ADMISSIBILITY IN Σ^0_n ENUMERATIONS

A. A. SOSKOVA AND I. N. SOSKOV

Abstract. In the paper we introduce the notion of Σ^0_n *partial enumeration* of an abstract structure $\mathfrak{A}$. Given a $k \leq n$ we obtain a characterization of the subsets of $\mathfrak{A}$ possessing Σ^0_k pullbacks in all Σ^0_n partial enumerations of $\mathfrak{A}$.

А. А. Соскова и И. Н. Сосков Допустимост в Σ^0_n номерации.

В статията се въвежда понятието Σ^0_n частична номерация на абстрактна структура $\mathfrak{A}$. Получена е характеризация на подмножествата на $\mathfrak{A}$, притежаващи Σ^0_k първообрази във всички Σ^0_n частични номерации, при $k \leq n$.

1. Introduction

Let $\mathfrak{A} = (A; R_1, R_2, \dots, R_l)$ be a countable abstract structure, where each R_i is an a_i-ary predicate on A.

A total mapping f of the set of the natural numbers N onto A is called a *total enumeration* of $\mathfrak{A}$. Every total enumeration f of $\mathfrak{A}$ determines a unique structure $\mathfrak{B}_f = (N; R^f_1, R^f_2, \dots, R^f_l)$ of the same relational type as $\mathfrak{A}$ where

$$R^f_i(x_1, \dots, x_{a_i}) \iff R_i(f(x_1), \dots, f(x_{a_i})).$$

Let $\alpha < \omega_1^{CK}$. A subset M of A^a is said to be Σ^0_α - *admissible* in $\mathfrak{A}$ if for every total enumeration f of $\mathfrak{A}$ the pullback $f^{-1}(M)$ of M is Σ^0_α in the diagram $D(\mathfrak{B}_f)$ of $\mathfrak{B}_f$.

The notion of Σ^0_1-admissibility with respect to injective total enumerations was introduced in 1964 by Lacombe [3] under the name $\forall$-admissibility. Several modifications and generalizations of this notion appear since 1964. Among them we would like to mention the Σ^0_1-admissibility in partial enumerations introduced in [5] and the relatively intrinsically Σ^0_α sets introduced in [1] and [2] which are defined by means of Σ^0_α-admissibility with respect to injective total enumerations.

1991 *Mathematics Subject Classification.* 03D70, 03D75.

Key words and phrases. definability, enumerations, forcing.

This work was partially supported by the Ministry of education and science, Contract I 604/96.

In [5] the second author made the observation that the sets on an abstract structure which are Σ^0_1-admissible with respect to partial enumerations with relatively recursively enumerable (r. e.) domains coincide with the sets which are Σ^0_1-admissible with respect to total enumerations.

In the present paper we are going to study further the interplay between admissibility in total and partial enumerations. For we introduce the notion of Σ^0_n-admissibility in partial enumerations with relatively Σ^0_n domains and more generally, for $k \leq n$, Σ^0_k-admissibility with respect to partial enumerations with relatively Σ^0_n domains. A normal form of the admissible sets is obtained. It turns out that for $k < n$ the admissible sets coincide with those which are Σ^0_k-admissible in all partial enumerations and are described by means of quantifier free recursive Σ^0_k formulas. If $k = n$, then our notion of admissibility leads to a class of sets, described by means of a simple kind of recursive Σ^0_n formulas on the abstract structure in which the quantifiers ranging over the domain of the structure are existential and appear only on the last level.

The arguments use the machinery of the so called regular enumerations, which seems to have a wide range of other applications.

2. Preliminaries

Consider again the countable structure $\mathfrak{A} = (A; R_1, R_2, \dots, R_l)$, which we shall from now on suppose fixed.

2.1. Definition. An *enumeration* of $\mathfrak{A}$ is an ordered pair $\langle f, \mathfrak{B}_f \rangle$, where f is a partial surjective mapping of N onto A with an infinite domain, $\mathfrak{B}_f = (N; \sigma_1, \sigma_2, \dots, \sigma_l)$ is a structure of the same relational type as $\mathfrak{A}$ and the following condition holds for every $i \in [1, l]$ and all $x_1, \dots, x_{a_i} \in dom(f)$:

$$\sigma_i(x_1, \dots, x_{a_i}) \iff R_i(f(x_1), \dots, f(x_{a_i})).$$

2.2. Definition. Let $n \geq 1$. The enumeration $\langle f, \mathfrak{B}_f \rangle$ is called Σ^0_n if the domain of f is Σ^0_n in the diagram $D(\mathfrak{B}_f)$ of $\mathfrak{B}_f$.

2.3. Definition. Let $k \geq 1$. A subset M of A^a is Σ^0_k-*admissible* in $\langle f, \mathfrak{B}_f \rangle$ if there exists a Σ^0_k in $D(\mathfrak{B}_f)$ subset W of N^a such that for all $x_1, \dots, x_a \in dom(f)$,

$$(x_1, \dots, x_a) \in W \iff (f(x_1), \dots, f(x_a)) \in M.$$

As stated in the introduction our goal is to obtain an explicit characterization of the sets which are Σ^0_k-admissible in all Σ^0_n enumerations, $k \leq n$. For we consider two kinds of recursive Σ^0_k formulas in the language $\mathcal{L}_{\omega_1 \omega}$ of the structure $\mathfrak{A}$, which we call "quantifier-free" and "existential" respectively.

The Σ^0_k, the Π^0_k and the Δ^0_{k+1} quantifier-free formulas are defined simultaneously with their indices by induction on k. We shall suppose that a coding of the formulas in $\mathcal{L}$ is fixed. Given an index v, by Φ^v we shall denote the formula having index v. For every formula Φ, by $\Phi(X_1, \dots, X_a)$ we shall denote that the free variables in Φ are among $X_1, \dots, X_a$.

As usual by $W_0, \ldots, W_e, \ldots$ we shall denote the standard enumeration of the r.e. sets of natural numbers.

2.4. Definition.

(i) The logical constant $\mathbb{T}$ and all atomic formulas in $\mathfrak{L}$ are Σ_0^0 quantifier-free formulas.
The logical constant $\mathbb{F}$ and all negated atomic formulas in $\mathfrak{L}$ are Π_0^0 quantifier-free formulas.
The Δ_1^0 quantifier-free formulas are finite conjunctions of Σ_0^0 and Π_0^0 quantifier-free formulas.
The indices of the Σ_0^0, Π_0^0 and Δ_1^0 quantifier-free formulas are their respective codes as formulas in $\mathfrak{L}$.

(ii) If every element of W_e is index of some Δ_{k+1}^0 quantifier-free formula with variables among $X_1, \ldots, X_a$, then

$$\bigvee_{v \in W_e} \Phi^v(X_1, \ldots, X_a),$$

is a Σ_{k+1}^0 quantifier-free formula with index $\langle 0, k+1, e\rangle$.
If Φ is a Σ_{k+1}^0 quantifier-free formula, then $\neg\Phi$ is a Π_{k+1}^0 quantifier-free formula. For every index $\langle 0, k+1, e\rangle$ of Φ, the triple $\langle 1, k+1, e\rangle$ is an index of $\neg\Phi$.
If $\Phi_1, \ldots, \Phi_b$ are Σ_r^0 or Π_r^0, $r \leq k+1$, then $\chi = \Phi_1 \& \ldots \& \Phi_b$ is a Δ_{k+2}^0 quantifier-free formula. If $v_1, \ldots, v_b$ are indices of $\Phi_1, \ldots, \Phi_b$ respectively, then $\langle 2, v_1, \ldots, v_b\rangle$ is an index of χ.

2.5. Definition. A Σ_k^0 *existential formula*, $k \geq 1$, is a formula of the form

$$\bigvee_{v \in V} \exists Y_1 \cdots \exists Y_{q_v} \Phi^v(Y_1, \ldots, Y_{q_v}, X_1, \ldots, X_a),$$

where V is an r.e. set of indices of Δ_k^0 formulas.

Let $M \subseteq A^a$ and $\Phi(X_1, \ldots, X_a, Z_1, \ldots, Z_b)$ be a Σ_k^0 quantifier-free or existential formula.

2.6. Definition. The set M is *definable* by Φ on $\mathfrak{A}$ if for some $t_1, \ldots, t_b \in A$

$$(\forall s_1, \ldots, s_a \in A)((s_1, \ldots, s_a) \in M \iff \mathfrak{A} \models \Phi(s_1, \ldots, s_a, t_1, \ldots, t_b)).$$

In the rest of the paper we are going to prove the following two theorems.

2.7. Theorem. *Let $M \subseteq A^a$ and $1 \leq k < n$. The set M is Σ_k^0 - admissible in all Σ_n^0 enumerations of $\mathfrak{A}$ if and only if M is definable by some Σ_k^0 quantifier-free formula on $\mathfrak{A}$.*

2.8. Theorem. *The set M is Σ_n^0 - admissible in all Σ_n^0 enumerations of $\mathfrak{A}$ if and only if M is definable by some Σ_n^0 existential formula on $\mathfrak{A}$.*

3. Generic enumerations

The proofs of Theorem 2.7 and Theorem 2.8 use a forcing construction. In this section we shall describe the fundamentals of this construction.

3.1. Satisfaction relation.

To simplify the notations we shall consider only the subsets of the domain of the structure $\mathfrak{A}$. All results can be easily proved for subsets of A^a, $a > 1$.

Let $\langle f, \mathfrak{B}_f \rangle$ be a partial enumeration of the structure $\mathfrak{A} = (A; R_1, R_2, \ldots, R_l)$. And suppose that $\mathfrak{B}_f = (N; \sigma_1, \sigma_2, \ldots, \sigma_l)$. We shall identify the diagram $D(\mathfrak{B}_f)$ of $\mathfrak{B}_f$ with the set consisting of the codes of the atomic and the negated atomic formulas which are true on $\mathfrak{B}_f$. In other words, we shall assume that

$$D(\mathfrak{B}_f) = \{\langle i, x_1, \ldots, x_{a_i}, \varepsilon \rangle : \sigma_i(x_1, \ldots, x_{a_i}) = \varepsilon,\ i \in [1, l]\}.$$

If $u \in N$ then define

$$f \models\ u \iff u \in D(\mathfrak{B}_f).$$

If E is a finite subset of N then

$$f \models E \iff f \models u \text{ for each } u \in E.$$

Assume also fixed an effective coding of all finite sets of natural numbers. By E_v we shall denote the finite set with the code v.

Let us fix for every $n \geq 1$ and each $e \in N$ a unary predicate letter F^n_e. We adopt the notation $\neg^i F^n_e(x) = F^n_e(x)$ if $i = 0$ and $\neg^i F^n_e(x) = \neg F^n_e(x)$ if $i = 1$. We shall assume that the code of $\neg^i F^n_e(x)$ is $\langle i, n, e, x \rangle$.

For each $x \in N$ and every predicate letter F^n_e the satisfaction relation $f \models \neg^i F^n_e(x)$ is defined by induction on n. Given a finite set E of natural numbers and $n \geq 1$, by $f \models_n E$ we shall denote that every element u of E is of the form $\langle i, n, e, x \rangle$ and $f \models \neg^i F^n_e(x)$.

3.2. Definition.

(i) $f \models F^1_e(x) \iff \exists v(\langle v, x \rangle \in W_e \ \&\ f \models E_v)$,
$f \models \neg F^1_e(x) \iff f \not\models F^1_e(x)$.

(ii) $f \models F^{n+1}_e(x) \iff \exists v(\langle v, x \rangle \in W_e \ \&\ f \models_n E_v)$,
$f \models \neg F^{n+1}_e(x) \iff f \not\models F^{n+1}_e(x)$.

3.3. Proposition.

(1) *The sets $\{x : f \models F^n_e(x)\}$ coincide with the Σ^0_n in $D(\mathfrak{B}_f)$sets.*
(2) *The sets $\{x : f \models \neg F^n_e(x)\}$ coincide with the Π^0_n in $D(\mathfrak{B}_f)$sets.*

Proof. The proof is by induction on n.

For $n = 1$ note that from the definition of "$\models$" we have

$$f \models F^1_e(x) \iff x \in \Gamma_e(D(\mathfrak{B}_f)),$$

where Γ_e is the e-th enumeration operator, see [4].

Since $N \setminus D(\mathfrak{B}_f)$ is enumeration reducible to $D(\mathfrak{B}_f)$, the r.e. in $D(\mathfrak{B}_f)$ sets coincide with the sets which are enumeration reducible to $D(\mathfrak{B}_f)$.

The step from n to $n+1$ follows easily by the Strong hierarchy theorem, see [4]. $\square$

3.4. Corollary. *A set $M \subseteq A$ is Σ^0_n-admissible in $\langle f, \mathfrak{B}_f\rangle$ iff there exists an $e \in N$ such that for all $x \in dom(f)$,*

$$f \models F^n_e(x) \iff f(x) \in M.$$

3.5. Finite parts and forcing.

The conditions of the forcing are finite mappings of N into A with some additional properties which we call *finite parts*. We use δ, τ, ρ to denote finite parts.

Let $[0, q]$ be an initial segment of N.

3.6. Definition. A *finite part* δ on $[0, q]$ is an ordered triple $\langle \alpha_\delta, H_\delta, D_\delta\rangle$ with the following properties:

(1) α_δ is a partial mapping of $[0, q]$ into A;
(2) $H_\delta \subseteq [0, q]$;
(3) $dom(\alpha_\delta) \cup H_\delta = [0, q]$ and $dom(\alpha_\delta) \cap H_\delta = \emptyset$;
(4) D_δ is the diagram of a finite structure of the same relational type as $\mathfrak{A}$ and domain $[0, q]$, and such that if $x_1, \ldots, x_{a_i} \in dom(\alpha_\delta)$, then

$$\langle i, x_1, \ldots, x_{a_i}, \varepsilon\rangle \in D_\delta \iff R_i(\alpha_\delta(x_1), \ldots, \alpha_\delta(x_{a_i})) = \varepsilon.$$

Let Δ be the set of all finite parts.

3.7. Definition. Given finite parts δ and τ, let
$\delta \subseteq \tau \iff \alpha_\delta \subseteq \alpha_\tau \ \&\ H_\delta \subseteq H_\tau \ \&\ D_\delta \subseteq D_\tau$.
If $\langle f, \mathfrak{B}_f\rangle$ is an enumeration, then let
$\delta \subseteq \langle f, \mathfrak{B}_f\rangle \iff \alpha_\delta \subseteq f \ \&\ H_\delta \cap dom(f) = \emptyset \ \&\ D_\delta \subseteq D(\mathfrak{B}_f)$.

Let $\delta \in \Delta$.
If $u \in N$ then $\delta \Vdash u$ iff $u \in D_\delta$.
If $E = \{u_1, \ldots, u_r\}$ is a finite subset of N, then let

$$\delta \Vdash E \iff \delta \Vdash u_1 \ \&\ \ldots\ \&\ \delta \Vdash u_r.$$

Now we are ready to define the forcing relation $\delta \Vdash F^n_e(x)$ for all $e, x \in N$ by induction on $n \geq 1$. As before we shall denote by $\delta \Vdash_n E$ that every element u of the finite set E is in the form $\langle i, n, e, x\rangle$ and $\delta \Vdash \neg^i F^n_e(x)$.

3.8. Definition.

(i) $\delta \Vdash F^1_e(x) \iff \exists v(\langle v, x\rangle \in W_e \ \&\ \delta \Vdash E_v)$;
$\delta \Vdash \neg F^1_e(x) \iff \forall \rho(\rho \supseteq \delta \Longrightarrow \rho \nVdash F^1_e(x))$.
(ii) $\delta \Vdash F^{n+1}_e(x) \iff \exists v(\langle v, x\rangle \in W_e \ \&\ \delta \Vdash_n E_v)$;
$\delta \Vdash \neg F^{n+1}_e(x) \iff \forall \rho(\rho \supseteq \delta \Longrightarrow \rho \nVdash F^{n+1}_e(x))$.

From the definition above it follows immediately the monotonicity of the forcing, i.e. if $\delta \Vdash F^n_e(x)$ and $\delta \subseteq \tau$, then $\tau \Vdash F^n_e(x)$.

3.9. Definition. Let $Y \subseteq \Delta$. The enumeration $\langle f, \mathfrak{B}_f \rangle$ *meets* Y if for some $\delta \in Y$, $\delta \subseteq f$.

3.10. Definition. A subset $Y \subseteq \Delta$ is *dense in the enumeration* $\langle f, \mathfrak{B}_f \rangle$ if

$$(\forall \delta \subseteq f)(\exists \tau \in Y)(\delta \subseteq \tau).$$

3.11. Definition. Let $\mathcal{F}$ be a family of subsets of Δ. An enumeration $\langle f, \mathfrak{B}_f \rangle$ is $\mathcal{F}$-*generic* if whenever $Y \in \mathcal{F}$ and Y is dense in $\langle f, \mathfrak{B}_f \rangle$, then $\langle f, \mathfrak{B}_f \rangle$ meets Y.

As usual we have that for every countable family $\mathcal{F}$ of subsets of Δ and every $\delta \in \Delta$ there exists an $\mathcal{F}$-generic enumeration $\langle f, \mathfrak{B}_f \rangle$ such that $f \supseteq \delta$.

Let $\mathcal{F}_0 = \{\emptyset\}$. For $n \geq 1$ set $Y^n_{e,x} = \{\tau : \tau \Vdash F^n_e(x)\}$ and let $\mathcal{F}_n = (\bigcup_{e,x} Y^n_{e,x}) \cup \mathcal{F}_{n-1}$. The following Truth lemma can be proved by induction on n.

3.12. Lemma. *Let $\langle f, \mathfrak{B}_f \rangle$ be an enumeration, $n \geq 0$. Then for all $e, x \in N$*

(1) *If $\langle f, \mathfrak{B}_f \rangle$ is $\mathcal{F}_n$-generic, then*

$$f \models F^{n+1}_e(x) \iff (\exists \delta \subseteq f)(\delta \Vdash F^{n+1}_e(x)).$$

(2) *If $\langle f, \mathfrak{B}_f \rangle$ is $\mathcal{F}_{n+1}$-generic, then*

$$f \models \neg F^{n+1}_e(x) \iff (\exists \delta \subseteq f)(\delta \Vdash \neg F^{n+1}_e(x)).$$

3.13. Definition. Let $\delta \subseteq \tau$. Then τ/δ is the finite part $\langle \alpha_\delta, H_\tau \cup (dom(\alpha_\tau) \setminus dom(\alpha_\delta)), D_\tau \rangle$.

By $\delta \preceq \tau$ we shall denote that $dom(\alpha_\delta) = dom(\alpha_\tau)$ and $\delta \subseteq \tau$.

3.14. Lemma.

(1) *If $\delta \subseteq \tau$, then $\delta \preceq \tau/\delta$;*
(2) *If $\delta \subseteq \tau_1 \subseteq \tau_2$, then $\tau_1/\delta \preceq \tau_2/\delta$;*
(3) *If $\delta \subseteq \tau$ and $\tau/\delta \preceq \rho$, then there exists a finite part ρ' such that $\tau \preceq \rho'$ and $\rho'/\delta = \rho$.*

Proof. (3) Let $\delta \subseteq \tau$ and $\tau/\delta \preceq \rho$. Then $\tau/\delta = \langle \alpha_\delta, H_\tau \cup (dom(\alpha_\tau) \setminus dom(\alpha_\delta)), D_\tau \rangle$. $\tau/\delta \preceq \rho$ implies $\rho = \langle \alpha_\delta, H_\tau \cup (dom(\alpha_\tau) \setminus dom(\alpha_\delta)) \cup H', D_\rho \rangle$, where $D_\tau \subseteq D_\rho$ and $H' \cap (dom(\alpha_\tau) \cup H_\tau) = \emptyset$.

Let $\rho' = \langle \alpha_\tau, H_\tau \cup H', D_\rho \rangle$. Then $\tau \preceq \rho'$ and $\rho'/\delta = \langle \alpha_\delta, H_\tau \cup (dom(\alpha_\tau) \setminus dom(\alpha_\delta)) \cup H', D_\rho \rangle = \rho$. $\square$

3.15. Stared forcing. We define a stared forcing relation $\delta \Vdash^* F^n_e(x)$ for all $n \geq 1$, $e, x \in N$ by means of the following inductive definition:

3.16. Definition.

(i) $\delta \Vdash^* F^1_e(x) \iff \delta \Vdash F^1_e(x)$;
$\delta \Vdash^* \neg F^1_e(x) \iff \forall \rho(\rho \succeq \delta \Longrightarrow \rho \not\Vdash^* F^1_e(x))$.
(ii) $\delta \Vdash^* F^{n+1}_e(x) \iff \exists v(\langle v, x \rangle \in W_e \ \& \ \delta \Vdash^*_n E_v)$;
$\delta \Vdash^* \neg F^{n+1}_e(x) \iff \forall \rho(\rho \succeq \delta \Longrightarrow \rho \not\Vdash^* F^{n+1}_e(x))$.

Here $\delta \Vdash_n^* E_v$ means as before that every element of E_v is in the form $\langle i, n, e, x\rangle$ and $\delta \Vdash^* \neg^i F_e^n(x)$.

From the definition above it follows immediately that the stared forcing is monotone with respect to "$\preceq$", i.e. $\delta \Vdash^* F_e^n(x)$ & $\delta \preceq \tau \Longrightarrow \tau \Vdash^* F_e^n(x)$.

3.17. Lemma. *Let $\delta \subseteq \tau$. Then for all $e, x \in N$, $n \geq 1$*

(1) $\tau \Vdash F_e^n(x) \iff \tau/\delta \Vdash^* F_e^n(x)$;
(2) $\tau \Vdash \neg F_e^n(x) \iff \tau/\delta \Vdash^* \neg F_e^n(x)$.

Proof. The proof is by induction on n.

Since $D_\tau = D_{\tau/\delta}$, (1) holds for $n = 1$.

Suppose now that (1) is true for some $n \geq 1$.

(2) ($\Rightarrow$). Let $\tau \Vdash \neg F_e^n(x)$. Assume that $\tau/\delta \nVdash^* \neg F_e^n(x)$. Then there is a finite part $\rho \succeq \tau/\delta$ such that $\rho \Vdash^* F_e^n(x)$. By Lemma 3.14 there exists a finite part ρ' such that $\rho' \succeq \tau$ and $\rho'/\delta = \rho$. Then $\rho'/\delta \Vdash^* F_e^n(x)$ and by induction $\rho' \Vdash F_e^n(x)$. Clearly $\rho' \supseteq \tau$. A contradiction.

(2) ($\Leftarrow$). Let $\tau/\delta \Vdash^* \neg F_e^n(x)$. Assume that $\tau \nVdash \neg F_e^n(x)$. Then there exists $\rho \supseteq \tau$ such that $\rho \Vdash F_e^n(x)$. By induction $\rho/\delta \Vdash^* F_e^n(x)$. By Lemma 3.14 $\rho/\delta \succeq \tau/\delta$. A contradiction.

Now, using the respective definitions we get immediately that

$$\tau \Vdash F_e^{n+1}(x) \iff \tau \Vdash^* F_e^{n+1}(x). \quad \square$$

3.18. Lemma. *Let δ be a finite part, $n \geq 1$, $e, x \in N$. Then*

(1) $\delta \Vdash F_e^n(x) \iff \delta \Vdash^* F_e^n(x)$;
(2) $(\exists \tau \supseteq \delta)(\tau \Vdash F_e^n(x)) \iff (\exists \rho \succeq \delta)(\rho \Vdash^* F_e^n(x))$.

Proof. Since $\delta/\delta = \delta$, (1) follows from the previous lemma. By the same argument $\delta \Vdash \neg F_e^n(x) \iff \delta \Vdash^* \neg F_e^n(x)$. From here (2) follows by contraposition. $\square$

4. Regular enumerations

Given a finite part δ defined on $[0, q]$, we shall call q the length of δ and denote it by $|\delta|$. If $p \leq q$ then by $\delta\lceil p$ we shall denote the restriction of δ on $[0, p]$, i.e. $\delta\lceil p = \langle \alpha_\delta\lceil[0,p], H_\delta\lceil[0,p], D_\delta\lceil[0,p]\rangle$. Clearly $\delta\lceil p$ is a finite part and $\delta\lceil p \subseteq \delta$.

Given finite parts τ_1 and τ_2, say that τ_1 is *shorter* than τ_2 if

(a) $|\tau_1| < |\tau_2|$ or
(b) $|\tau_1| = |\tau_2|$ and the code of the finite set D_{τ_1} is less than the code of D_{τ_2}.

Notice that "being shorter than" is a recursive relation and for every finite part δ it is a well ordering on the set $\{\tau | \delta \preceq \tau\}$.

Let $\mathcal{F}_n^*$ be the sequence $\{X_0^n, X_1^n, \dots, X_i^n, \dots\}$ of sets of finite parts, where $X_i^0 = \emptyset$ and $X_i^n = \{\tau : \tau \Vdash^* F_{(i)_0}^n((i)_1)\}$ for $n \geq 1$.

The finite part τ *decides* X_i^n if $\tau \in X_i^n$ or $(\forall \rho \succeq \tau)(\rho \notin X_i^n)$. Clearly for every δ and i, there exists a $\tau \succeq \delta$ such that τ decides X_i^n. By Lemma 3.18, if τ decides X_i^n and $\tau \subseteq \rho$, then ρ also decides X_i^n.

Let

$$\mu_n(i,\delta) = \begin{cases} \delta & \text{if } (\forall \tau \succeq \delta)(\tau \notin X_i^n), \\ (\textit{the shortest } \tau)(\delta \preceq \tau \ \& \ \tau \in X_i^n) & \text{otherwise.} \end{cases}$$

Clearly, $\mu_n(i,\delta)$ decides X_i^n. Notice also that the length of $\mu_n(i,\delta)$ depends only on the length $|\delta|$ of δ and on its diagram D_δ. Moreover, there exists a recursive in $\emptyset^{(n)}$ function λ_n such that

$$\forall i \forall \delta (\lambda_n(i, |\delta|, D_\delta) = |\mu_n(i,\delta)|).$$

4.1. Definition. Let δ be a finite part on $[0,q]$. Then δ is *n-regular* if $0 \in dom(\alpha_\delta)$ and if $q_0 < q_1 < \cdots < q_r$ are the elements of $dom(\alpha_\delta)$, then

(a) $(\forall i < r)(\delta\lceil(q_{i+1}-1) = \mu_n(i, \delta\lceil q_i))$.
(b) $\delta = \mu_n(r, \delta\lceil q_r)$.

We shall denote the number r from the definition above by $\|\delta\|$.

4.2. Lemma. *Let δ be an n-regular finite part, where $dom(\alpha_\delta) = \{q_0 < q_1 < \cdots < q_r\}$. Then for each $i < r$, $\delta\lceil(q_{i+1}-1)$ is n-regular.*

4.3. Definition. An enumeration $\langle f, \mathfrak{B}_f\rangle$ of $\mathfrak{A}$ is called *n-regular* if for each finite part $\delta \subseteq f$ there exists an n-regular finite part τ such that $\delta \subseteq \tau \subseteq f$.

4.4. Lemma. *Let $\langle f, \mathfrak{B}_f\rangle$ be an n-regular enumeration of $\mathfrak{A}$. Then for each natural number r there exists an n-regular finite part $\delta \subseteq f$ such that $\|\delta\| = r$.*

Proof. Given an r, consider the first $r+1$ elements $q_0 < q_1 < \cdots < q_r$ of $dom(f)$. Let δ be the shortest n-regular finite part such that $\{q_0, \ldots, q_r\} \subseteq dom(\alpha_\delta)$ and $\delta \subseteq f$. Assume that $\|\delta\| > r$. Then there exists an element q_{r+1} of $dom(\alpha_\delta)$ such that $q_r < q_{r+1}$. By Lemma 4.2 $\delta\lceil(q_{r+1}-1)$ is n-regular. Clearly $\delta\lceil(q_{r+1}-1)$ is shorter than δ and $\{q_0,\ldots,q_r\} \subseteq dom(\alpha_{\delta\lceil(q_{r+1}-1)})$. The last contradicts the choice of δ. $\square$

Recall the family $\mathcal{F}_n$. Notice that by Lemma 3.18, $\mathcal{F}_n = \mathcal{F}_n^*$.

4.5. Proposition. *Let $\langle f, \mathfrak{B}_f\rangle$ be an n-regular enumeration of $\mathfrak{A}$. Then $\langle f, \mathfrak{B}_f\rangle$ is $\mathcal{F}_n$-generic.*

Proof. Skipping the trivial case $n = 0$, suppose that $n \geq 1$. We shall show that $\langle f, \mathfrak{B}_f\rangle$ is generic with respect to the family $\mathcal{F}_n^*$. Suppose that X_i^n is dense in $\langle f, \mathfrak{B}_f\rangle$. We have to prove that $\langle f, \mathfrak{B}_f\rangle$ meets X_i^n, i.e. there is a $\delta \subseteq f$ such that $\delta \in X_i^n$. By the previous Lemma there exists an n-regular $\delta \subseteq f$, such that $\|\delta\| = i$. Clearly δ decides X_i^n. Assume that $\delta \notin X_i^n$. Then $\delta \Vdash^* \neg F^n_{(i)_0}((i)_1)$ and hence, by Lemma 3.17, $\delta \Vdash \neg F^n_{(i)_0}((i)_1)$. The last contradicts the density of X_i^n. $\square$

4.6. Proposition. *Let $\langle f, \mathfrak{B}_f\rangle$ be an n-regular enumeration of $\mathfrak{A}$. Then $dom(f)$ is Δ^0_{n+1} relative to $D(\mathfrak{B}_f)$.*

Proof. We have the following recursive in $D(\mathfrak{B}_f) \oplus \emptyset^{(n)}$ procedure, which lists the elements of $dom(f)$ in an increasing order.

We start by printing out 0. Suppose that the first $r+1$ elements $q_0, \dots, q_r$ of $dom(f)$ are listed. Consider the finite part $\delta_r \subseteq f$ on $[0, q_r]$. Using the oracle $D(\mathfrak{B}_f)$ we can obtain the diagram D_{δ_r}. Let q_{r+1} be the first element of $dom(f)$ greater than q_r. Clearly there exists an n-regular finite part τ such that $\delta_r \subseteq \tau$ and $q_{r+1} \in dom(\alpha_\tau)$. By Definition 4.1, $q_{r+1} = \lambda_n(r, q_r, D_{\delta_r}) + 1$. $\square$

5. The normal form theorems

In this section we shall obtain a normal form of the Σ^0_k-admissible in all Σ^0_n enumerations of $\mathfrak{A}$ sets, for $k \leq n$. We start with the case $k = n$.

Let δ be a finite part, $x = |\delta| + 1$ and $s \in A$. By $\delta * s$ we shall denote the finite part $\langle \alpha', H_\delta, D\rangle$, where $dom(\alpha') = dom(\alpha_\delta) \cup \{x\}$, $\alpha_\delta \subseteq \alpha'$, $\alpha'(x) \simeq s$, and D is the appropriate extension of the diagram D_δ.

5.1. Theorem. *Let $M \subseteq A$, $n \geq 1$ and M be a Σ^0_n-admissible in all Σ^0_n enumerations of $\mathfrak{A}$ set. Then there exists a finite part δ and a natural number e such that for each $s \in A$ if $x = |\delta| + 1$, then*

$$s \in M \iff (\exists \tau \supseteq \delta * s)(\tau \text{ is } (n-1)\text{-regular } \ \& \ \tau \Vdash^* F^n_e(x)). \tag{5.1}$$

Proof. Assume the opposite. We shall construct an $(n-1)$-regular enumeration $\langle f, \mathfrak{B}_f\rangle$ of $\mathfrak{A}$ such that M is not admissible in it.

The construction of $\langle f, \mathfrak{B}_f\rangle$ will be carried out by steps. On each step j we shall define $(n-1)$-regular finite part δ_j so that $\delta_j \subseteq \delta_{j+1}$ and take $f = \bigcup \alpha_{\delta_j}$ and $\mathfrak{B}_f$ to be the structure with diagram $\bigcup D_{\delta_j}$.

On the even steps we shall ensure that f is onto A. On the odd steps we shall ensure that M is not admissible in $\langle f, \mathfrak{B}_f\rangle$.

Let $t_0, t_1, \dots, t_i, \dots$ be a fixed enumeration of the elements of A.

Let δ_0 be the shortest $(n-1)$-regular finite part such that $\alpha_{\delta_0}(0) = t_0$.

Step $j = 2e+1$. Let $x = |\delta_{2e}| + 1$. By the assumption there exists $s \in A$ such that

$$\neg[s \in M \iff (\exists \tau \supseteq \delta_{2e} * s)(\tau \text{ is } (n-1)\text{-regular } \ \& \ \tau \Vdash^* F^n_e(x))].$$

We have two possibilities.

Case (i). $s \in M$ and $(\forall \tau \supseteq \delta_{2e} * s)(\tau \text{ is } (n-1)\text{-regular} \Rightarrow \tau \not\Vdash^* F^n_e(x))$. In this case let δ_{2e+1} be the shortest $(n-1)$-regular finite part τ such that $\tau \supseteq \delta_{2e} * s$.

Case (ii). $s \notin M$ and $(\exists \tau \supseteq \delta_{2e} * s)(\tau \text{ is } (n-1)\text{-regular and } \tau \Vdash^* F^n_e(x))$. In this case let δ_{2e+1} be the shortest such τ.

Step $j = 2e+2$. Let t be the first $t_i \in A$ such that $t \notin range(\alpha_{\delta_{2e+1}})$. Let δ_{2e+2} be the shortest $(n-1)$-regular finite part τ such that $\tau \supseteq \delta_{2e+1} * t$.

Clearly the enumeration $\langle f, \mathfrak{B}_f\rangle$ is $(n-1)$-regular and hence $dom(f)$ is Σ^0_n relative to $D(\mathfrak{B}_f)$ and $\langle f, \mathfrak{B}_f\rangle$ is $\mathcal{F}_{n-1}$-generic.

Towards a contradiction assume that M is Σ^0_n-admissible in $\langle f, \mathfrak{B}_f\rangle$. Then there exists an $e \in N$ such that for all $x \in dom(f)$

$$f(x) \in M \iff f \models F^n_e(x).$$

Consider the stage $j = 2e+1$ of the construction. Let $x = |\delta_{2e}| + 1$. Using the Truth lemma (Lemma 3.12), we get that

$$f(x) \in M \iff (\exists \tau)(\delta_{2e+1} \subseteq \tau \subseteq f \ \&\ \tau \Vdash F^n_e(x)).$$

On the other hand, according to our construction this is not the case. So, M is not Σ^0_n-admissible in $\langle f, \mathfrak{B}_f\rangle$. $\square$

5.2. Theorem. *Let $k < n$, $M \subseteq A$ and let M be Σ^0_k-admissible in all Σ^0_n enumerations of $\mathfrak{A}$. Then there exists a finite part δ and a natural number e such that for each $s \in A$ if $x = |\delta| + 1$, then*

$$s \in M \iff (\exists \tau \succeq \delta * s)(\tau \Vdash^* F^k_e(x)). \tag{5.2}$$

Proof. Assume the contrary. We shall construct an enumeration $\langle f, \mathfrak{B}_f\rangle$ of $\mathfrak{A}$ with the following properties:

(1) $\langle f, \mathfrak{B}_f\rangle$ is $\mathcal{F}_{n-1}$-generic;
(2) $dom(f)$ is Σ^0_n relative to $D(\mathfrak{B}_f)$;
(3) the set M is not Σ^0_k-admissible in $\langle f, \mathfrak{B}_f\rangle$.

The construction of the enumeration $\langle f, \mathfrak{B}_f\rangle$ is very similar to that used in the proof of the previous theorem. Again it will be carried out by steps. On steps $j = 3e+1$ we shall satisfy that $\langle f, \mathfrak{B}_f\rangle$ is a $\mathcal{F}_{n-1}$-generic enumeration. On steps $j = 3e+2$ – that M is not Σ^0_k-admissible in $\langle f, \mathfrak{B}_f\rangle$. And on steps $j = 3e+3$ we shall ensure that f is a mapping onto A.

Let $t_0, t_1, \dots, t_i, \dots$ be a fixed enumeration of the elements of A and let δ_0 be the shortest $(n-1)$-regular finite part such that $\alpha_{\delta_0}(0) = t_0$.

Step $j = 3e+1$. Let $\delta_{3e+1} = \mu_{n-1}(e, \delta_{3e})$.

Step $j = 3e+2$. Let $x = |\delta_{3e+1}| + 1$. According to the assumption there exists a $s \in A$ such that

$$\neg[s \in M \iff (\exists \tau \succeq \delta_{3e+1} * s)(\tau \Vdash^* F^k_e(x))].$$

Case (i). $s \in M$ and $(\forall \tau \succeq \delta_{3e+1} * s)(\tau \nVdash^* F^k_e(x))$.

Put $\delta_{3e+2} = \delta_{3e+1} * s$.

Case (ii). $s \notin M$ and $(\exists \tau \succeq \delta_{3e+1} * s)(\tau \Vdash^* F^k_e(x))$.

In this case let δ_{3e+2} be the shortest such τ.

Step $j = 3e+3$. Find the first $t \in A$ such that $t \notin range(\alpha_{\delta_{3e+2}})$. Let $\delta_{3e+3} = \delta_{3e+2} * t$.

The enumeration $\langle f, \mathfrak{B}_f\rangle$ is constructed as in Theorem 5.1, i.e. $f = \bigcup \alpha_{\delta_j}$ and $D(\mathfrak{B}_f) = \bigcup D_{\delta_j}$.

Arguments very similar to those used in the previous section show that $\langle f, \mathfrak{B}_f\rangle$ is $\mathcal{F}_{n-1}$-generic and $dom(f)$ is Δ^0_n in $D(\mathfrak{B}_f)$.

Assume that M is Σ_k^0-admissible in $\langle f, \mathfrak{B}_f\rangle$. Then there is an $e \in N$ such that for all $x \in dom(f)$

$$f \models F_e^k(x) \iff f(x) \in M.$$

Consider the stage $j = 3e+2$ of our construction and let $x = |\delta_{3e+1}| + 1$. There exists $s \in A$ such that

Case (i). $s \in M$ and $(\forall \tau \succeq \delta_{3e+1} * s)(\tau \not\Vdash^* F_e^k(x))$.

Since $\delta_{3e+2} \subseteq f$, $f(x) \in M$. Then $f \models F_e^k(x)$. Clearly $\langle f, \mathfrak{B}_f\rangle$ is $\mathfrak{F}_{k-1}$-generic. By Lemma 3.12 and Lemma 3.18 there exists a finite part τ such that $\delta_{3e+1} * s \preceq \tau$ & $\tau \Vdash^* F_e^k(x)$. A contradiction.

Case (ii). $s \notin M$ and $(\exists \tau \succeq \delta_{3e+1} * s)(\tau \Vdash^* F_e^k(x))$. Since $\delta_{3e+2} \subseteq f$, $f(x) = s$. Using again Lemma 3.12 and Lemma 3.18, we get $f \models F_e^k(x)$. A contradiction. □

6. The proofs of Theorem 2.7 and Theorem 2.8

In this section we shall prove Theorem 2.7 and Theorem 2.8.

If a subset M of A is definable by a Σ_k^0 quantifier-free formula on $\mathfrak{A}$ then it is clear that M is Σ_k^0-admissible in all enumerations of $\mathfrak{A}$. It is easy to verify also that if a set M is definable by a Σ_n^0 existential formula on $\mathfrak{A}$ then M is Σ_n^0-admissible in all Σ_n^0 enumerations of $\mathfrak{A}$.

The proofs of both theorems in the non-trivial directions make use of the respective normal form theorems.

Suppose that the first order language $\mathfrak{L}$ consists of the predicate letters $\{P_1, \ldots, P_l\}$ and let *var* be a recursive one to one mapping of the natural numbers onto the set of all variables.

6.1. Lemma. *Let K, H, D be finite sets and $K = \{z_1, \ldots, z_r\}$. Let $Z_1 = var(z_1)$, ..., $Z_r = var(z_r)$. There exists a uniform effective way to define a Δ_1^0 quantifier-free formula $\Pi_{K,H,D}(Z_1, \ldots, Z_r)$ such that for all $t_1, \ldots, t_r \in A$,*

$$\mathfrak{A} \models \Pi_{K,H,D}(Z_1/t_1, \ldots, Z_r/t_r) \iff \exists \delta(dom(\alpha_\delta) = K \;\&\; H_\delta = H \;\&\; D_\delta = D \;\&\; \alpha_\delta(z_i) \simeq t_i).$$

Proof. If $K \cap H \neq \emptyset$ or $K \cup H$ is not an initial segment $[0, q]$ or D is not a diagram of a finite structure of the language $\mathfrak{L}$ with domain $K \cup H$, then set $\Pi_{K,H,D} = \mathbb{F}$. Otherwise, let $\{u_1, \ldots, u_v\}$ be all elements of D such that if $u_j = \langle i, x_1, \ldots, x_{a_i}, \varepsilon\rangle$, $i \in [1, l]$, then $\{x_1, \ldots, x_{a_i}\} \subseteq K$. For every such u_j let $L_j = \neg^\varepsilon P_i(var(x_1), \ldots, var(x_{a_i}))$ and define $\Pi_{K,H,D} = L_1 \& \ldots \& L_v$. □

6.2. Corollary. *There exists a uniform effective way given finite sets K, H, D and E to define a Δ_1^0 quantifier-free formula $\Pi_{K,H,D,E}$ with free variables among $\{var(z) : z \in K\}$ such that if $K = \{z_1, \ldots, z_r\}$ and $var(z_i) = Z_i$, then for all $t_1, \ldots, t_r \in A$,*

$$\mathfrak{A} \models \Pi_{K,H,D,E}(Z_1/t_1, \ldots, Z_r/t_r) \iff \exists \delta(dom(\alpha_\delta) = K \;\&\; H_\delta = H \;\&\; D_\delta = D \;\&\; (\forall i \in [1, r])(\alpha_\delta(z_i) \simeq t_i) \;\&\; \delta \Vdash^* E).$$

Proof. Set $\Pi_{K,H,D,E} = \mathbb{F}$ if $E \not\subseteq D$ and let $\Pi_{K,H,D,E} = \Pi_{K,H,D}$ otherwise. □

6.3. Lemma. *Let $k \geq 0$, $\delta = \langle \alpha_\delta, H_\delta, D_\delta \rangle$ be a finite part, $dom(\alpha_\delta) = \{z_1, \dots, z_r\}$ and $\alpha_\delta(z_1) \simeq t_1, \dots, \alpha_\delta(z_r) \simeq t_r$. Suppose that $var(z_i) = Z_i$. Then there exists a uniform in $dom(\alpha_\delta), H_\delta, D_\delta$ effective way, given natural numbers e, x and finite set E of natural numbers, to define:*

(1) *A Δ^0_{k+1} quantifier-free formula $\Gamma^k_{dom(\alpha_\delta),H_\delta,D_\delta,E}(Z_1, \dots, Z_r)$ such that*

$$\mathfrak{A} \models \Gamma^k_{dom(\alpha_\delta),H_\delta,D_\delta,E}(Z_1/t_1, \dots, Z_r/t_r) \iff \delta \Vdash^*_k E;$$

(2) *A Σ^0_{k+1} quantifier-free formula $\Theta^{k+1}_{dom(\alpha_\delta),H_\delta,D_\delta,e,x}(Z_1, \dots, Z_r)$ such that*

$$\mathfrak{A} \models \Theta^{k+1}_{dom(\alpha_\delta),H_\delta,D_\delta,e,x}(Z_1/t_1, \dots, Z_r/t_r) \iff \delta \Vdash^* F^{k+1}_e(x);$$

(3) *A Σ^0_{k+1} quantifier-free formula $\Psi^{k+1}_{dom(\alpha_\delta),H_\delta,D_\delta,e,x}(Z_1, \dots, Z_r)$ such that*

$$\mathfrak{A} \models \Psi^{k+1}_{dom(\alpha_\delta),H_\delta,D_\delta,e,x}(Z_1/t_1, \dots, Z_r/t_r) \iff (\exists \tau \succeq \delta)(\tau \Vdash^* F^{k+1}_e(x));$$

(4) *A Π^0_{k+1} quantifier-free formula $\Phi^{k+1}_{dom(\alpha_\delta),H_\delta,D_\delta,e,x}(Z_1, \dots, Z_r)$ such that*

$$\mathfrak{A} \models \Phi^{k+1}_{dom(\alpha_\delta),H_\delta,D_\delta,e,x}(Z_1/t_1, \dots, Z_r/t_r) \iff \delta \Vdash^* \neg F^{k+1}_e(x);$$

Proof. Induction on k. Using Corollary 6.2, we shall suppose that (1) is true for k and proceed to prove (2), (3) and (4). After that we shall show the validity of (1) for $k+1$. Let $R_{e,x} = \{v : \langle v, x \rangle \in W_e\}$. Following the definition of the stared forcing, we get

$$\Theta^{k+1}_{dom(\alpha_\delta),H_\delta,D_\delta,e,x} = \bigvee_{v \in R_{e,x}} \Gamma^k_{dom(\alpha_\delta),H_\delta,D_\delta,E_v}.$$

$$\Psi^{k+1}_{dom(\alpha_\delta),H_\delta,D_\delta,e,x} = \bigvee_{H \supseteq H_\delta, D \supseteq D_\delta} \Pi_{dom(\alpha_\delta),H,D} \ \& \ \Theta^{k+1}_{dom(\alpha_\delta),H,D,e,x}.$$

$$\Phi^{k+1}_{dom(\alpha_\delta),H_\delta,D_\delta,e,x} = \neg \Psi^{k+1}_{dom(\alpha_\delta),H_\delta,D_\delta,e,x}.$$

So it remains to construct $\Gamma = \Gamma^{k+1}_{dom(\alpha_\delta),H_\delta,D_\delta,E}$. Set $\Gamma = \mathbb{F}$ if not all elements u of E are of the form $\langle i, k+1, e, x \rangle$, $i \in \{0, 1\}$. Otherwise, for every element $u = \langle i, k+1, e, x \rangle$ of E let $L^u = \Theta^{k+1}_{dom(\alpha_\delta),H_\delta,D_\delta,e,x}$ if $i = 0$ and let $L^u = \Phi^{k+1}_{dom(\alpha_\delta),H_\delta,D_\delta,e,x}$ if $i = 1$. Put $\Gamma = \bigwedge_{u \in E} L^u$. $\square$

As a corollary we obtain the proof of Theorem 2.7. Indeed, suppose that $M \subseteq A$, $1 \leq k < n$ and M be Σ^0_k-admissible in all Σ^0_n enumerations. Using Theorem 5.2 we obtain that there exists δ and e such that if $x = |\delta| + 1$, then for all $s \in A$,

$$s \in M \iff (\exists \tau \succeq \delta * s)(\tau \Vdash^* F^k_e(x)).$$

Let $dom(\alpha_\delta) = \{z_1, \dots, z_r\}$, $var(z_i) = Z_i$, $var(x) = X$. Denote by K the finite set $dom(\alpha_\delta) \cup \{x\}$. Put $\Psi = \Psi^k_{K,H_\delta,D_\delta,e,x}$. Clearly the variables of Ψ are among $\{Z_1, \dots, Z_r, X\}$. Let $\alpha_\delta(z_i) \simeq t_i$. Notice that $\alpha_{\delta * s}(x) \simeq s$ for all $s \in A$. Then

$$s \in M \iff \mathfrak{A} \models \Psi(Z_1/t_1, \dots, Z_r/t_r, X/s).$$

Using Lemma 6.3 and the definition of the regular finite parts, one can easily prove the following:

6.4. Lemma. *For every $n \geq 0$ there exists a uniform effective way to construct, given finite sets $K = \{z_1, \dots, z_r\}$, H and D, a finite disjunction $\Omega^n_{K,H,D}$ of Δ^0_{n+1} quantifier-free formulas with variables among $var(z_1), \dots, var(z_r)$ such that if $var(z_i) = Z_i$ and $t_1, \dots, t_r$ are elements of A, then*

$$\mathfrak{A} \models \Omega^n_{K,H,D}(Z_1/t_1, \dots, Z_r/t_r) \iff \exists\delta(\delta \text{ is } n\text{-regular } \& \ dom(\alpha_\delta) = K \ \& \ H_\delta = H \& \\ D_\delta = D \ \& \ (\forall i \in [1, r])(\alpha_\delta(z_i) \simeq t_i)).$$

Now we are ready to prove Theorem 2.8. Let $n \geq 1$, $M \subseteq A$. Suppose that M is Σ^0_n admissible in all Σ^0_n enumerations. By Theorem 5.1 there exist δ and e such that if $x = |\delta| + 1$, then for all $s \in A$,

$$s \in M \iff (\exists\tau \supseteq \delta * s)(\tau \text{ is } (n-1)\text{-regular } \& \ \tau \Vdash^* F^n_e(x)).$$

Let $dom(\alpha_\delta) = \{z_1, \dots, z_r\}$ and $\alpha_\delta(z_i) = t_i$. Let $var(z_i) = Z_i$ and $var(x) = X$. Given any formula Φ and finite set $K = \{y_1 < \cdots < y_q\}$, by $\exists(y \in K)\Phi$ we shall denote the formula $\exists var(y_1) \dots \exists var(y_q)\Phi$. Let $K_\delta = dom(\alpha_\delta) \cup \{x\}$. Define

$$\Phi(Z_1, \dots, Z_r, X) = \bigvee_{K \supseteq K_\delta, H \supseteq H_\delta, D \supseteq D_\delta} \exists(y \in K \setminus K_\delta)(\Omega^{(n-1)}_{K,H,D} \& \ \Theta^n_{K,H,D,e,x}).$$

Clearly Φ is a Σ^0_n existential formula and

$$\mathfrak{A} \models \Phi(Z_1/t_1, \dots, Z_r/t_r, X/s) \iff (\exists\tau \supseteq \delta * s)(\tau \text{ is } (n-1)\text{-regular } \& \ \tau \Vdash^* F^n_e(x)).$$

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Department of Mathematics and Computer Science, Sofia University, Blvd. "James Bourchier" 5, 1126 Sofia, Bulgaria

E-mail address: asoskova@fmi.uni-sofia.bg