THE ω-ENUMERATION DEGREES

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Abstract. In the present paper we initiate the study of the partial ordering of the ω-enumeration degrees. This ordering is a semi-lattice which extends the semi-lattice of the enumeration degrees. The main results include a jump inversion theorem, a density theorem for the ω-enumeration degrees below the first jump of the least degree and the lack of minimal ω-enumeration degrees.

1. Introduction

In the theory of the degrees of unsolvability one often considers the so called *jump hierarchies*. As the name would suggest these are classes whose definitions are based on properties of the jumps of the degrees in the class. So we say, for example, that a degree $\mathbf{a}$ is low if $\mathbf{a}' = \mathbf{0}'$ and the degree $\mathbf{a} \leq \mathbf{0}'$ is high if $\mathbf{a}' = \mathbf{0}''$.

In the present paper we consider a general notion of jump class. Let us denote by $\mathcal{S}$ the set of all sequences $\mathcal{B} = \{B_k\}_{k<\omega}$ of sets of natural numbers. Consider an element $\mathcal{B}$ of $\mathcal{S}$ and let the *jump class* $J_{\mathcal{B}}$ defined by $\mathcal{B}$ be the set of the Turing degrees of all $X \subseteq \mathbb{N}$ such that $(\forall k)(B_k$ is r.e. in $X^{(k)}$ uniformly in $k)$.

Given two sequences $\mathcal{A}$ and $\mathcal{B}$ let $\mathcal{A} \leq_u \mathcal{B}$ ($\mathcal{A}$ is *uniformly reducible to* $\mathcal{B}$) if $J_{\mathcal{B}} \subseteq J_{\mathcal{A}}$ and $\mathcal{A} \equiv_u \mathcal{B}$ if $J_{\mathcal{B}} = J_{\mathcal{A}}$. Clearly "$\leq_u$" is a reflexive and transitive relation on $\mathcal{S}$ and "$\equiv_u$" is an equivalence relation on $\mathcal{S}$.

For every sequence $\mathcal{B}$ let $d_\omega(\mathcal{B}) = \{\mathcal{A} : \mathcal{A} \equiv_u \mathcal{B}\}$ and let $\mathcal{D}_\omega = \{d_\omega(\mathcal{B}) : \mathcal{B} \in \mathcal{S}\}$. We call the elements of $\mathcal{D}_\omega$ the *ω-enumeration degrees*.

The ω-enumeration degrees can be ordered in the usual way. Given two elements $\mathbf{a} = d_\omega(\mathcal{A})$ and $\mathbf{b} = d_\omega(\mathcal{B})$ of $\mathcal{D}_\omega$, let $\mathbf{a} \leq_\omega \mathbf{b}$ if $\mathcal{A} \leq_u \mathcal{B}$. Clearly $\mathcal{D}_\omega = (\mathcal{D}_\omega, \leq_\omega)$ is a partial ordering with least element $\mathbf{0}_\omega = d_\omega(\emptyset_\omega)$, where all members of $\emptyset_\omega$ are equal to $\emptyset$.

Given two sequences $\mathcal{A} = \{A_k\}$ and $\mathcal{B} = \{B_k\}$ of sets of natural numbers let $\mathcal{A} \oplus \mathcal{B} = \{A_k \oplus B_k\}$. Is it easy to see that $J_{\mathcal{A}\oplus\mathcal{B}} = J_{\mathcal{A}} \cap J_{\mathcal{B}}$ and hence every two elements $\mathbf{a} = d_\omega(\mathcal{A})$ and $\mathbf{b} = d_\omega(\mathcal{B})$ of $\mathcal{D}_\omega$ have a least upper bound $\mathbf{a} \cup \mathbf{b} = d_\omega(\mathcal{A} \oplus \mathcal{B})$.

Finally we would like to mention that there is a natural embedding of the enumeration degrees into the ω-enumeration degrees. Given a set A of natural numbers denote by $A \uparrow \omega$ the sequence $\{A_k\}_{k<\omega}$, where $A_0 = A$ and for all $k \geq 1$, $A_k = \emptyset$.

1.1. Proposition. *For every $A, B \subseteq \mathbb{N}$, $A \uparrow \omega \leq_u B \uparrow \omega \iff A \leq_e B$.*

Proof. Suppose that $A \uparrow \omega \leq_u B \uparrow \omega$. Then $J_{B\uparrow\omega} \subseteq J_{A\uparrow\omega}$ and hence for every $X \subseteq \mathbb{N}$, B is r.e. in X implies A is r.e. in X. By the Selman's Theorem [6], $A \leq_e B$.

The implication $A \leq_e B \Rightarrow J_{B\uparrow\omega} \subseteq J_{A\uparrow\omega}$ is obvious. □

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So we may define the embedding $\kappa : \mathcal{D}_e \to \mathcal{D}_\omega$ by $\kappa(d_e(A)) = d_\omega(A \uparrow \omega)$.

In the present paper we initiate the study of the ω-enumeration degrees. We define a jump operation $\mathcal{A} \to \mathcal{A}'$ on sequences of sets which is degree invariant and such that for every sequence $\mathcal{B}$, $J_{\mathcal{B}'} = \{\mathbf{a}' : \mathbf{a} \in J_{\mathcal{B}}\}$ and prove a jump inversion theorem.

After that we show that the ω-degrees below $d_\omega(\emptyset'_\omega)$ are dense and that there are no minimal ω-enumeration degrees. The proofs of these results require some non-trivial generalizations of the techniques of Gutteridge and Cooper [1] and Lachlan and Shore [4] and the systematic use of the Recursion Theorem.

The paper is organized as follows. After some preliminaries in section 3 we give an explicit characterization of the uniform reducibility and define the jump operation. In section 4 we generalize the good approximations from [4] for sequences of sets. Section 5 contains the proof of the density theorem and in section 6 we show that there are no minimal ω-enumeration degrees.

2. Preliminaries

In this section we shall summarize some facts about the enumeration reducibility and the enumeration jump needed in the rest of the paper.

We shall assume that an effective coding of all finite sets of natural numbers is fixed and shall identify the finite sets and their codes. Finite sets will be denoted by the letters D, F and S.

2.1. Definition. *Given sets A and B of natural numbers, let*

$$A(B) = \{x : (\exists D)(\langle x, D\rangle \in A \;\&\; D \subseteq B\}$$

Let $W_0, \dots, W_a, \dots$ be a Gödel enumeration of the recursively enumerable (r.e.) sets of natural numbers.

The operators $\lambda B.W_a(B)$ are called *enumeration operators*. For $A, B \subseteq \mathbb{N}$, $A \leq_e B$ (A *is enumeration reducible to* B) if there exists an r.e. set W such that $A = W(B)$. Let $A \equiv_e B \iff A \leq_e B \;\&\; B \leq_e A$. The relation $\equiv_e$ is an equivalence relation and the respective equivalence classes are called enumeration degrees. For an introduction to the enumeration degrees the reader might consult [3].

The enumeration jump operator is defined in [2] and further studied in [5]. Here we shall use the following definition of the enumeration jump which is m-equivalent to the original one, see [5].

For every set A of natural numbers let $A^+ = A \oplus (\mathbb{N} \setminus A)$. Clearly a set B is r.e. in A if and only if $B \leq_e A^+$. Moreover there exist recursive functions λ and μ such that for all $a \in \mathbb{N}$ and $A \subseteq \mathbb{N}$, $W_a^A = W_{\lambda(a)}(A^+)$ and $W_a(A^+) = W^A_{\mu(a)}$.

2.2. Definition. *Given a set A of natural numbers, set $L_A = \{\langle a, x\rangle : x \in W_a(A)\}$ and let the* enumeration jump *of A be the set L_A^+.*

Here we shall prove some simple properties of the enumeration jump which will be used in the rest of the paper. To avoid any misunderstanding from now on we shall use the notation A' only to denote the enumeration jump of A.

For every set A let $\langle \chi_A\rangle = \{\langle x, 1\rangle : x \in A\} \cup \{\langle x, 0\rangle : x \in \mathbb{N} \setminus A\}$.

It is easy to see that A^+ is uniformly enumeration equivalent to $\langle \chi_A\rangle$, i.e. there exists enumeration operators Φ_1 and Φ_2 such that $A^+ = \Phi_1(\langle \chi_A\rangle)$ and $\langle \chi_A\rangle = \Phi_2(A^+)$.

The next proposition shows that $\langle \chi_A \rangle$ and A^+ are uniformly e-reducible to A'.

2.3. Proposition. *There exists an enumeration operator W_a such that $\langle \chi_A \rangle = W_a(A')$ for all $A \subseteq \mathbb{N}$.*

Proof. Let a_0 be an index of the r.e. set $\{\langle x, \{x\}\rangle : x \in \mathbb{N}\}$. Clearly $W_{a_0}(A) = A$ for every $A \subseteq \mathbb{N}$. Then

$$2\langle a_0, x\rangle \in A' \iff x \in W_{a_0}(A) \iff x \in A \text{ and}$$
$$2\langle a_0, x\rangle + 1 \in A' \iff x \notin W_{a_0}(A) \iff x \notin A.$$

Let a be an index of the r.e set

$$\{\langle\langle x, 1\rangle, \{2\langle a_0, x\rangle\}\rangle : x \in \mathbb{N}\} \cup \{\langle\langle x, 0\rangle, \{2\langle a_0, x\rangle + 1\}\rangle : x \in \mathbb{N}\}.$$

Clearly $W_a(A') = \langle \chi_A \rangle$. □

Given a set A of natural numbers, denote by K_A the Turing jump of A. Recall that

$$K_A = \{\langle e, x\rangle : \{e\}^A(x) \text{ is defined}\}.$$

2.4. Proposition. *There exists an enumeration operator Φ such that $K_A = \Phi(A')$ for all $A \subseteq \mathbb{N}$.*

Proof. Rewriting the definition of K_A we get

$$K_A = \{\langle e, x\rangle : (\exists \text{ finite function } \theta)(\{e\}^\theta(x) \text{ is defined and } \theta \subseteq \chi_A\}.$$

From here we get immediately that there exists an enumeration operator Φ_0 such that for all $A \subseteq \mathbb{N}$, $K_A = \Phi_0(\langle \chi_A \rangle)$. By Proposition 2.3 there exists an enumeration operator Φ_1 such that for all $A \subseteq \mathbb{N}$, $\langle \chi_A \rangle = \Phi_1(A')$. So we may define $\Phi(A) = \Phi_0(\Phi_1(A))$. □

The following Proposition shows that the jump is preserved under the Roger's embedding ι of the Turing degrees into the enumeration degrees defined by $\iota(d_T(A)) = d_e(A^+)$.

2.5. Proposition. *For all $A \subseteq \mathbb{N}$, $(A^+)' \equiv_e K_A^+$ uniformly in A.*

Proof. Let us fix recursive functions λ, μ such that for all $a \in \mathbb{N}$ and $A \subseteq \mathbb{N}$, $W_a^A = W_{\lambda(a)}(A^+)$ and $W_a(A^+) = W_{\mu(a)}^A$. Now we have

$$\langle a, x\rangle \in L_{A^+} \iff x \in W_a(A^+) \iff x \in W_{\mu(a)}^A \iff \langle \mu(a), x\rangle \in K_A.$$

From here one can easily construct an enumeration operator Φ_0 such that $(A^+)' = L_{A^+}^+ = \Phi_0(K_A^+)$ for all $A \subseteq \mathbb{N}$.

Similarly, we have that for all $a \in \mathbb{N}$ and for all $A \subseteq \mathbb{N}$,

$$\langle a, x\rangle \in K_A \iff x \in W_a^A \iff x \in W_{\lambda(a)}(A^+) \iff \langle \lambda(a), x\rangle \in L_{A^+}.$$

and hence there exists an enumeration operator Φ_1 such that $K_A^+ = \Phi_1((A^+)')$ for all $A \subseteq \mathbb{N}$. □

2.6. Proposition. *There exists a recursive function g such that for all $e \in \mathbb{N}$ and $B \subseteq \mathbb{N}$, $W_e(B)' = W_{g(e)}(B')$.*

Proof. Consider a recursive function λ such that for every a and e, $W_a(W_e(B)) = W_{\lambda(a,e)}(B)$. Then

$$2\langle a, x\rangle \in W_e(B)' \iff 2\langle \lambda(a, e), x\rangle \in B' \text{ and}$$
$$2\langle a, x\rangle + 1 \in W_e(B)' \iff 2\langle \lambda(a, e), x\rangle + 1 \in B'.$$

Let g be a recursive function yielding for every e an index of the r.e set

$$\{\langle 2\langle a, x\rangle, \{2\langle \lambda(a, e), x\rangle\}\rangle : a, x \in \mathbb{N}\} \cup \{\langle 2\langle a, x\rangle + 1, \{2\langle \lambda(a, e), x\rangle + 1\}\rangle : a, x \in \mathbb{N}\}.$$

Then for all e, $W_e(B)' = W_{g(e)}(B')$. □

Notice that using the enumeration jump and the enumeration reducibility we can express the relations "r.e. in" and "r.e. in the k-th Turing jump of". For example X is r.e. in the k-th Turing jump of A if $X \leq_e (A^+)^{(k)}$ and hence for every sequence $\mathcal{A} = \{A_k\}_{k<\omega}$,

$$J_{\mathcal{A}} = \{d_T(X) : (\forall k)(A_k \leq_e (X^+)^{(k)} \text{ uniformly in } k)\}.$$

We conclude the preliminaries with a Jump inversion theorem proved in [7].

Given a sequence $\mathcal{B} = \{B_k\}_{k<\omega}$ of sets of natural numbers we define the respective *jump sequence* $\mathcal{P}(\mathcal{B}) = \{\mathcal{P}_k(\mathcal{B})\}_{k<\omega}$ by induction on k:

(i) $\mathcal{P}_0(\mathcal{B}) = \emptyset \oplus B_0$;
(ii) $\mathcal{P}_{k+1}(\mathcal{B}) = \mathcal{P}_k(\mathcal{B})' \oplus B_{k+1}$.

Clearly for every k the set $\mathcal{P}_k(\mathcal{B})$ depends only on the first $k + 1$ members $B_0, \ldots, B_k$ of the sequence $\mathcal{B}$. So given $k+1$ sets $B_0, \ldots, B_k$ by $\mathcal{P}_k(B_0, \ldots, B_k)$ we shall denote the k-th jump set of any sequence with first $k + 1$ members equal to $B_0, \ldots, B_k$ respectively.

2.7. Theorem. *Let $\mathcal{B} = \{B_k\}_{k<\omega}$ be a sequence of sets of natural numbers. Suppose that for some $X \subseteq \mathbb{N}$ and for some $n \in \mathbb{N}$, $\mathcal{P}_n(\mathcal{B}) \leq_e X^+$. Then there exists $F \subseteq \mathbb{N}$ satisfying the following conditions:*

(1) $(\forall k \leq n)(B_k \leq_e (F^+)^{(k)})$
(2) $(\forall k < n)((F^+)^{(k+1)} \equiv_e (F^+) \oplus \mathcal{P}_k(\mathcal{B})')$.
(3) $(F^+)^{(n)} \equiv_e X^+$.

3. The uniform reducibility

Given a set W of natural numbers and $k \in \mathbb{N}$, let $W[k] = \{u : \langle k, u\rangle \in W\}$.

3.1. Definition. *For every $W \subseteq \mathbb{N}$ and every sequence $\mathcal{B} = \{B_k\}_{k<\omega}$ of sets of natural numbers, let $W(\mathcal{B}) = \{W[k](B_k)\}_{k<\omega}$.*

3.2. Definition. *Let $\mathcal{A} = \{A_k\}_{k<\omega}$ and $\mathcal{B} = \{B_k\}_{k<\omega}$ be elements of $\mathcal{S}$. Then $\mathcal{A} \leq_e \mathcal{B}$ (*$\mathcal{A}$ is enumeration reducible to $\mathcal{B}$*) if $\mathcal{A} = W(\mathcal{B})$ for some r.e. set W.*

A simple application of the S^m_n-Theorem shows that $\mathcal{A} \leq_e \mathcal{B}$ if and only if there exists a recursive function h such that $(\forall k)(A_k = W_{h(k)}(B_k))$.

Let $\mathcal{A} \equiv_e \mathcal{B}$ if $\mathcal{A} \leq_e \mathcal{B}$ and $\mathcal{B} \leq_e \mathcal{A}$.

The following facts follow easily from the definitions.

3.3. Proposition. *Let $\mathcal{A}, \mathcal{B} \in \mathcal{S}$. Then the following assertions hold:*

(1) $\mathcal{A} \leq_e \mathcal{P}(\mathcal{A})$
(2) $\mathcal{P}(\mathcal{P}(\mathcal{A})) \leq_e \mathcal{P}(\mathcal{A})$.
(3) $\mathcal{A} \leq_e \mathcal{B} \Rightarrow \mathcal{P}(\mathcal{A}) \leq_e \mathcal{P}(\mathcal{B})$.

The following Theorem from [8] gives an explicit characterization of the uniform reducibility.

3.4. Theorem. *For every two sequences $\mathcal{A}$ and $\mathcal{B}$ of sets of natural numbers*

$$\mathcal{A} \leq_u \mathcal{B} \iff \mathcal{A} \leq_e \mathcal{P}(\mathcal{B}.).$$

3.5. Corollary.

(1) *For all $\mathcal{A} \in \mathcal{S}$, $\mathcal{A} \equiv_u \mathcal{P}(\mathcal{A})$.*
(2) *For all $\mathcal{A}, \mathcal{B} \in \mathcal{S}$, $\mathcal{A} \leq_e \mathcal{B} \Rightarrow \mathcal{A} \leq_u \mathcal{B}$.*

We shall need some properties of the sequences uniformly equivalent to $\emptyset_\omega$ which are in some sense an analog of the r.e. sets.

Call a sequence $V = \{V[k]\}_{k<\omega}$ *recursively enumerable (r.e.)* if $V \leq_u \emptyset_\omega$, i.e if there exists an r.e set W such that $V = W(\{\emptyset^{(k)}\}_{k<\omega})$.

We have a natural enumeration $V_0, \ldots, V_a, \ldots$ of the *r.e.* sequences, where for each a, $V_a = W_a(\{\emptyset^{(k)}\}_{k<\omega})$. If $V = V_a$, then we shall say that a is an index of V.

Given two sequences $V = \{V[k]\}_{k<\omega}$ and $\mathcal{A} = \{A_k\}_{k<\omega}$ of sets of natural numbers, let

$$V(\mathcal{A}) = \{V[k](A_k)\}_{k<\omega}.$$

3.6. Proposition. *There exists a recursive function μ such that for all a, and for all sequences $\mathcal{A} = \{A_k\}_{k<\omega}$,*

$$V_a(\mathcal{A}) = W_{\mu(a)}(\{\emptyset^{(k)} \oplus A_k\}_{k<\omega}).$$

Proof. Clearly for every k we have that

$$x \in V_a[k](A_k) \iff (\exists D)(\langle x, D\rangle \in V_a[k] \ \&\ D \subseteq A_k) \iff$$
$$(\exists D_1)(\exists D)(\langle\langle x, D\rangle, D_1\rangle \in W_a[k] \ \&\ D_1 \subseteq \emptyset^{(k)} \ \&\ D \subseteq A_k).$$

Let $W_{\mu(a)} = \{\langle k, \langle x, D_1 \oplus D\rangle\rangle : \langle k, \langle\langle x, D\rangle, D_1\rangle\rangle \in W_a\}$. □

From here we get immediately the following corollary:

3.7. Corollary. *For every r.e. sequence V and every sequence $\mathcal{A}$, $V(\mathcal{A}) \leq_u \mathcal{A}$.*

Proof. Let V be a r.e. sequence and $\mathcal{A} \in \mathcal{S}$. Then for some r.e. set W, $V(\mathcal{A}) = W(\{\emptyset^{(k)}\} \oplus \mathcal{A})$ and hence $V(\mathcal{A}) \leq_e \{\emptyset^{(k)}\} \oplus \mathcal{A}$. On the other hand $\{\emptyset^{(k)}\} \oplus \mathcal{A} \leq_e \mathcal{P}(\mathcal{A})$. So $V(\mathcal{A}) \leq_e \mathcal{P}(\mathcal{A})$. Hence $V(\mathcal{A}) \leq_u \mathcal{A}$. □

The r.e sequences of sets share several properties with the r.e sets. For our purposes we shall need the following analog of the Recursion Theorem.

Given a r.e sequence of sets $R = \{R[k]\}_{k<\omega}$ and $e \in \mathbb{N}$, by R_e we shall denote the sequence $\{\{x : \langle e, x\rangle \in R[k]\}\}_{k<\omega}$.

3.8. Proposition. *Let R be a r.e sequence of sets. Then there exists an $e \in \mathbb{N}$ such that $R_e = V_e$.*

Proof. Clearly there exists a recursive function g such that for all $a \in \mathbb{N}$, $R_a = W_{g(a)}(\{\emptyset^{(k)}\}_{k<\omega})$. By the Recursion Theorem there exists an e such that $W_e = W_{g(e)}$. □

Next we turn to the definition of the jump operator on sequences of sets.

3.9. Definition. *For every $\mathcal{A} \in \mathcal{S}$ let $\mathcal{A}' = \{\mathcal{P}_{k+1}(\mathcal{A})\}_{k<\omega}$.*

3.10. Proposition. *Let $\mathcal{A} = \{A_k\}_{k<\omega} \in \mathcal{S}$. Then $J_{\mathcal{A}'} = \{\mathbf{a}' : \mathbf{a} \in J_{\mathcal{A}}\}$.*

Proof. Let $\mathbf{a} \in J_{\mathcal{A}}$. Since $\mathcal{P}(\mathcal{A}) \equiv_u \mathcal{A}$, $\mathbf{a} \in J_{\mathcal{P}(\mathcal{A})}$ and hence for some $X \in \mathbf{a}$ we have that for all k, $\mathcal{P}_k(\mathcal{A}) \leq_e (X^+)^{(k)}$ uniformly in k. From here it follows that for all k, $\mathcal{P}_{k+1}(\mathcal{A}) \leq_e ((X^+)')^{(k)}$. Thus $\mathbf{a}' \in J_{\mathcal{A}'}$.

Suppose now that $\mathbf{b} \in J_{\mathcal{A}'}$. Then for some $X \in \mathbf{b}$ and for all k, $\mathcal{P}_{k+1}(\mathcal{A}) \leq_e (X^+)^{(k)}$ uniformly in k. In particular $\mathcal{P}_1(\mathcal{A}) \leq_e X^+$. By Theorem 2.7 there exists $F \subseteq \mathbb{N}$ such that $A_0 \leq_e F^+$ and $(F^+)' \equiv_e X^+$. Let $\mathbf{a} = d_T(F)$. Then $\mathbf{a} \in J_{\mathcal{A}}$ and $\mathbf{a}' = \mathbf{b}$. □

3.11. Proposition. *Let $\mathcal{A}, \mathcal{B} \in \mathcal{S}$. Then the following assertions are true:*

(J0) $\mathcal{A} \lneq_u \mathcal{A}'$
(J1) $\mathcal{A} \leq_u \mathcal{B} \Rightarrow \mathcal{A}' \leq_u \mathcal{B}'$

Proof. Clearly $\mathcal{A} \leq_e \mathcal{P}(\mathcal{A}) \leq_e \mathcal{A}'$. Hence $\mathcal{A} \leq_u \mathcal{A}'$. Assume that $\mathcal{A}' \leq_u \mathcal{A}$. Then $\mathcal{A}' \leq_e \mathcal{P}(\mathcal{A})$ and hence $\mathcal{P}_1(\mathcal{A}) = \mathcal{P}_0(\mathcal{A})' \oplus A_1 \leq_e \mathcal{P}_0(\mathcal{A})$. By the properties of the enumeration jump the last is not possible.

The condition (J1) follows by Proposition 3.10. □

From (J1) it follows that $\mathcal{A} \equiv_u \mathcal{B} \Rightarrow \mathcal{A}' \equiv_u \mathcal{B}'$. So we may define a jump operation on the ω-enumeration degrees by $d_\omega(\mathcal{A})' = d_\omega(\mathcal{A}')$. As usual by $\mathcal{A}^{(n)}$ we shall denote the n-th iteration of the jump operation on $\mathcal{A}$. So, $\mathcal{A}^{(0)} = \mathcal{A}$ and $\mathcal{A}^{(n+1)} = (\mathcal{A}^{(n)})'$. Notice that $\mathcal{A}^{(n)} \equiv_e \{\mathcal{P}_{n+k}(\mathcal{A})\}_{k<\omega}$.

Next follows a jump inversion theorem which shows that the range of the jump operation on the ω-enumeration degrees is equal to the set of all ω-enumeration degrees greater than or equal to $d_\omega(\emptyset_\omega)'$.

3.12. Theorem. *Let $n \geq 1$. For every two sequences $\mathcal{B} = \{B_k\}_{k<\omega}$ and $\mathcal{Q} = \{Q_k\}_{k<\omega}$ such that $\mathcal{B}^{(n)} \leq_u \mathcal{Q}$ there exists a sequence $\mathcal{F}$ such that $\mathcal{B} \leq_u \mathcal{F}$ and $\mathcal{F}^{(n)} \equiv_u \mathcal{Q}$.*

Proof. Suppose that $\mathcal{B}^{(n)} \equiv_e \{\mathcal{P}_{n+k}(\mathcal{B})\}_{k<\omega}$ is uniformly reducible to $\mathcal{Q}$.

By Theorem 2.7 there exists a set F such that $(\forall k < n)(B_k \leq_e (F^+)^{(k)})$ and $(F^+)^{(n)} \equiv_e \mathcal{P}_{n-1}(\mathcal{B})'$. Set $F_0 = F^+$, $F_k = \emptyset$, $1 \leq k \leq n-1$, and $F_k = Q_{n-k}$ for $k \geq n$. Let $\mathcal{F} = \{F_k\}_{k<\omega}$.

Clearly $(\forall k < n)(\mathcal{P}_k(\mathcal{F}) \equiv_e F_0^{(k)})$. Notice that $\mathcal{P}_n(\mathcal{B}) \leq_e \mathcal{P}_0(\mathcal{Q}) \leq_e Q_0$. Then

$$\mathcal{P}_n(\mathcal{F}) = \mathcal{P}_{n-1}(\mathcal{F})' \oplus F_n \equiv_e F_0^{(n)} \oplus Q_0 \equiv_e \mathcal{P}_{n-1}(\mathcal{B})' \oplus Q_0 \equiv_e Q_0.$$

From here it follows directly that $\mathcal{F}^{(n)} \equiv_u \mathcal{Q}$. Clearly $\mathcal{P}(\mathcal{B}) \leq_e \mathcal{P}(\mathcal{F})$. Hence $\mathcal{B} \leq_u \mathcal{F}$. □

In what follows we shall study the ω-enumeration degrees below $\emptyset'_\omega$. By an analogy with the enumeration degrees we call these degrees Σ^0_2. Notice that a sequence $\mathcal{A}$ is uniformly reducible to $\emptyset'_\omega$ if and only if $\mathcal{A} \leq_e \{\emptyset^{(k+1)}\}_{k<\omega}$.

4. Good Approximations

The notions of good and better approximation of a set of natural numbers are introduced by Lachlan and Shore in [4].

4.1. Definition. *A recursive sequence of finite sets $\{B_s\}$ is a* good *approximation of the set B if it satisfies the following two conditions:*

(G1) $(\forall n)(\exists s)(B \upharpoonright n \subseteq B_s \subseteq B)$.
(G2) $(\forall n)(\exists s)(\forall t \geq s)(B_t \subseteq B \Rightarrow B \upharpoonright n \subseteq B_t)$.

The stages s such that $B_s \subseteq B$ are called *good stages*. From (G1) it follows that for every n there exists a good stage s such that $B \upharpoonright n \subseteq B_s$. Clearly, if the set B is infinite, then there exist arbitrary large good stages.

Notice that every r.e. set W_a has a good approximation $\{W_{a,s}\}$, where

$$W_{a,s} = \{x : x \leq s \ \&\ \{a\}(x) \text{ halts in less than } s \text{ steps}\}.$$

4.2. Definition. *A recursive sequence of finite characteristic functions* $\{\alpha_s\}$ *is a* better *approximation of the set* B *if the following two conditions are true:*

(B1) $(\forall n)(\exists s)(B \upharpoonright n \subseteq \alpha_s \subseteq B)$.

(B2) $(\forall n)(\exists s)(\forall t \geq s)(\{x : \alpha_t(x) \simeq 1\} \subseteq B \Rightarrow B \upharpoonright n \subseteq \alpha_t)$.

The following lemma stated in [4] is important for our construction. Since the proof in [4] is not entirely correct we present one here.

4.3. Lemma.([4]) *If a set* B *is infinite and has a good approximation, then it has also a better approximation.*

Proof. Suppose that $\{B_s\}$ is a good approximation of the set B. We may assume that $B_0 = \emptyset$. Set $\alpha_0 = \emptyset$. Suppose that $s > 0$. Let $t < s$ be the largest number such that $B_t \subseteq B_s$ and let $m_s = \min(B_s \setminus B_t)$ (set $m_s = s$ if $B_s = B_t$). Let

$$\alpha_s(k) \simeq \begin{cases} 1, & \text{if } k \in B_s, \\ 0, & \text{if } k \notin B_s \ \&\ k \leq m_s, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Denote by G the set of all good stages of the approximation $\{B_s\}$ and let

$$c(n) \simeq \mu s[s \in G \ \&\ B \upharpoonright n = B_s \upharpoonright n].$$

Since $\{B_s\}$ is a good approximation, the function c is total.

First we shall show that for every n, $\alpha_{c(n)} \subseteq B$. Indeed, set $s = c(n)$. Then $B_s \subseteq B$ and $B \upharpoonright n = B_s \upharpoonright n$. Let $t < s$ be the largest number such that $B_t \subseteq B_s$. Then $t \in G$ and there exists a $m < n$ such that $m \in B$ and $m \notin B_t$. Then $m \in B_s \setminus B_t$ and hence $m_s \leq m < n$. Let $\alpha_s(k) \simeq 1$. Then $k \in B_s$ and hence $B(k) \simeq 1$. Assume that $\alpha_s(k) \simeq 0$. Then $k \leq m_s < n$ and $k \notin B_s$ and hence $k \notin B$. So, $B(k) \simeq 0$.

Next we show that the function c obtains arbitrary large values, i.e. for every s there exists an n such that $s < c(n)$. Indeed, let $D = \bigcup_{t \leq s; t \in G} B_t$. Let $n \in B \setminus D$. Clearly if $t \leq s$ and $t \in G$, then $B_t \upharpoonright (n+1) \neq B \upharpoonright (n+1)$. Hence $c(n+1) > s$.

Now we are ready to show that $\{\alpha_s\}$ satisfies the properties (B1) and (B2) of the definition of better approximation of B. We start with the proof of (B2).

Let us fix a natural number n. By (G2) there exists a v_0 such that if $s \geq v_0$ and $s \in G$, then $B \upharpoonright n \subseteq B_s$. Let $s_0 \geq v_0$ and $s_0 \in G$. Again by (G2) there exists a $v > s_0$ such that if $v \leq s \in G$, then $B_{s_0} \subseteq B_s$.

Now let $v \leq s$ and $\{x : \alpha_s(x) \simeq 1\} = B_s \subseteq B$. Then $s \in G$ and hence $B_{s_0} \subseteq B_s$. Let $t < s$ be the largest number such that $B_t \subseteq B_s$. Clearly $s_0 \leq t$ and hence $v_0 \leq t$. Therefore $B \upharpoonright n \subseteq B_t$. Then $m_s \geq n$. Let $k < n$. If $k \in B$, then $k \in B_s$ and hence $\alpha_s(k) \simeq 1$. Suppose that $k \notin B$. Then $k < m_s$ and $k \notin B_s$ and hence $\alpha_s(k) \simeq 0$. So, $B \upharpoonright n \subseteq \alpha_s$.

To prove (B1) fix again a natural number n. By (B2) there exists a v such that if $s \geq v$ and $s \in G$, then $B \upharpoonright n \subseteq \alpha_s$. As shown above there exists a m such that $v \leq c(m)$. Then $B \upharpoonright n \subseteq \alpha_{c(m)} \subseteq B$. $\square$

Notice that from the proof above it follows that we have an uniform way to obtain from every good approximation of B a better one. If a better approximation $\{\alpha_s\}$ is obtained in this way from a good approximation $\{B_s\}$, then we shall call $\{\alpha_s\}$ *consistent with* $\{B_s\}$.

If α is a partial function and $a, s \in \mathbb{N}$, then by $W^{\alpha}_{a,s}$ we shall denote the set of all $x \leq s$ such that the computation $\{a\}^{\alpha}(x)$ halts successfully in less than s steps. We shall assume that if during a computation the oracle α is called with an argument outside it's domain, then the computation halts unsuccessfully.

4.4. Lemma.([4]) *Let $\{B_s\}$ be a good approximation of B and $\{\alpha_s\}$ be a better approximation of B consistent with $\{B_s\}$. Let $A = W^B_a$. Then $\{W^{\alpha_s}_{a,s} \oplus B_s\}$ is a good approximation of $A \oplus B$.*

Next we turn to good approximations of sequences of sets.

4.5. Definition. *Let $\mathcal{B} = \{B_k\}_{k<\omega}$ be a sequence of sets of natural numbers. A sequence $\{B^s_k\}$ of finite sets recursive in k and s is a* good approximation of $\mathcal{B}$ *if the following three conditions are satisfied:*

(i) $(\forall s)(\forall k)[B^s_k \subseteq B_k \Rightarrow (\forall r \leq k)(B^s_r \subseteq B_r)]$.
(ii) $(\forall n)(\forall k)(\exists s)(\forall r \leq k)(B_r \restriction n \subseteq B^s_r \subseteq B_r)$.
(iii) $(\forall n)(\forall k)(\exists s)(\forall t \geq s)[B^t_k \subseteq B_k \Rightarrow (\forall r \leq k)(B_r \restriction n \subseteq B^t_r)]$.

If $\{B^s_k\}$ is a good approximation of the sequence $\mathcal{B} = \{B_k\}_{k<\omega}$, then by G_k we shall denote the set of all *k-good stages*, i.e the set of all s such that $B^s_k \subseteq B_k$. Clearly $G_r \supseteq G_k$ for all $r \leq k$.

4.6. Definition. *Let $\mathcal{A} = \{A_k\}_{k<\omega}$ and $\mathcal{B} = \{B_k\}_{k<\omega}$ be sequences of sets of natural numbers and let $\{B^s_k\}$ be a good approximation of $\mathcal{B}$. A sequence $\{A^s_k\}$ of finite sets recursive in s and k is a* correct (with respect to $\{B^s_k\}$) approximation of $\mathcal{A}$ *if the following two conditions hold:*

(C1) $(\forall k, s)(B^s_k \subseteq B_k \Rightarrow (\forall r \leq k)(A^s_r \subseteq A_r))$.
(C2) *For all natural numbers k, n there exists a v such that if $s \geq v$ and $B^s_k \subseteq B_k$, then $(\forall r \leq k)(A_r \restriction n \subseteq A^s_r)$.*

The following lemma is an analogue of Lemma 2.2 from [4] and can be proved by similar arguments:

4.7. Lemma. *Let $\{B^s_k\}$ be a good approximation of the sequence $\mathcal{B}$. Let W be an r.e. set. Then $\{W_s[k](B^s_k)\}$ is a correct approximation of $W(\mathcal{A})$.*

Let $K_0 = K = K_\emptyset$ and $K_{k+1} = K_{K_k}$. In other words, let K_k be the $k+1$-th iteration of the Turing jump of $\emptyset$.

Using Proposition 2.4 we get immediately the following

4.8. Lemma. *For every sequence $\mathcal{B}$, $\{K_k\}_{k<\omega} \leq_e \mathcal{P}(\mathcal{B})$.*

Given a sequence $\mathcal{P} = \{P_k\}_{k<\omega}$, define the sequence $\mathcal{P}^* = \{P^*_k\}_{k<\omega}$ by induction:

(i) $P^*_0 = P_0 \oplus K_0$;
(ii) $P^*_{k+1} = P_{k+1} \oplus P^*_k \oplus K_{k+1}$.

Clearly $\mathcal{P} \leq_e \mathcal{P}^*$. Using the previous Lemma we obtain immediately that for jump sequences the reverse inequality is also true.

4.9. Lemma. *Let $\mathcal{P} = \mathcal{P}(\mathcal{B})$ for some sequence $\mathcal{B}$. Then $\mathcal{P}^* \equiv_e \mathcal{P}$.*

4.10. Proposition. *Let $\mathcal{P} = \{P_k\}_{k<\omega}$ be a sequence of sets of natural numbers. Suppose that $\mathcal{P} \leq_u \emptyset'_\omega$. Then the sequence $\mathcal{P}^*$ has a good approximation.*

Proof. By Proposition 2.5 $\{\emptyset^{(k+1)}\}_{k<\omega} \leq_e \{K_k^+\}_{k<\omega}$. Hence there exists a recursive function g such that $(\forall k)(P_k = W_{g(k)}^{K_k})$.

We shall define for every k a recursive sequence $\{P_k^s\}_{s<\omega}$ of finite sets of natural numbers by recursion on k.

Let $\{K_0^s\}$ be a good approximation of K_0 and let $\{\alpha_0^s\}$ be a better approximation of K_0 consistent with $\{K_0^s\}$. Set $P_0^s = W_{g(0),s}^{\alpha_0^s} \oplus K_0^s$. Since $P_0 = W_{g(0)}^{K_0}$, the sequence $\{P_0^s\}$ is a good approximation of $P_0 \oplus K_0 = P_0^*$.

Suppose that we have defined a good approximation $\{P_k^s\}$ of P_k^*. Clearly we can find affectively an index a such that $K_{k+1} = W_a^{P^*_k}$. Let $\{\beta_k^s\}$ be a better approximation of P_k^* consistent with $\{P_k^s\}$. Then $\{P_k^s \oplus W_{a,s}^{\beta_k^s}\}$ is a good approximation of $P_k^* \oplus K_{k+1}$. Let $\{\alpha_k^s\}$ be a better approximation of $P_k^* \oplus K_{k+1}$ consistent with it.

Since $P_{k+1} = W_{g(k+1)}^{K_{k+1}}$, we can find effectively an index b such that $P_{k+1} = W_b^{P_k \oplus K_{k+1}}$. Set $P_{k+1}^s = W_{b,s}^{\alpha_k^s} \oplus P_k^s \oplus W_{a,s}^{\beta_k^s}$.

Clearly $\{P_{k+1}^s\}$ is a good approximation of P_{k+1}^*

It is easy to check that $\{P_k^s\}_{s<\omega,k<\omega}$ is a good approximation of the sequence $\{P_k^*\}_{k<\omega}$. □

Next we turn to the following problem. Suppose that the sequence $\mathcal{P}(\mathcal{B})$ has a good approximation $\{P_k^s\}$ and let W be an r.e. set. How to define a correct approximation of $\mathcal{P}(W(\mathcal{P}(\mathcal{B})))$? It turns out that we can do that provided that we know an index of the r.e. set W.

4.11. Proposition. *There exists a recursive function $\rho(e,k)$ such that for every sequence $\mathcal{B}$, $(\forall k)(\mathcal{P}_k(W_e(\mathcal{P}(\mathcal{B}))) = W_{\rho(e,k)}(\mathcal{P}_k(\mathcal{B})) \oplus W_e[k](\mathcal{P}_k(\mathcal{B})))$.*

Proof. We shall define $\rho(e,k)$ by recursion on k.

Let a_0 be an index of $\emptyset$. Set $\rho(e,0) = a_0$. Clearly for every sequence $\mathcal{B}$,

$$\mathcal{P}_0(W_e(\mathcal{P}(\mathcal{B})) = \emptyset \oplus W_e[0](\mathcal{P}_0(\mathcal{B})) = W_{\rho(e,0)}(\mathcal{P}_0(\mathcal{B})) \oplus W_e[0](\mathcal{P}_0(\mathcal{B})).$$

Suppose that $k \geq 0$ and $\rho(e,k)$ is defined. Then for every sequence $\mathcal{B}$,

$$\mathcal{P}_{k+1}(W_e(\mathcal{P}(\mathcal{B}))) = \mathcal{P}_k(W_e(\mathcal{P}(\mathcal{B})))' \oplus W_e[k+1](\mathcal{P}_{k+1}(\mathcal{B})) =$$
$$[W_\rho(e,k)(\mathcal{P}_k(\mathcal{B})) \oplus W_e[k](\mathcal{P}_k(\mathcal{B}))]' \oplus W_e[k+1](\mathcal{P}_{k+1}(\mathcal{B})).$$

Let $\lambda(a,b)$ be a recursive function such that for all $a, b \in \mathbb{N}$ and $P \subseteq \mathbb{N}$,

$$W_{\lambda(a,b)}(P) = W_a(P) \oplus W_b(P).$$

Let $\kappa(e,k)$ be a recursive function such that for all $e, k \in \mathbb{N}$, $W_{\kappa(e,k)} = W_e[k]$. Then

$$\mathcal{P}_{k+1}(W_e(\mathcal{P}(\mathcal{B}))) = W_{\lambda(\rho(e,k),\kappa(e,k))}(\mathcal{P}_k(\mathcal{B}))' \oplus W_e[k+1](\mathcal{P}_{k+1}(\mathcal{B})).$$

By Proposition 2.6 there exists a recursive function g such that if $a \in \mathbb{N}$ and $P, B \subseteq \mathbb{N}$, then $W_a(P)' = W_{g(a)}(P' \oplus B)$.

Set $\rho(e,k+1) = g(\lambda(\rho(e,k),\kappa(e,k)))$. We have

$$W_{\rho(e,k+1)}(\mathcal{P}_{k+1}(\mathcal{B})) = W_{\rho(e,k+1)}(\mathcal{P}_k(\mathcal{B})' \oplus B_{k+1}) =$$
$$[W_{\lambda(\rho(e,k),\kappa(e,k))}(\mathcal{P}_k(\mathcal{B})]' = [W_{\rho(e,k)}(\mathcal{P}_k(\mathcal{B})) \oplus W_e[k](\mathcal{P}_k(\mathcal{B}))]'.$$

Hence $W_{\rho(e,k+1)}(\mathcal{P}_{k+1}(\mathcal{B})) \oplus W_e[k+1](\mathcal{P}_{k+1}(\mathcal{B})) = \mathcal{P}_{k+1}(W_e(\mathcal{P}(\mathcal{B})))$. □

The following proposition can be proved by almost the same arguments.

4.12. Proposition. *There exists a recursive function $\rho^*(e,k)$ such that for every sequence $\mathcal{B}$, $(\forall k)(\mathcal{P}_k(W_e(\mathcal{P}(\mathcal{B})^*)) = W_{\rho^*(e,k)}(\mathcal{P}_k(\mathcal{B})^*) \oplus W_e[k](\mathcal{P}_k(\mathcal{B})^*))$.*

4.13. Corollary. *Let $\{P_k^s\}$ be a good approximation of $\mathcal{P}(\mathcal{B})^*$. Then the sequence $\{W_{\rho^*(e,k),s}(P_k^s) \oplus W_{e,s}[k](P_k^s)\}$ is a correct approximation of the sequence $\mathcal{P}(W_e(\mathcal{P}(\mathcal{B})^*))$.*

5. The density theorem

In this section we are going to prove the following theorem:

5.1. Theorem. *The Σ_2^0 ω-enumeration degrees are dense.*

Let us fix two sequences $\mathcal{A} = \{A_k\}_{k<\omega}$ and $\mathcal{B} = \{B_k\}_{k<\omega}$. Suppose that $\mathcal{A} <_u \mathcal{B} \leq_u \emptyset'_\omega$. Our goal is to construct a sequence $\mathcal{C}$ such that $\mathcal{A} <_u \mathcal{C} <_u \mathcal{B}$.

Clearly, $\mathcal{B} \equiv_u \mathcal{A}\oplus\mathcal{B}$ and hence $\mathcal{P}(\mathcal{A}\oplus\mathcal{B}) \leq_e \{\emptyset^{(k+1)}\}_{k<\omega}$ and $\mathcal{P}(\mathcal{A}) <_e \mathcal{P}(\mathcal{A}\oplus\mathcal{B})$.

Assume that there exists a $k < \omega$ such that $\mathcal{P}_k(\mathcal{A}) \not\equiv_e \mathcal{P}_k(\mathcal{A} \oplus \mathcal{B})$.

In this case the Theorem follows by the density of the n-rea enumeration degrees [4]. To see that we need the following

5.2. Lemma. *Let $k \in \mathbb{N}$ and $A \subseteq \mathbb{N}$. Suppose that $\emptyset^{(k)} \leq_e A \leq_e \emptyset^{(k+1)}$. Then the enumeration degree of A contains a $(k+2)$-rea set.*

Proof. First of all notice that for every k the set $X_k = K_0 \oplus \cdots \oplus K_k$ is $k+1$-rea.

If $A \leq_e \emptyset'$ then A is r.e. in K_0 and hence A is 2-rea.

Let $k \geq 1$. Suppose that $\emptyset^{(k)} \leq_e A \leq_e \emptyset^{(k+1)}$. Then $K_{k-1}^+ \leq_e A \leq_e K_k^+$. It follows from here that A is r.e. in K_k and hence A is r.e. in X_k. Then $X \oplus A$ is $(k+2)$-rea. On the other hand K_k is r.e. in K_{k-1} and hence $K_k \leq_e K_{k-1}^+ \leq_e A$. Clearly for all $i < k$,

$$K_i \leq_e K_i^+ \equiv_e \emptyset^{(i+1)} \leq_e \emptyset^{(k)} \leq_e A.$$

So, $X_k \leq_e A$ and hence $X \oplus A \equiv_e A$. □

By the Lemma the enumeration degrees of $\mathcal{P}_k(\mathcal{A})$ and $\mathcal{P}_k(\mathcal{A} \oplus \mathcal{B})$ are $(k+2)$-rea. Hence there exists a set C such that $\mathcal{P}_k(\mathcal{A}) <_e C <_e \mathcal{P}_k(\mathcal{A} \oplus \mathcal{B})$. Define the sequence $\mathcal{C}$ by $C_n = A_n$ if $n \neq k$ and $C_k = C$. Clearly $\mathcal{A} <_u \mathcal{C} <_u \mathcal{B}$ and the theorem follows.

Now, assume that

$$\text{(E)} \qquad (\forall k)(\mathcal{P}_k(\mathcal{A}) \equiv_e \mathcal{P}_k(\mathcal{A} \oplus \mathcal{B})).$$

Set $\mathcal{P} = \mathcal{P}(\mathcal{A}\oplus\mathcal{B})^*$. Clearly $\mathcal{P} = \{P_k\}_{k<\omega}$ is enumeration equivalent to $\mathcal{P}(\mathcal{A}\oplus\mathcal{B})$ and hence $(\forall k)(\mathcal{P}_k(\mathcal{A}) \equiv_e P_k)$.

We shall construct a r.e. set V, such that $\mathcal{P}(\mathcal{A}) <_e \mathcal{P}(\mathcal{A} \oplus V(\mathcal{P})) <_e \mathcal{P}$. Then we will have that $\mathcal{A} <_u \mathcal{A} \oplus V(\mathcal{P}) <_u \mathcal{P}$. Clearly $\mathcal{P} \equiv_u \mathcal{A} \oplus \mathcal{B} \equiv_u \mathcal{B}$. Then $\mathcal{A} <_u \mathcal{A} \oplus V(\mathcal{P}) <_u \mathcal{B}$.

Our construction will be similar to that one used in [4].

Evidently for every r.e. set V,

$$\mathcal{P}(\mathcal{A}) \leq_e \mathcal{P}(\mathcal{A} \oplus V(\mathcal{P})) \leq_e \mathcal{P}.$$

So, to make these inequalities strong we shall construct V satisfying the following conditions for every $i \in \mathbb{N}$:

$P_i)$ $W_i(\mathcal{P}(\mathcal{A})) \neq V(\mathcal{P})$.

$Q_i)$ $W_i(\mathcal{P}(\mathcal{A} \oplus V(\mathcal{P}))) \neq \mathcal{P}$.

For the construction we shall need a good approximation of the sequence $\mathcal{P}$, and correct with respect to this approximation approximations of the sequences $\mathcal{P}(\mathcal{A})$, $\mathcal{P}(\mathcal{A} \oplus V(\mathcal{P}))$ and $V(\mathcal{P})$.

By Proposition 4.10 there exists a good approximation $\{P_k^s\}$ of the sequence $\mathcal{P}$. From now on we shall call an approximation *correct* if it is correct with respect to $\{P_k^s\}$.

Since $\mathcal{P}(\mathcal{A}) \leq_e \mathcal{P}$, there exists an r.e. set W such that $\mathcal{P}(\mathcal{A}) = W(\mathcal{P})$. Given $s, k \in \mathbb{N}$, set $P_k^s(\mathcal{A}) = W_s[k](P_k^s)$. By Lemma 4.7 $\{P_k^s(\mathcal{A})\}$ is a correct approximation of $\mathcal{P}(\mathcal{A})$.

Next we turn to the problem of finding a correct approximation of the sequence $\mathcal{P}(\mathcal{A} \oplus V(\mathcal{P}))$.

Fix a recursive function $\lambda(a, b)$ such that for all natural numbers a and b,

$$W_{\lambda(a,b)} = \{\langle k, \langle u, D\rangle\rangle : (\exists x)((u = 2x \ \&\ \langle k, \langle x, D\rangle\rangle \in W_a) \vee \\ (u = 2x + 1 \ \&\ \langle k, \langle x, D\rangle\rangle \in W_b))\}.$$

Then for every $k < \omega$ and every $P \subseteq \mathbb{N}$, $W_{\lambda(a,b)}[k](P) = W_a[k](P) \oplus W_b[k](P)$.

Clearly $\mathcal{A} \leq_e \mathcal{P}$ and hence there exists a r.e. set W_a such that $\mathcal{A} = W_a(\mathcal{P})$. Set

$$A_k^s = W_{a,s}[k](P_k^s).$$

By Lemma 4.7 $\{A_k^s\}$ is a correct approximation of $\mathcal{A}$.

By Proposition 4.12 for every e,

$$(\forall k)(\mathcal{P}_k(\mathcal{A} \oplus W_e(\mathcal{P})) = \mathcal{P}_k(W_{\lambda(a,e)}(\mathcal{P})) = W_{\rho^*(\lambda(a,e),k)}(P_k) \oplus W_{\lambda(a,e)}[k](P_k)).$$

Set $\mu(e, k) = \rho^*(\lambda(a, e), k)$. Then for all e,

$$(\forall k)(\mathcal{P}_k(\mathcal{A} \oplus W_e(\mathcal{P})) = W_{\mu(e,k)}(P_k) \oplus A_k \oplus W_e[k](P_k)).$$

Now suppose that V_s is a recursive sequence of finite sets such that $V_s \subseteq V_{s+1}$ and $V = \bigcup_s V_s$. Suppose also that e is an index of V, i.e. $V = W_e$. Let

$$Q_{e,k}^s = W_{\mu(e,k),s}(P_k^s) \oplus A_k^s.$$

Clearly $\{Q_{e,k}^s \oplus V_s[k](P_k^s)\}$ is a recursive in e, s, k sequence of finite sets which is a correct approximation of $\mathcal{P}(\mathcal{A} \oplus V(\mathcal{P}))$. The problem here is that to define this approximation we need to know an index e of V. Thus during the construction of V we need to know an index of it. To resolve this problem we shall use the Recursion Theorem in the following form:

5.3. Theorem.(Recursion Theorem) *Let $R \subseteq \mathbb{N}^2$ be a r.e. set. Then there exists an e such that $\{x : (e, x) \in R\} = W_e$.*

For every $R \subseteq \mathbb{N}^2$, let $R_e = \{x : (e, x) \in R\}$.

The general plan of our construction is the following. At every stage s we shall construct effectively a finite subset R^s of $\mathbb{N}^2$ so that $R^s \subseteq R^{s+1}$ and let $R = \bigcup R_s$. After that using the Recursion Theorem we shall find an e_0 such that $R_{e_0} = W_{e_0}$ and set $V = R_{e_0}$. Clearly $V = \bigcup R_{e_0}^s$.

While deciding whether to put a pair (e, x) in R we shall use $\{Q_{e,k}^s \oplus R_e^s[k](P_k^s)\}$ as approximation of $\mathcal{P}(\mathcal{A} \oplus V(\mathcal{P}))$ and $R_e^s(\{P_k^s\})$ as approximation of $V(\mathcal{P})$. Clearly for $e = e_0$ these approximations will be correct.

Now we are ready to describe the construction of R. We start with the definition of the length of agreement function:

5.4. Definition. *Given two sequences* $\mathcal{A} = \{A_k\}_{k<\omega}$ *and* $\mathcal{B} = \{B_k\}_{k<\omega}$ *of sets of natural numbers and* $s \in \mathbb{N}$, *let*

$$l_s(\mathcal{A}, \mathcal{B}) = \max\{u < s : (\forall \langle k, x\rangle < u)(A_k(x) = B_k(x))\}.$$

Let $R^0 = \emptyset$. Suppose that R^s is defined. For every $e \leq s$ and for every $i \leq s$ we act for the requirements $P_i)$ and $Q_i)$ as follows:

$P_i)$ Let $l_{e,s} = l_s(W_{i,s}(\{P^s_k(\mathcal{A})\}), R^s_e(\{P^s_k\}))$.
For every $\langle k, x\rangle < l_{e,x}$ if $x \in P^s_k$ and $\langle i, x\rangle \notin R^s_e[k](P^s_k)$, then we enumerate the axiom $\langle\langle i, x\rangle, P^s_k\rangle$ in $R_e[k]$, i.e. we add $(e, \langle k, \langle\langle i, x\rangle, P^s_k\rangle\rangle)$ to R^{s+1}.

$Q_i)$ Let $m_{e,s} = l_s(W_{i,s}(\{Q^s_{e,k} \oplus R^s_e[k](P^s_k)\}), \{P^s_k\})$.
For every $\langle k, x\rangle < m_{e,s}$ if $x \notin W_{i,s}[k](Q^s_{e,k} \oplus R^s_e[k](P^s_k))$ but

$$x \in W_{i,s}[k](Q^s_{e,k} \oplus (R^s_e[k](P^s_k)^{[\leq i]} \cup \mathbb{N}^{[>i]}))$$

via an axiom $\langle x, F \oplus L\rangle \in W_{i,s}[k]$, then for every $y \in L^{[>i]}$ we enumerate the axiom $\langle y, P^s_k\rangle$ in $R_e[k]$.

End of the construction

Now let $R = \bigcup R^s$ and let e be an index such that $V = R_e = W_e$. Set $V_s = R^s_e$. Notice that the sequence $\{V_s[k](A^s_k)\}$ is a correct approximation of $V(\mathcal{A})$ and the sequence $\{Q^s_{e,k} \oplus V_s[k](P^s_k)\}$ is a correct approximation of $\mathcal{P}(\mathcal{A} \oplus V(P))$.

Call s a k-good stage of the construction if $P^s_k \subseteq P_k$. Clearly if s is a k-good stage, then s is a r-good stage for all $r \leq k$.

5.5. Lemma. *For all i the following conditions are true:*

1. $W_i(\mathcal{P}(\mathcal{A})) \neq V(\mathcal{P})$.
2. *There exists a natural number p_i such that only finitely many axioms are enumerated in V by the actions of $P_i)$ on the p_i-good stages of the construction.*
3. $W_i(\mathcal{P}(\mathcal{A} \oplus V(\mathcal{P}))) \neq \mathcal{P}$.
4. *There exists a natural number q_i such that only finitely many axioms are enumerated in V by the actions of $Q_i)$ on the q_i-good stages of the construction.*

Proof. We shall prove the conditions (1) - (4) simultaneously by induction on i. Suppose that for $j < i$ the conditions (1) - (4) are true. Then there exists a p such that if $r > p$, then for $j < i$ the actions of $P_j)$ and $Q_j)$ do not contribute anything to $V[r]$ at r-good stages and hence $V[r](P_r)^{[<i]} = \emptyset$.

Towards a contradiction assume that $W_i(\mathcal{P}(\mathcal{A})) = V(\mathcal{P})$. Then

$$(\forall k)(W_i[k](\mathcal{P}_k(\mathcal{A})) = V[k](P_k)).$$

We shall show that for all $r > p$,

$$(\forall x)(\langle i, x\rangle \in V[r](P_r) \iff x \in P_r).$$

Indeed, fix a $r > p$.

Suppose that $\langle i, x\rangle \in V[r](P_r)$. Then there exists an axiom $\langle\langle i, x\rangle, D\rangle \in V[r]$ such that $D \subseteq P_r$. Let s be the stage at which this axiom has been enumerated in $V[r]$. Then $D = P^s_r$ and hence s is a r-good stage. Since $r > p$ the axiom has been enumerated in $V[r]$ by $P_i)$. Therefore $x \in P^s_r$ and hence $x \in P_r$.

Let $x \in P_r$. Clearly there exists a r-good stage s such that $e < s$, $i < s$, $x \in P_r^s$ and $\langle r, x\rangle < l_{e,s}$. Then either $\langle i, x\rangle \in V_s[r](P_r^s)$ or, by the action of $P_i)$, $\langle i, x\rangle \in V_{s+1}[r](P_r^s)$. In both cases $\langle i, x\rangle \in V[r](P_r)$.

Thus we obtain that $(\forall r > p)(\forall x)(\langle i, x\rangle \in W_i[r](\mathcal{P}_r(\mathcal{A})) \iff x \in P_k)$. From here since $P_k(\mathcal{A}) \equiv_e P_k$ for all $k \leq p$ it follows that $\mathcal{P} \leq_e \mathcal{P}(\mathcal{A})$. A contradiction.

Let us turn to the proof of (2). By (1) $W_i(\mathcal{P}(A)) \neq V(\mathcal{P})$. Then there exists a pair $\langle k, x\rangle$ such that $W_i[k](\mathcal{P}_k(\mathcal{A}))(x) \neq V[k](P_k)(x)$. Clearly there exists a v such that if $s > v$ is a k-good stage, then $W_{i,s}[k](P_k^s(\mathcal{A}))(x) = W_i[k](\mathcal{P}_k(\mathcal{A}))(x)$ and $V_s[k](P_k^s)(x) = V[k](P_k)(x)$. Then for all $s > v$, if s is a k-good stage, then $l_{e,s} < \langle k, x\rangle$. Hence all axioms enumerated in V by $P_i)$ at k-good stages $s > v$ are of the form $\langle l, \langle\langle i, y\rangle, P_l^s\rangle\rangle$, where $\langle l, y\rangle < \langle k, x\rangle$. Let

$$p_i = \max\{l : (\exists y)(\langle l, y\rangle \leq \langle k, x\rangle)\}.$$

Assume that $P_i)$ enumerates infinitely many axioms at p_i-good stages in V. Since every p_i-good stage is also a k-good stage, there exists a pair $\langle l, y\rangle < \langle k, x\rangle$ such that infinitely many axioms $\langle l, \langle\langle i, y\rangle, P_l^{t_n}\rangle\rangle$, where for all n, t_n is a p_i-good stage, are enumerated by $P_i)$ in V. Since every p_i-good stage is also a l-good stage, there exist n_1 and n_2 such that $t_{n_1} < t_{n_2}$ and $P_l^{t_{n_1}} \subseteq P_l^{t_{n_2}}$. Then

$$\langle i, y\rangle \in V_{t_{n_1}+1}[l](P_l^{t_{n_1}}) \subseteq V_{t_{n_2}}[l](P_l^{t_{n_2}})$$

So at stage t_{n_2} the conditions under which $P_i)$ enumerates the axiom $\langle l, \langle\langle i, y\rangle, P_l^{t_{n_2}}\rangle\rangle$ in V are not satisfied. A contradiction.

Now we start with the proof of (3). Let us assume that

$$W_i(\mathcal{P}(\mathcal{A} \oplus V(\mathcal{P}))) = \mathcal{P}.$$

We shall get a contradiction by showing that $\mathcal{P} \leq_e \mathcal{P}(\mathcal{A})$. For we shall define a recursive function $\lambda(k)$ so that $(\forall k)(P_k = W_{\lambda(k)}(\mathcal{P}_k(\mathcal{A})))$.

By the induction hypothesis and by (2) there exists a q such that if $r \geq q$, then $V[r](P_r)^{[\leq i]} = \emptyset$. Thus if $r \geq q$, then $V[r](P_r) \subseteq \mathbb{N}^{[>i]}$.

By (E), $(\forall k)(P_k \equiv_e \mathcal{P}_k(\mathcal{A}))$. Hence there exist indices $a_0, \dots, a_q$, such that for all $k \leq q$, $P_k = W_{a_k}(\mathcal{P}_k(\mathcal{A}))$. Let for $k \leq q$, $\lambda(k) \simeq a_k$.

Suppose that $\lambda(r)$ is defined for some $r \geq q$. Since in the definition of $\mathcal{P}_r(\mathcal{A} \oplus V(\mathcal{P}))$ only the sets A_k and $V[k](P_k)$, $k \leq r$, are used, using the values of $\lambda(k)$, $k \leq r$, we can define effectively an index c such that $\mathcal{P}_r(\mathcal{A} \oplus V(\mathcal{P})) = W_c(\mathcal{P}_r(\mathcal{A}))$. Then

$$\begin{aligned}\mathcal{P}_{r+1}(\mathcal{A} \oplus V(\mathcal{P})) =& \mathcal{P}_r(\mathcal{A} \oplus V(\mathcal{P}))' \oplus A_{r+1} \oplus V[r+1](P_{r+1}) = \\ & W_c(\mathcal{P}_r(\mathcal{A}))' \oplus A_{r+1} \oplus V[r+1](P_{r+1}).\end{aligned}$$

Let g be a recursive function such that for all $a \in \mathbb{N}$ and for all $P, B \subseteq \mathbb{N}$,

$$W_a(P)' = W_{g(a)}(P' \oplus B).$$

Then

$$\mathcal{P}_{r+1}(\mathcal{A} \oplus V(\mathcal{P})) = W_{g(c)}(\mathcal{P}_{r+1}(\mathcal{A})) \oplus A_{r+1} \oplus V[r+1](P_{r+1}).$$

Set $Q = W_{g(c)}(\mathcal{P}_{r+1}(\mathcal{A})) \oplus A_{r+1}$. We shall show that

$$W_i[r+1](Q \oplus V[r+1](P_{r+1})) = W_i[r+1](Q \oplus \mathbb{N}^{[>i]}).$$

Since $V[r+1](P_{r+1}) \subseteq \mathbb{N}^{[>i]}$ we obtain immediately that

$$W_i[r+1](Q \oplus V[r+1](P_{r+1})) \subseteq W_i[r+1](Q \oplus \mathbb{N}^{[>i]}).$$

Let us turn to the proof of the reverse inclusion. Recall that $\mathcal{P}_{r+1}(\mathcal{A}\oplus V(\mathcal{P})) = W_{\mu(e,r+1)}(P_{r+1}) \oplus A_{r+1} \oplus V[r+1](P_{r+1})$. Then $Q = W_{\mu(e,r+1)}(P_{r+1}) \oplus A_{r+1}$ and hence $\{Q^s_{e,r+1}\}_{s<\omega}$ is a correct approximation of Q.

Let $x \in W_i[r+1](Q \oplus \mathbb{N}^{[>i]})$. Let s be a $r+1$-good stage such that $e, i < s$, $\langle r+1, x\rangle < m_{e,s}$ and $x \in W_{i,s}[r+1](Q^s_{e,r+1} \oplus \mathbb{N}^{[>i]})$. Then by the construction $x \in W_{i,s}[r+1](Q^s_{e,r+1} \oplus V_{s+1}[r+1](P^s_{r+1}))$ and hence $x \in W_i(Q \oplus V[r+1](P_{r+1}))$.

So,

$$\begin{aligned} P_{r+1} &= W_i[r+1](\mathcal{P}_{r+1}(\mathcal{A}\oplus V(\mathcal{P}))) = \\ &W_i[r+1](W_{g(c)}(\mathcal{P}_{r+1}(\mathcal{A})) \oplus A_{r+1} \oplus \mathbb{N}^{[>i]}). \end{aligned}$$

From here one can obtain effectively a value $\lambda(r+1)$ such that

$$P_{r+1} = W_{\lambda(r+1)}(\mathcal{P}_{r+1}(\mathcal{A})).$$

The proof of (4) is similar to the proof of (2). By (2) $W_i(\mathcal{P}(\mathcal{A}\oplus V(\mathcal{A}))) \neq \mathcal{P}$ and hence there exists a pair $\langle k, x\rangle$ and a v such that for all k-good stages $s > v$, $m_{e,s} \leq \langle k, x\rangle$. Choose q_i as in the proof of (2) i.e.

$$q_i = \max\{l : (\exists z)(\langle l, z\rangle \leq \langle k, x\rangle)\}.$$

Assume that infinitely many axioms are enumerated by Q_i) at p_i-good stages in V. Then there exists a pair $\langle l, z\rangle < \langle k, x\rangle$ such that for infinitely many p_i-good stages t_n, $z \notin W_{i,t_n}(Q^{t_n}_{e,l} \oplus V_{t_n}[l](P^{t_n}_l))$ but $z \in W_{i,t_n}(Q^{t_n}_{e,l} \oplus V_{t_n+1}[l](P^{t_n}_l))$. It follows from the properties of the approximations $\{Q^s_{e,k}\}$ and $\{P^s_k\}$ that this is not possible. A contradiction. □

6. There is no minimal ω-enumeration degree

The following theorem, proved in [1], combined with the density of the Σ^0_2 enumeration degrees shows that there are no minimal enumeration degrees:

6.1. Theorem. *There is an enumeration operator V such that for any enumeration operator Ψ:*

(1) *If $V(B)$ is r.e. then B is recursive in $\emptyset'$.*
(2) *If $\Psi(V(B)) = B$ then B is r.e.*

A generalization of the methods used in the proof of this theorem leads to the following result.

Given a sequence $\mathcal{B} = \{B_k\}_{k<\omega}$ and $a \in \mathbb{N}$, denote by $\mathcal{B} \upharpoonright a$ the sequence $\{B_k \upharpoonright a\}_{k<\omega}$.

6.2. Theorem. *There is an r.e. sequence V such that the following conditions hold for every sequence $\mathcal{B}$:*

(1) *If $V(\mathcal{B})$ is r.e., then $\mathcal{B} \leq_u \emptyset'_\omega$.*
(2) *If for some a, $W_a(\mathcal{P}(V(\mathcal{B}))) = \mathcal{B}$ then $\mathcal{B} \leq_u \mathcal{B} \upharpoonright a$.*

From the last theorem it follows that if the ω-enumeration degree $\mathbf{b}$ of a sequence $\mathcal{B}$ is minimal, then for some a the degree $\mathbf{b}$ contains the sequence $\mathcal{B} \upharpoonright a$. Indeed, consider the r.e. sequence V from the theorem. Clearly $V(\mathcal{B}) \leq_u \mathcal{B}$ and hence $V(\mathcal{B}) \leq_u \emptyset_\omega$ or $\mathcal{B} \leq_u V(\mathcal{B})$. The first is impossible by (1) and by the Density Theorem. So it follows that $\mathcal{B} \leq_u V(\mathcal{B})$. Then $\mathcal{B} \leq_e \mathcal{P}(V(\mathcal{B}))$. Let $\mathcal{B} = W_a(\mathcal{P}(V(\mathcal{B})))$. By (2), $\mathcal{B} \equiv_u \mathcal{B} \upharpoonright a$.

Since for every $a \geq 1$ there exist $2^{\aleph_0}$ sequences of subsets of $\mathbb{N} \upharpoonright a$, not all of them are uniformly equivalent to $\emptyset_\omega$. So to conclude the proof that there is no minimal ω-enumeration degree we shall show also the following:

6.3. Theorem. *For every natural number c there exists an r.e. sequence V satisfying the following conditions for every sequence $\mathcal{B}$ of subsets of $\mathbb{N} \upharpoonright c$:*

1. *If $V(\mathcal{B})$ is r.e., then $\mathcal{B} \leq_u \emptyset'_\omega$.*
2. *If for some r.e. set W, $W(\mathcal{P}(V(\mathcal{B}))) = \mathcal{B}$ then $\mathcal{B} \leq_u \emptyset_\omega$.*

From the last two theorems we get easily

6.4. Theorem. *There are no minimal ω-enumeration degrees.*

In the rest of the paper we shall give the proofs of Theorem 6.2 and Theorem 6.3.

We start with a simple proposition which is important for the strategies used in the construction.

6.5. Proposition. *Let $\Psi : \mathcal{P}(\mathbb{N})^{k+1} \to \mathcal{P}(\mathbb{N})$ be monotone. Let $X_1, \dots, X_k, B$ be subsets of $\mathbb{N}$, $\Psi(X_1, \dots, X_k, B) = B$ and for some $a \in \mathbb{N}$,*

$$(\forall n \geq a)(n \in \Psi(X_1, \dots, X_k, B) \Rightarrow n \in \Psi(X_1, \dots, X_k, B \upharpoonright n)).$$

Then $B = \bigcup C_k$, where $C_0 = B \upharpoonright a$ and $C_{k+1} = \Psi(X_1, \dots . X_k, C_k) \cup C_k$.

Proof. Using the equality $\Psi(X_1, \dots, X_k, B) = B$ one can show easily by induction on k that for all k, $C_k \subseteq B$. To prove the reverse inclusion assume that n is the least element of $B \setminus \bigcup C_k$. Then $B \upharpoonright n \subset \bigcup C_k$ and hence $B \upharpoonright n \subseteq C_k$ for some k. Clearly $n \geq a$. Then

$$n \in B \Rightarrow n \in \Psi(X_1, \dots, X_k, B \upharpoonright n) \subseteq \Psi(X_1, \dots, X_k, C_k) \subseteq C_{k+1}.$$

A contradiction. □

6.6. Proposition. *There exists a recursive function $\lambda(a, b)$ such that for every $X, Y, Z \subseteq \mathbb{N}$, if $C_0 = Z$, $C_{k+1} = W_a(X \oplus W_b(Y \oplus C_k)) \cup C_k$ then*

$$\bigcup C_k = W_{\lambda(a,b)}(X \oplus Y \oplus Z).$$

6.7. Proposition. *There exists a recursive function $\mu(a, b, k)$ such that for every sequence $\mathcal{B} = \{B_k\}_{k<\omega}$ of sets of natural numbers, every r.e. set W_a and every r.e. sequence of sets $V_b = \{V[k]\}_{k<\omega}$ satisfying the conditions:*

1. $W_a(\mathcal{P}(V(\mathcal{B})) = \mathcal{B}$.
2. $(\forall n \geq a)(n \in W_a[0](\emptyset \oplus V_b[0](B_0)) \Rightarrow n \in W_a[0](\emptyset \oplus V_b[0](B_0 \upharpoonright n)))$.
3. $(\forall k \geq 1)(\forall n \geq a)(n \in W_a[k](\mathcal{P}_{k-1}(V_b(\mathcal{B}))' \oplus V_b[k](B_k)) \Rightarrow$

$$n \in W_a[k](\mathcal{P}_{k-1}(V_b(\mathcal{B}))' \oplus V_b[k](B_k \upharpoonright n))).$$

we have that $(\forall k)(\mathcal{P}_k(V_b(\mathcal{B})) = W_{\mu(a,b,k)}(\mathcal{P}_k(\mathcal{B} \upharpoonright a))$.

Proof. We shall define the function μ by recursion on k.

Let us fix the natural numbers a and b. Suppose that W is an r.e set with index a and V is an r.e. sequence of sets with index b. Let $\mathcal{B}$ be a sequence of sets such that the conditions (1) - (3) are satisfied.

We can find effectively an index c such that $(\forall k)(V[k](B_k) = W_c[k](\emptyset^{(k)} \oplus B_k))$.

Then $B_0 = W_a[0](\emptyset \oplus W_c[0](\emptyset \oplus B_0))$. Let a_0 be an index of $W_a[0]$ and c_0 be an index of $W_c[0]$. By (2) and by the previous two propositions $B_0 = W_{\lambda(a_0,c_0)}(\emptyset \oplus \emptyset \oplus B_0 \upharpoonright a)$. From here one can construct easily an index $\mu(a,b,0)$ such that

$$\mathcal{P}_0(V(\mathcal{B})) = \emptyset \oplus V[0](B_0) = \emptyset \oplus W_c[0](\emptyset \oplus B_0) = W_{\mu(a,b,0)}(\emptyset \oplus B_0 \upharpoonright a).$$

Suppose that $\mu(a,b,k)$ is defined.

Clearly $V[k+1](B_{k+1}) = W_c[k+1](\emptyset^{(k+1)} \oplus B_{k+1})$. Let a_{k+1} be an index of $W_a[k+1]$ and c_{k+1} be an index of $W_c[k+1]$. By (3) and by the previous two propositions

$$B_{k+1} = W_{\lambda(a_{k+1},c_{k+1})}(\mathcal{P}_k(V(\mathcal{B}))' \oplus \emptyset^{(k+1)} \oplus B_{k+1} \upharpoonright a).$$

Since $\mathcal{P}_k(V(\mathcal{B})) = W_{\mu(a,b,k)}(\mathcal{P}_k(\mathcal{B} \upharpoonright a))$, we can find an index p such that

$$\mathcal{P}_k(V(\mathcal{B}))' = W_p(\mathcal{P}_k(\mathcal{B} \upharpoonright a)').$$

Then

$$B_{k+1} = W_{\lambda(a_{k+1},c_{k+1})}(W_p(\mathcal{P}_k(\mathcal{B} \upharpoonright a)') \oplus \emptyset^{(k+1)} \oplus B_{k+1} \upharpoonright a).$$

From here we can easily find an index q such that $V[k+1](B_{k+1}) = W_q(\mathcal{P}_{k+1}(\mathcal{B} \upharpoonright a))$ and define the function $\mu(a,b,k+1)$ so that

$$\mathcal{P}_{k+1}(V(\mathcal{B})) = \mathcal{P}_k(V(\mathcal{B}))' \oplus V[k+1](B_{k+1}) = W_{\mu(a,b,k+1)}(\mathcal{P}_{k+1}(\mathcal{B} \upharpoonright a)).$$

□

6.8. Corollary. *There exists a recursive function $\nu(a,b,k)$ such that $W_{\nu(a,b,0)}(\emptyset) = \emptyset$ and for every sequence $\mathcal{B} \in \mathcal{S}$, every r.e. set W_a, every r.e. sequence V_b and every $k \geq 1$, if*

(1) $(\forall l < k)(W_a[l](\mathcal{P}_l(V(\mathcal{B}))) = \mathcal{B}_l)$.
(2) $(\forall n \geq a)(n \in W_a[0](\emptyset \oplus V_b[0](B_0)) \Rightarrow n \in W_a[0](\emptyset \oplus V_b[0](B_0 \upharpoonright n)))$.
(3) $(\forall n \geq a)(\forall l \in [1,k))(n \in W_a[l](\mathcal{P}_{l-1}(V_b(\mathcal{B}))' \oplus V_b[l](B_l)) \Rightarrow$

$$n \in W_a[l](\mathcal{P}_{l-1}(V_b(\mathcal{B}))' \oplus V_b[l](B_k \upharpoonright n))).$$

then $W_{\nu(a,b,k)}(\mathcal{P}_{k-1}(\mathcal{B} \upharpoonright a)') = \mathcal{P}_{k-1}(V_b(\mathcal{B}))'$.

We need one more proposition to conclude the preparation for the proof of Theorem 6.2.

For every sequence $v_0, \dots, v_k$ by $\langle v_0, \dots, v_k \rangle$ we shall denote it's canonical code. We shall assume that the code of the empty sequence $\langle\rangle$ is 0.

6.9. Proposition. *There exists a recursive function $\pi(v)$ such that*

(1) *$W_{\pi(0)}(X) = \emptyset$ for every $X \subseteq \mathbb{N}$.*
(2) *If $k \geq 1$ and $v = \langle v_0, \dots, v_{k-1} \rangle$, then $W_{\pi(v)}(\emptyset^{(k)}) = \mathcal{P}_{k-1}(D_{v_0}, \dots, D_{v_{k-1}})'$.*

Proof of Theorem 6.2. We shall construct an r.e. sequence $R = \{R[k]\}_{k<\omega}$ of sets of natural numbers. The desired sequence V will be defined as R_e for some e such that $R_e = V_e$. The existence of such an e follows from Proposition 3.8. So during the construction we may assume that we know an index e of V.

In the construction we shall use the functions $\nu(a,b,k)$ and $\pi(v)$, defined in Corollary 6.8 and Proposition 6.9.

The construction of R will be performed by stages. At every stage s we shall define the sequence $R^s[k], k < \omega$ of finite sets, so that for every k and s, $R^s[k] \subseteq R^{s+1}[k]$ and let $R[k] = \bigcup_s R^s[k]$.

Let for all k, $R^0[k] = \emptyset$. Suppose that all sets $R^s[k]$, $k < \omega$, are defined.

For every $e, k \leq s$ we do the following. For every triple (a, n, F) where $a, n \leq s$ and F is a subset of $\mathbb{N} \upharpoonright n$ and for every sequence $D_{v_0}, \dots, D_{v_{k-1}}$ of subsets of $\mathbb{N} \upharpoonright a$, let $v = \langle v_0, \dots, v_{k-1} \rangle$ and check whether there exists a $S \subseteq \{\langle i, j \rangle : j \geq a, j \geq n\}$ such that

$$(*) \qquad n \in W_{a,s}[k](W_{\nu(a,e,k),s}(W_{\pi(v),s}(\emptyset^{(k)})) \oplus (R^s_e[k](F) \cup S)).$$

If such S exists then let S_0 be the one with least canonical code and for every pair $\langle i, j \rangle \in S_0$ enumerate $\langle \langle i, j \rangle, \emptyset \rangle$ in $R_e[k]$, i.e. enumerate $\langle e, \langle \langle i, j \rangle, \emptyset \rangle \rangle$ in $R[k]$.

When done with all triples (a, n, F) find for every $j \leq s$ the least pair $\langle i, j \rangle$ such that $\langle \langle i, j \rangle, \emptyset \rangle$ is not yet enumerated in $R_e[k]$ and enumerate $\langle \langle i, j \rangle, \{j\} \rangle$ in $R_e[k]$.

End of construction

From the construction it follows immediately that the sequence R is r.e. Let us fix an e such that $R_e = V_e$ and set $V = V_e$.

6.10. Lemma. *For every k and every j there are finitely many i such that $\langle \langle i, j \rangle, \emptyset \rangle$ belongs to $V[k]$.*

Proof. Let us fix a k and a j. Suppose that $\langle \langle i, j \rangle, \emptyset \rangle \in V[k]$. Then there exists a triple (a, n, F) such that $a \leq j$, $n \leq j$ and $F \subseteq \mathbb{N} \upharpoonright n$ and some sequence $D_{v_0}, \dots, D_{v_{k-1}}$ of subsets $\mathbb{N} \upharpoonright a$ satisfying (*) for some s and some $S \subseteq \{\langle i, j \rangle : j \geq a, j \geq n\}$ such that $\langle i, j \rangle \in S$.

There are finitely many such triples (a, n, F) and finitely many sequences of length k of subsets of $\mathbb{N} \upharpoonright a$ and and by the construction every such triple and every such sequence will enumerate finitely many pairs $\langle \langle i, j \rangle, \emptyset \rangle$ in $V[k]$. $\square$

Now we are ready to show that V satisfies the conditions (1) and (2) of the Theorem. Let $\mathcal{B} = \{B_k\}_{k<\omega}$ be a sequence of sets of natural numbers. Assume that $V(\mathcal{B}) \leq_u \emptyset_\omega$. Then $V(\mathcal{B}) \leq_e \{\emptyset^{(k)}\}_{k<\omega}$.

By the lemma above for every k and every j there exists an i such that $\langle \langle i, j \rangle, \{j\} \rangle \in V[k]$ but $\langle \langle i, j \rangle, \emptyset \rangle \notin V[k]$. Hence for every k,

$$B_k = \{j : (\exists i)(\langle \langle i, j \rangle, \{j\} \rangle \in V[k] \;\&\; \langle \langle i, j \rangle, \emptyset \rangle \notin V[k] \;\&\; \langle i, j \rangle \in V[k](B_k)\}.$$

From here one gets immediately that $\mathcal{B} \leq_e \{\emptyset^{(k+1)}\}_{k<\omega}$ and hence $\mathcal{B} \leq_u \emptyset'_\omega$.

Assume now that for some a we have that $W_a(\mathcal{P}(V(\mathcal{B})) = \mathcal{B}$. We shall show that $\mathcal{B}$, W_a and V satisfy the conditions (1) - (3) of Proposition 6.7. Clearly the first condition is true.

Next we shall show that

$$(\forall n \geq a)(n \in W_a[0](\emptyset \oplus V[0](B_0)) \Rightarrow n \in W_a[0](\emptyset \oplus V[0](B_0 \upharpoonright a))).$$

Indeed, let $n \geq a$ and $n \in W_a[0](\emptyset \oplus V[0](B_0))$. Let D be a finite subset of B_0 such that $n \in W_a[0](\emptyset \oplus V[0](D))$. Since $W_{\pi(\langle\rangle)}(\emptyset) = \emptyset$ and $W_{\nu(a,e,0)}(\emptyset) = \emptyset$, we have that

$$W_a[0](\emptyset \oplus V[0](D)) = W_a(W_{\nu(a,e,0)}(W_{\pi(\langle\rangle)}(\emptyset)) \oplus R_e[0](D)).$$

Then there exists a stage s such that

$$n \in W_{a,s}[0](W_{\nu(a,e,0),s}(W_{\pi(\langle\rangle),s}(\emptyset)) \oplus R^s_e[0](D)).$$

We may assume that $e, n \leq s$. Let $F = D \upharpoonright n$ and $S = R^s_e[0](D) \setminus R^s_e[0](F)$. Consider an element $\langle i, j \rangle$ of S. Clearly $\langle \langle i, j \rangle, \{j\} \rangle \in R^s_e[0]$ and $j > n$. Since $n \geq a$ we have also that $j \geq a$. So, $S \subseteq \mathbb{N}^{\geq \max(a,n)}$. So we have found a S such that

$$n \in W_{a,s}(W_{\nu(a,e,0),s}(W_{\pi(\langle\rangle),s}(\emptyset)) \oplus (R^s_e[0](F) \cup S)).$$

By the construction, $n \in W_{a,s}(W_{\nu(a,e,0),s}(W_{\pi(\langle\rangle),s}(\emptyset)) \oplus R_e^{s+1}[0](F)))$. And hence since $F \subseteq B \restriction n$, $n \in W_a(\emptyset \oplus V[0](B \restriction n))$.

To prove the third condition we shall show by induction on k that for all $k \geq 1$,
,

$$(\forall n \geq a)(n \in W_a[k](\mathcal{P}_{k-1}(V(\mathcal{B}))' \oplus V[k](B_k)) \Rightarrow \\ n \in W_a[k](\mathcal{P}_{k-1}(V(\mathcal{B}))' \oplus V[k](B_k \restriction n))).$$

Suppose that $k \geq 1$ and for $l < k$ the condition holds. By Corollary 6.8 we have that $\mathcal{P}_{k-1}(V(\mathcal{B}))' = W_{\nu(a,e,k)}(\mathcal{P}_{k-1}(\mathcal{B} \restriction a)')$. Let $D_{v_0} = B_0 \restriction a, \ldots, D_{v_{k-1}} = B_{k-1} \restriction a$ and $v = \langle v_0, \ldots, v_{k-1}\rangle$. Then

$$\mathcal{P}_{k-1}(\mathcal{B} \restriction a)' = \mathcal{P}_{k-1}(D_{v_0}, \ldots, D_{v_{k-1}})' = W_{\pi(v)}(\emptyset^{(k)}).$$

Thus

$$W_a[k](\mathcal{P}_{k-1}(V(\mathcal{B}))' \oplus V[k](B_k)) = W_a[k](W_{\nu(a,e,k)}(W_{\pi(v)}(\emptyset^{(k)})) \oplus R_e[k](B_k)).$$

Suppose that $n \geq a$ and $n \in W_a[k](\mathcal{P}_{k-1}(V(\mathcal{B}))' \oplus V[k](B_k))$. Then there exists a finite subset D of B_k and a stage s such that

$$n \in W_{a,s}[k](W_{\nu(a,e,k),s}(W_{\pi(v),s}(\emptyset^{(k)})) \oplus R_e^s[k](D)).$$

As in the proof above set $F = D \restriction n$ and $S = R_e^s[k](D) \setminus R_e^s[k](F)$. Then $S \subseteq \{\langle i,j\rangle : j \geq a, j \geq n\}$ and hence by the construction

$$n \in W_{a,s}[k](W_{\nu(a,e,k),s}(W_{\pi(v),s}(\emptyset^{(k)})) \oplus R_e^{s+1}[k](F)).$$

From here it follows easily that $n \in W_a[k](\mathcal{P}_{k-1}(V(\mathcal{B}))' \oplus V[k](B_k \restriction n))$.

Now from Proposition 6.7 it follows that $\mathcal{P}(V(\mathcal{B})) \leq_e \mathcal{P}(\mathcal{B} \restriction a)$. Since $\mathcal{B} \leq_e \mathcal{P}(V(\mathcal{B}))$ we get that $\mathcal{B} \leq_e \mathcal{P}(\mathcal{B} \restriction a)$ and hence $\mathcal{B} \leq_u \mathcal{B} \restriction a$. □

Now we turn to the proof of Theorem 6.3. We shall need the following versions of Proposition 6.7 and Corollary 6.8 which can be proved in a similar way.

6.11. Proposition. *There exists a recursive function $\mu^*(a,b,k)$ such that for every sequence $\mathcal{B} = \{B_k\}_{k<\omega}$ of sets of natural numbers, every r.e. set W_a and every r.e. sequence of sets $V_b = \{V[k]\}_{k<\omega}$ satisfying the conditions:*

(1) $W_a(\mathcal{P}(V(\mathcal{B})) = \mathcal{B}$.
(2) $(\forall n)(\forall k > a)(n \in W_a[k](\mathcal{P}_{k-1}(V_b(\mathcal{B}))' \oplus V_b[k](B_k)) \Rightarrow$

$$n \in W_a[k](\mathcal{P}_{k-1}(V_b(\mathcal{B}))' \oplus V_b[k](B_k \restriction n))).$$

we have that $(\forall k)(\mathcal{P}_k(V_b(\mathcal{B})) = W_{\mu^*(a,b,k)}(\mathcal{P}_a(\mathcal{B})' \oplus \emptyset^{(k)}))$.

6.12. Corollary. *There exists a recursive function $\nu^*(a,b,k)$ such that for every sequence $\mathcal{B} \in \mathcal{S}$, every r.e. set W_a, every r.e. sequence V_b and every $k > a$, if*

(1) $(\forall l < k)(W_a[l](\mathcal{P}_l(V(\mathcal{B}))) = \mathcal{B}_l)$.
(2) $(\forall n)(\forall l \in (a,k))(n \in W_a[l](\mathcal{P}_{l-1}(V_b(\mathcal{B}))' \oplus V_b[l](B_l)) \Rightarrow$

$$n \in W_a[l](\mathcal{P}_{l-1}(V_b(\mathcal{B}))' \oplus V_b[l](B_k \restriction n))).$$

then $W_{\nu^*(a,b,k)}(\mathcal{P}_a(\mathcal{B})' \oplus \emptyset^{(k)}) = \mathcal{P}_{k-1}(V_b(\mathcal{B}))'$.

Proof of Theorem 6.3. The general idea of the proof is very similar to that used in the proof of Theorem 6.2.

Let us fix a natural number c. We shall define the r.e. sequence V by means of the Recursion Theorem. So we shall construct an r.e. sequence $R = \{R[k]\}_{k<\omega}$ and let $V = R_e$ for some index e such that $R_e = V_e$.

The construction of R will be performed by stages. Let for all k, $R^0[k] = \emptyset$ and suppose that for some s all sets $R^s[k]$ are defined.

For every $e, k \le s+1$ we do the following. For every triple (a, n, F) where $a < k$, $n \le s$ and F is a subset of $\mathbb{N} \restriction n$ and for every sequence $D_{v_0}, \dots, D_{v_a}$ of subsets of $\mathbb{N} \restriction c$, let $v = \langle v_0, \dots, v_a \rangle$ and check whether there exists a $S \subseteq \{\langle i, j \rangle : j \ge n\}$ such that

(**) $$n \in W_{a,s}[k](W_{\nu^*(a,e,k),s}(W_{\pi(v),s}(\emptyset^{(a+1)}) \oplus \emptyset^{(k)}) \oplus (R^s_e[k](F) \cup S)).$$

If such S exists then let S_0 be the one with least canonical code and for every pair $\langle i, j \rangle \in S_0$ enumerate $\langle \langle i, j \rangle, \emptyset \rangle$ in $R_e[k]$..

When done with all triples (a, n, F) find for every $j \le s$ the least pair $\langle i, j \rangle$ such that $\langle \langle i, j \rangle, \emptyset \rangle$ is not yet enumerated in $R_e[k]$ and enumerate $\langle \langle i, j \rangle, \{j\} \rangle$ in $R_e[k]$.

End of construction

It follows directly from the construction that the sequence $\{R[k]\}$ is r.e. Let e be a natural number such that $R_e = V_e$. Set $V = V_e$

6.13. Lemma. *For every k and every j there are finitely many i such that $\langle \langle i, j \rangle, \emptyset \rangle$ belongs to $V[k]$.*

Proof. Let us fix a k and a j. Suppose that $\langle \langle i, j \rangle, \emptyset \rangle \in V[k]$. Then there exists a triple (a, n, F) such that $a < k$, $n \le j$ and $F \subseteq \mathbb{N} \restriction n$ and some sequence $D_{v_0}, \dots, D_{v_a}$ of subsets of $\mathbb{N} \restriction c$ satisfying (**) for some s and some $S \subseteq \mathbb{N}^{\ge n}$ such that $\langle i, j \rangle \in S$.

There are finitely many such triples (a, n, F) and for every a finitely many sequences of length $a+1$ of subsets of $\mathbb{N} \restriction c$ and by the construction every such triple and every such sequence will enumerate finitely many pairs $\langle \langle i, j \rangle, \emptyset \rangle$ in $V[k]$. $\square$

Let $\mathcal{B} = \{B_k\}_{k<\omega}$ be a sequence of subsets of $\mathbb{N} \restriction c$.

As in the proof of Theorem 6.2 by the last Lemma we have that if $V(B)$ is an r.e. sequence then $\mathcal{B} \le_u \emptyset'_\omega$.

Suppose now that for some a, $W_a(\mathcal{P}(V(\mathcal{B}))) = \mathcal{B}$. We shall show by induction on k that

$$(\forall k > a)(\forall n)(n \in W_a[k](\mathcal{P}_{k-1}(V(\mathcal{B}))' \oplus V[k](B_k)) \Rightarrow$$
$$n \in W_a[k](\mathcal{P}_{k-1}(V(\mathcal{B}))' \oplus V[k](B_k \restriction n))).$$

Suppose that for all l, $a < l < k$ the assertion holds. By Corollary 6.12,

$$\mathcal{P}_{k-1}(V(B))' = W_{\nu^*(a,e,k)}(\mathcal{P}_a(\mathcal{B})' \oplus \emptyset^{(k)})$$

Let $B_0 = D_{v_0}, \dots, B_a = D_{v_a}$ and $v = \langle v_0, \dots, v_a \rangle$. Then

$$\mathcal{P}_a(\mathcal{B})' = \mathcal{P}_a(D_{v_0}, \dots, D_{v_a})' = W_{\pi(v)}(\emptyset^{(a+1)}).$$

Assume that $n \in W_a[k](\mathcal{P}_{k-1}(V(\mathcal{B}))' \oplus V[k](B_k))$. Then for some finite $D \subseteq B_k$ and for some $s > e, k$,

$$n \in W_{a,s}[k](W_{\nu^*(a,e,k),s}(W_{\pi(v),s}(\emptyset^{(a+1)}) \oplus \emptyset^{(k)}) \oplus R^s_e[k](D))$$

Let $F = D \restriction n$ and $S = R^s_e[k](D) \setminus R^s_e[k](F)$. Consider an element $\langle i, j \rangle$ of S. Then $\langle \langle i, j \rangle, \{j\} \rangle \in R^s_e[k]$ and $j \ge n$. Hence $S \subseteq \{\langle i, j \rangle : j \ge n\}$. By the construction,

$$n \in W_{a,s}[k](W_{\nu^*(a,e,k),s}(W_{\pi(v),s}(\emptyset^{(a+1)}) \oplus \emptyset^{(k)}) \oplus R^{s+1}_e[k](F))$$

and hence $n \in W_a[k](\mathcal{P}_{k-1}(V(\mathcal{B}))' \oplus V[k](B_k \restriction n))$.

Now, by Proposition 6.11, $(\forall k)(\mathcal{P}_k(V(\mathcal{B})) = W_{\mu^*(a,e,k)}(\mathcal{P}_a(\mathcal{B})' \oplus \emptyset^{(k)}))$.

Since $\mathcal{P}_a(\mathcal{B})' \leq_e \emptyset^{(a+1)}$, there exists a recursive function h such that

$$(\forall k > a)(\mathcal{P}_k(\mathcal{B}) = W_{h(k)}(\emptyset^{(k)})).$$

From here since for all $k \leq a$, $B_k \leq_e \emptyset^{(k)}$ it follows that $\mathcal{P}(\mathcal{B}) \leq_e \{\emptyset^{(k)}\}_{k<\omega}$ and hence $\mathcal{B} \leq_u \emptyset_\omega$. □

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