dots, t_r$ of B such that for all $s \in B$*

$$s \in A \iff \mathfrak{A} \models F(W_1/t_1, \dots, W_r/t_r, X/s).$$

Proof. Suppose that A is relatively intrinsically Σ on $\mathfrak{A}$. From Theorem 4.3 it follows that there exist finite part δ and $u \in Ind$ such that for all $x \in N$ and $s \in \mathcal{R}(\delta, x)$

$$s \in A \iff (\exists\tau \supseteq \delta)(\tau(x) = s \;\&\; \tau \Vdash J_u(x)).$$

Let $dom(\delta) = \{w_1, \dots, w_r\}$ and $\delta(w_i) = t_i$, $i = 1, \dots, r$. Fix a $x \notin \{w_1, \dots, w_r\}$. Clearly $\mathcal{R}(\delta, x) = B \setminus \{t_1, \dots, t_r\}$. From here we obtain the following representation of A:

$$s \in A \iff (\exists\tau \supseteq \delta)(\tau(x) = s \;\&\; \tau \Vdash J_u(x)) \text{ or}$$
$$\bigvee_{i=1}^{r} s = t_i \;\&\; (\exists\tau \supseteq \delta)(\tau \Vdash J_u(w_i)).$$

Let $var(w_i) = W_i$, $var(x) = X$ and $D = \{w_1, \dots, w_r\}$. Let

$$F = [(\bigwedge_{i=1}^{r} W_i \neq X) \wedge F^D_{x,u}] \vee [\bigvee_{i=1}^{r} X = W_i \wedge F^D_{w_i,u}].$$

Using the previous lemma, one can easily see that for all $s \in B$

$$s \in A \iff \mathfrak{A} \models F(W_1/t_1, \dots, W_r/t_r, X/s). \quad \square$$

It is obvious that every definable by means of a recursive Σ formula set is relatively intrinsically Σ on $\mathfrak{A}$. So we have the following corollary:

4.8. Corollary. *Let $A \subseteq B$. Then A is relatively intrinsically Σ on $\mathfrak{A}$ iff it is definable by means of some recursive Σ formula on $\mathfrak{A}$.*

In [1] and [5] the relatively intrinsically Σ_α, $\alpha < \omega_1^{CK}$, sets are studied and is proved that those sets coincide with the sets definable by means of recursive Σ_α formulas. Using this result we obtain also the following:

4.9. Corollary. *A subset A of B is relatively intrinsically Σ iff it is relatively intrinsically Σ_α for some $\alpha < \omega_1^{CK}$.*

5. Applications

Let us call a structure $\mathfrak{A}$ *hyperarithmetical* if the domain of $\mathfrak{A}$ is equal to N and all predicates of $\mathfrak{A}$ are hyperarithmetical. The following theorem is an effective version of Kueker's theorem [7].

5.1. Theorem. *Let $\mathfrak{A} = (N; R_1, \dots, R_k)$ be a hyperarithmetical structure. A subset A of N is definable by means of a recursive Σ formula on $\mathfrak{A}$ iff A is hyperarithmetical and the family $S = \{X : X \subseteq N \ \& \ (\mathfrak{A}, X) \cong (\mathfrak{A}, A)\}$ has fewer than $2^{\aleph_0}$ members.*

Proof. In the one direction the theorem is obvious. Clearly, if A is definable on $\mathfrak{A}$ by means of some recursive infinitary formula, then A is hyperarithmetical and the family S is countable.

Suppose now, that A is a hyperarithmetical set and the family S contains less than $2^{\aleph_0}$ elements. First we shall show that A is relatively intrinsically hyperarithmetical on $\mathfrak{A}$, i.e. for every enumeration f of $\mathfrak{A}$, $f^{-1}(A)$ is hyperarithmetical in $D(\mathfrak{B}_f)$. For we are going to use the Perfect set theorem [10]. Let f be an enumeration of $\mathfrak{A}$. Consider the family S_f of subsets of N defined by the equivalence:

$$Y \in S_f \iff \exists g (g \text{ is an isomorphism from } \mathfrak{B}_f \text{ to } \mathfrak{A} \text{ and } g^{-1}(A) = Y).$$

Since $\mathfrak{A}$ is a hyperarithmetical structure and the set A is hyperarithmetical, the family S_f is Σ_1^1 in $D(\mathfrak{B}_f)$. It is not hard to see that S_f is of the same cardinality as the family S. So, S_f has less than $2^{\aleph_0}$ elements. From here, by the Perfect set theorem, all members of S_f are hyperarithmetical in $D(\mathfrak{B}_f)$. Obviously $f^{-1}(A) \in S_f$. Hence $f^{-1}(A)$ is hyperarithmetical in $D(\mathfrak{B}_f)$.

So, A is relatively intrinsically HYP on $\mathfrak{A}$. By Proposition 3.11, A is relatively intrinsically Σ on $\mathfrak{A}$. Hence, by Theorem 4.7, A is definable by means of some recursive Σ formula on $\mathfrak{A}$. □

Now we are ready to show that the relatively intrinsically hyperarithmetical sets on a recursive structure $\mathfrak{A}$ coincide with the intrinsically hyperarithmetical sets on $\mathfrak{A}$.

5.2. Theorem. *Let $\mathfrak{A}$ be a recursive structure and $A \subseteq |\mathfrak{A}|$. Then the following are equivalent:*

(1) *A is relatively intrinsically HYP on $\mathfrak{A}$;*
(2) *A is relatively intrinsically Σ on $\mathfrak{A}$;*
(3) *A is definable by means of some recursive Σ formula on $\mathfrak{A}$;*
(4) *A is intrinsically HYP on $\mathfrak{A}$.*

Proof. Since the assertions (1) – (4) are invariant with respect to isomorphisms, we may assume that $|\mathfrak{A}| = N$ and the underlined predicates of $\mathfrak{A}$ are recursive. Now (1) $\Rightarrow$ (2) follows from Proposition 3.11. The implication (2) $\Rightarrow$ (3) follows from Theorem 4.7. The implications (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (4) are obvious. So it remains to show that (4) $\Rightarrow$ (3).

Let A be intrinsically HYP on $\mathfrak{A}$. Since the identity I is an effective enumeration of $\mathfrak{A}$, $A = I^{-1}(A)$ is hyperarithmetical. Consider the family $S = \{X : X \subseteq N \ \& \ (\mathfrak{A}, X) \cong (\mathfrak{A}, A)\}$. Assume that S is not countable. Then there exist an enumeration f of $\mathfrak{A}$, such that $f^{-1}(\mathfrak{A}) = \mathfrak{A}$ and $f^{-1}(A)$ is not hyperarithmetical. A contradiction. So, by the previous theorem, A is definable by means of some recursive Σ formula on $\mathfrak{A}$. □

Let $\alpha < \omega_1^{CK}$. A subset A of the domain of a recursive structure $\mathfrak{A}$ is *intrinsically* Σ_α on $\mathfrak{A}$ if for every effective enumeration f of $\mathfrak{A}$, $f^{-1}(A)$ is Σ_α.

For the definition of the Σ_α hierarchy, the reader may consult [14].

5.3. Corollary. *Let $\alpha < \omega_1^{CK}$ and A be intrinsically Σ_α on the recursive structure $\mathfrak{A}$. Then A is relatively intrinsically Σ_β for some $\beta < \omega_1^{CK}$.*

Proof. Since A is intrinsically Σ_α, it is clearly intrinsically HYP on $\mathfrak{A}$. Then A is relatively intrinsically Σ on $\mathfrak{A}$. Hence, by Corollary 4.9, A is relatively intrinsically Σ_β for some $\beta < \omega_1^{CK}$. □

We don't know what is the relationship between the ordinals α and β above. The only known result is that of Chisholm [4], which shows that on existentially decidable structures every intrinsically r. e. (Σ_1) set is relatively intrinsically Π_2 and hence relatively intrinsically Σ_3.

References

1. C. Ash, J. Knight, M. Manasse, and T. Slaman, *Generic copies of countable structures*, Ann. Pure Appl. Logic **42** (1989), 195–205.
2. C. J. Ash, *Recursive labeling systems and stability of recursive structures in hyperarithmetical degrees*, Trans. Amer. Math. Soc. **298** (1986), 497–514.
3. C.J. Ash and A. Nerode, *Intrinsically recursive relations*, Aspects of Effective Algebra (Yarra Glen, Australia) (J. N. Crossley, ed.), U.D.A. Book Co., 1981, pp. 26–41.
4. J. Chisholm, *The complexity of intrinsically r.e. subsets of existentially decidable models*, J. Symbolic Logic **55** (1990), 1213–1232.
5. ———, *Effective model theory vs. recursive model theory*, J. Symbolic Logic **55** (1990), 1168–1191.
6. T. J. Grilliot, *Omitting types: applications to recursion theory*, J. Symbolic Logic **37** (1972), 81–89.
7. D. Kueker, *Definability, automorphisms and infinitary languages*, The syntax and semantics of infinitary logic. Lecture Notes in Mathematics (Berlin-Heidelberg-New York) (J. Barwise, ed.), Springer, 1968, pp. 152–165.
8. D. Lacombe, *Deux generalizations de la notion de recursivite relative*, C. R. de l'Academie des Sciences de Paris **258** (1964), 3410–3413.

9. M. S. Manasse, *Techniques and counterexamples in almost categorical recursive model theory*, Ph.D. thesis, University of Wisconsin, Madison, Wisconsin, 1982.
10. R. Mansfield, *Perfect subsets of definable sets of real numbers*, Pacific J. Math **35** (1970), 451–457.
11. Y. N. Moschovakis, *Abstract first order computability I*, Trans. Amer. Math. Soc. **138** (1969), 427–464.
12. ______, *Abstract first order computability II*, Trans. Amer. Math. Soc. **138** (1969), 465–504.
13. ______, *Elementary induction on abstract structures*, North - Holland, Amsterdam, 1974.
14. H. Rogers, *Theory of recursive functions and effective computability*, McGraw-Hill Book Company, New York, 1967.
15. J. R. Shoenfield, *Mathematical logic*, Addison-Wesley Publishing Company, 1967.
16. I. N. Soskov, *Intrinsically* Π^1_1 *relations*, to appear in Mathematical Logic Quarterly.
17. ______, *Definability via enumerations*, J. Symbolic Logic **54** (1989), 428–440.
18. ______, *Computability by means of effectively definable schemes and definability via enumerations*, Arch. Math. Logic **29** (1990), 187–200.

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