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The automorphism group and definability of the jump operator in the ω-enumeration degrees

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Abstract In the present paper, we show the first order definability of the jump operator in the upper semilattice of the ω-enumeration degrees. As a consequence, we derive the isomorphicity of the automorphism groups of the enumeration and the ω-enumeration degrees.

Keywords degree structures · enumeration reducibility · jump · definability

1 Introduction

When the classification of the notion of informational content of given objects is modelled by an algebraic degree structure, the question arises what is the extent of which the structure captures this notion. More particularly, is there a way to differentiate between objects with a different informational content within the algebraic structure itself? Does the informational content of an object uniquely determine its position in the structure, or can we find a reorganization of the structure which does not change any of its properties?

Formally, the above questions are described via the notion of *rigidity*. An algebraic structure is said to be *rigid* if and only if it has no nontrivial automorphisms. Rogers [10] is the first who raised the question of the rigidity of the structure $\mathcal{D}_T$ of the Turing degrees. Although the rigidity problem still remains unsolved, there is a sequence of results which restrict the behavior of the possible automorphisms of $\mathcal{D}_T$.

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Namely, in [14] Slaman and Woodin develop a coding method, which they use for the analysis of the automorphism group $\mathrm{Aut}(\mathcal{D}_T)$ of the Turing degrees. They succeed in showing that there are at most countably many automorphisms of $\mathcal{D}_T$, each of them is arithmetically definable and there is a Turing degree on which the action of any automorphism uniquely determines its global action. Also they prove that each automorphism of $\mathcal{D}_T$ fixes the upper cone above the second jump $\mathbf{0}''_T$ of the degree of the recursive sets. They derive in the same paper strong connections between the Turing degree structure and second-order arithmetic, leading to their famous *Bi-interpretability conjecture*, see Slaman [12] for a precise statement of the original conjecture. As shown in [13, 11], the bi-interpretability of $\mathcal{D}_T$ and second-order arithmetic is equivalent to the rigidity of $\mathcal{D}_T$. Also, if true, it clarifies the situation with the definable relations on $\mathcal{D}_T$: they are exactly those which are definable in second-order arithmetic.

An important consequence of the method of Slaman and Woodin is the definability of the double jump operator. Later, using this result, Shore and Slaman [11] prove the definability of the Turing jump operator itself.

Another model of computation is presented via the *enumeration reducibility* between sets. In contrast to Turing reducibility, where the auxiliary data is accessed via oracles, here the informational inputs are supplied by enumerations. More precisely, the set A is enumeration reducible to the set B if there is an effective way of transforming any enumeration of B into one of A. The world modulo enumeration reducibility is described by the algebraic structure $\mathcal{D}_e$ of the enumeration degrees. Just like $\mathcal{D}_T$, enumeration degrees structure is an upper semi-lattice with least element. Also an appropriate jump operation is defined in $\mathcal{D}_e$, [2, 9].

The Turing model of computation can be easily placed into the wider context of the enumeration reducibility via the so called *total* degrees. An enumeration degree is total if it contains as a representative some set B, for which there exists a set A such that there is an effective transformation between the enumerations of B and the oracle for A. More formally, the total degrees are exactly these which contain representative of the form $A \oplus (\omega \setminus A)$ for some A. The embedding of $\mathcal{D}_T$ into $\mathcal{D}_e$ can be done in a way that to preserve the order, least upper bound operation, and even the jump, simply by mapping $\deg_T(A) \longmapsto \deg_e(A \oplus (\omega \setminus A))$.

Since it is a much richer structure, one should expect that the automorphism group of the enumeration degrees is more restricted than $\mathrm{Aut}(\mathcal{D}_T)$. An interesting problem is if it is possible to restrict each automorphism of $\mathcal{D}_e$ to an automorphism of the substructure $\mathcal{D}_T$, ad vice versa – if each automorphism of $\mathcal{D}_T$ can be extended to such of $\mathcal{D}_e$? Overall, what are the connections between the groups $\mathrm{Aut}(\mathcal{D}_T)$ and $\mathrm{Aut}(\mathcal{D}_e)$?

Recent results by M. Soskova [19] reveal a similar picture in the automorphism group of the enumeration degrees as that in $\mathrm{Aut}(\mathcal{D}_T)$. Namely, we have that $\mathrm{Aut}(\mathcal{D}_e)$ has at most countably many elements, each of them is arithmetically definable and $\mathcal{D}_e$ has a singleton automorphism base. Using the analysis of $\mathrm{Aut}(\mathcal{D}_e)$, in [1] Cai, Ganchev, Lempp, Miller and Soskova find the answer of a long-standing open question: they prove that the total degrees are first-order definable in the structure of the enumeration degrees.

Combining the definability of the total degrees with the well known fact that they form an automorphism base for $\mathcal{D}_e$ leads us to the result that each automorphism of $\mathcal{D}_e$ induces an automorphism of $\mathcal{D}_T$. In other words, $\mathrm{Aut}(\mathcal{D}_e)$ is isomorphic to a subgroup of $\mathrm{Aut}(\mathcal{D}_T)$, and hence the rigidity of the Turing degrees implies that of the enumeration degrees. The converse, i.e. if each automorphism of $\mathcal{D}_T$ is induced, still remains an open problem.

The definability of the automorphism base of the total enumeration degrees also guarantees that each automorphism of $\mathcal{D}_e$ is the identity on the upper cone of the double jump $\mathbf{0}_e''$ of the enumeration degree of the computably enumerable sets, [1].

In this paper, we address the problem of the first-order definability of the jump operator in a proper extension of $\mathcal{D}_e$ – the ω-enumeration degree structure $\mathcal{D}_\omega$. Unlike the well known structures of the Turing and the enumeration degrees, which arise as a formal way to describe the corresponding notions for informational content comparing over the universe 2^ω of the sets of natural numbers, the structure $\mathcal{D}_\omega$ considered here algebraically describes a way of classification over the universe $(2^\omega)^\omega$ of ω-sequences of sets of natural numbers.

The study of this degree structure was initiated by Soskov in [16]. He introduced the ω-enumeration reducibility $\leq_\omega$, which compares the informational content of sequences of sets in a way that generalizes the Selman's characterization for the enumeration reducibility[1]. To be more precise, in Soskov's definition the informational content of each sequence $\mathcal{A} = \{A_k\}_{k<\omega}$ is described by the class $J_\mathcal{A}$ of the Turing degrees of all sets, that can compute in an uniform way, an enumeration of the k-th element of the sequence in their k-th Turing jump:

$$J_\mathcal{A} = \{\deg_T(X) \mid A_k \text{ is c.e. in } X^{(k)} \text{ uniformly in } k\}.$$

Thus $\mathcal{A} \leq_\omega \mathcal{B}$ iff $J_\mathcal{B} \subseteq J_\mathcal{A}$. This reducibility is a preorder on $(2^\omega)^\omega$, and hence it induces a degree structure in the usual way, denoted by $\mathcal{D}_\omega$ – the structure of the ω-enumeration degrees. Also a jump operation $'$ is defined in $\mathcal{D}_\omega$ is such a way that for each degree, the class that describes the jump of a degree consists exactly of the (Turing) jumps of the elements of the class describing the degree.

The mapping $\deg_e(A) \longmapsto \deg_\omega(A, \emptyset, \dots, \emptyset, \dots)$ is a natural embedding of the structure of the enumeration degrees into $\mathcal{D}_\omega$. It preserves the order, the least upper bound and the jump operations.

Both structures are closely related. More precisely, Soskov and Ganchev [17] show that the group $\mathrm{Aut}(\mathcal{D}_e)$ of the automorphisms of $\mathcal{D}_e$ and the group $\mathrm{Aut}(\mathcal{D}_\omega')$ of the jump preserving automorphisms of $\mathcal{D}_\omega$, are isomorphic. Again in [17] it is shown also that $\mathcal{D}_e$ is an automorphism base for $\mathcal{D}_\omega$, which is first order definable in $\mathcal{D}_\omega$ in the language of the structure order and the jump operation.

[1] Selman's Theorem states that $A \leq_e B \iff (\forall X \subseteq \omega)[B \leq_{c.e.} X \to A \leq_{c.e.} X]$.

The main our aim in this paper, is to prove the first-order definability of the ω-enumeration jump operator. Roughly, the proof can be divided into two parts. First, we prove that the (isomorphic copy of the) enumeration degrees structure is first-order definable in $\mathcal{D}_\omega$ in the language of the structure order only, in this way improving the above mentioned result of Soskov and Ganchev. This proof relies on the previously derived fact that the jump of the least element is first-order definable, Ganchev and Sariev [5]. In the second part, we prove the equivalence of the definabilities in $\mathcal{D}_\omega$ of the enumeration degrees and the jump operator.

Note, that the definability of the jump in $\mathcal{D}_\omega$ guarantees that all automorphisms of $\mathcal{D}_\omega$ preserve the jump, and hence that $\mathcal{D}_e$ and $\mathcal{D}_\omega$ have isomorphic automorphism groups. Thus the rigidity of the enumeration degrees is equivalent to the rigidity of the ω-enumeration degrees.

2 Preliminaries

2.1 Basic notions

We shall denote the set of natural numbers by ω. If not stated otherwise, a, b, c, ... shall stand for natural numbers, A, B, C, ... for sets of natural numbers, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, ... for degrees and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ... for sequences of sets of natural numbers. We shall further follow the following convention: whenever a sequence is denoted by a calligraphic Latin letter, then we shall use the roman style of the same Latin letter, indexed with a natural number, say k, to denote the k-th element of the sequence (we always start counting from 0). Thus, if not stated otherwise, $\mathcal{A} = \{A_k\}_{k<\omega}$, $\mathcal{B} = \{B_k\}_{k<\omega}$, $\mathcal{C} = \{C_k\}_{k<\omega}$, etc. We shall denote the set of all sequences (*of length* ω) of sets of natural numbers by $\mathcal{S}_\omega$.

As usual $A \oplus B$ shall stand for the set $\{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$. By A^+ we shall denote the set $A \oplus (\omega \setminus A)$.

For every natural number e and every set $A \subseteq \omega$, we denote by W_e^A the domain of the partial function computed by the oracle Turing machine with index e and using A as an oracle.

2.2 The enumeration degrees

We assume that the reader is familiar with the notion of enumeration reducibility, $\leq_e$, and with the structure of the enumeration degrees (for an introduction on the enumeration reducibility and the respective degree structure we refer the reader to [3,15]).

Intuitively, a set A is enumeration reducible (*e-reducible*) to a set B if there is an effective algorithm transforming each enumeration of B into an enumeration of A. It turns out that $A \leq_e B$ *iff* there is a c.e. set W such that

$$x \in A \iff (\exists u)[\langle x, u\rangle \in W \text{ and } D_u \subseteq B], \tag{1}$$

where $\langle x, u\rangle$ denotes the code of the tuple of natural numbers (x, u) under some fixed encoding and D_u is the finite set with canonical index u. We say that A is e-reducible to B via W, and we shall write $A = W(B)$.

The relation $\leq_e$ is a preorder on the powerset 2^ω of the natural numbers and induces a nontrivial equivalence relation $\equiv_e$. The equivalence classes under $\equiv_e$ are called enumeration degrees. The enumeration degree which contains the set A is denoted by $\deg_e(A)$. The set of all enumeration degrees is denoted by $\mathbf{D}_e$. The enumeration reducibility between sets induces a partial order $\leq_e$ on $\mathbf{D}_e$ by

$$\deg_e(A) \leq_e \deg_e(B) \iff A \leq_e B.$$

We denote by $\mathcal{D}_e$ the partially ordered set $(\mathbf{D}_e, \leq_e)$. The least element of $\mathcal{D}_e$ is the enumeration degree $\mathbf{0}_e$ of $\emptyset$. Also, the enumeration degree of $A \oplus B$ is the least upper bound of the degrees of A and B. Therefore $\mathcal{D}_e$ is an upper semi-lattice with least element.

The enumeration jump $J_e(A)$ of A is defined by $J_e(A) = \{x \mid x \in W_x(A)\}^+$. The jump operation preserves enumeration reducibility, so we can define

$$\deg_e(A)' = \deg_e(J_e(A)).$$

Since $A <_e J_e(A)$, we have $\mathbf{a} <_e \mathbf{a}'$ for every enumeration degree $\mathbf{a}$. The jump operator is uniform, i.e. there exists a recursive function j such that for all sets A and B, if $A = W_e(B)$ then $J_e(A) = W_{j(e)}(J_e(B))$.

The jump operation gives rise to the local substructure $\mathcal{G}_e$, consisting of all degrees bellow $\mathbf{0}'_e$ – the jump of the least enumeration degree. Cooper [3] proved that $\mathcal{G}_e$ is exactly the collection of all Σ^0_2 enumeration degrees.

2.3 The ω-enumeration degrees

ω-enumeration reducibility and the corresponding degree structure $\mathcal{D}_\omega$ were introduced by Soskov in [16]. An equivalent, but more approachable definition in terms of the uniform e-reducibility is derived in [18]. We shall focus our attention only on the latter. According to it, a sequence $\mathcal{A}$ is ω-enumeration reducible to a sequence $\mathcal{B}$, denoted by $\mathcal{A} \leq_\omega \mathcal{B}$, iff for every $n < \omega$,

$$A_n \leq_e P_n(\mathcal{B}) \text{ uniformly in } n.$$

Here, for each $\mathcal{X} \in \mathcal{S}_\omega$, $\mathcal{P}(\mathcal{X})$ is the so called *jump sequence* of $\mathcal{X}$ and is defined as the sequence $\{P_k(\mathcal{X})\}_{k<\omega}$ such that: $P_0(\mathcal{X}) = X_0$ and for each $k < \omega$, $P_{k+1}(\mathcal{X}) = J_e(P_k(\mathcal{X})) \oplus X_{k+1}$.

Clearly $\leq_\omega$ is a reflexive and transitive relation, and the relation $\equiv_\omega$ defined by

$$\mathcal{A} \equiv_\omega \mathcal{B} \iff \mathcal{A} \leq_\omega \mathcal{B} \text{ and } \mathcal{B} \leq_\omega \mathcal{A}$$

is an equivalence relation. The equivalence classes under this relation are called ω-enumeration degrees. In particular the equivalence class $\deg_\omega(\mathcal{A}) = \{\mathcal{B} \mid \mathcal{A} \equiv_\omega \mathcal{B}\}$ is called the ω-enumeration degree of $\mathcal{A}$. The relation $\leq_\omega$ defined by

$$\mathbf{a} \leq_\omega \mathbf{b} \iff \exists \mathcal{A} \in \mathbf{a} \exists \mathcal{B} \in \mathbf{b}(\mathcal{A} \leq_\omega \mathcal{B})$$

is a partial order on the set of all ω-enumeration degrees $\mathbf{D}_\omega$. By $\mathcal{D}_\omega$ we shall denote the structure $(\mathbf{D}_\omega, \leq_\omega)$. The ω-enumeration degree $\mathbf{0}_\omega$ of the sequence $\emptyset_\omega = \{\emptyset\}_{k<\omega}$ is the least element in $\mathcal{D}_\omega$. Further, the ω-enumeration degree of the sequence $\mathcal{A} \oplus \mathcal{B} = \{A_k \oplus B_k\}_{k<\omega}$ is the least upper bound $\mathbf{a} \vee \mathbf{b}$ of the pair of degrees $\mathbf{a} = \deg_\omega(\mathcal{A})$ and $\mathbf{b} = \deg_\omega(\mathcal{B})$. Thus $\mathcal{D}_\omega$ is an upper semi-lattice with least element.

It is not difficult to notice that each sequence and its jump sequence belong to the same ω-enumeration degree, i.e. for all $\mathcal{A} \in \mathcal{S}_\omega$,

$$\mathcal{A} \equiv_\omega \mathcal{P}(\mathcal{A}). \tag{2}$$

In this way, $\mathcal{P}(\mathcal{A})$ is an equivalent to $\mathcal{A}$ sequence, whose members are monotone with respect to $\leq_e$ and each of its members contains full information on previous members.

Given a set $A \subseteq \omega$, denote by $A \uparrow \omega$ the sequence $(A, \emptyset, \emptyset, \dots, \emptyset, \dots)$. The definition of $\leq_\omega$ and the uniformity of the jump operation imply that for all sets of natural numbers A and B,

$$A \uparrow \omega \leq_\omega B \uparrow \omega \iff A \leq_e B. \tag{3}$$

The last equivalence implies, that the mapping $\kappa : \mathcal{D}_e \to \mathcal{D}_\omega$, defined by

$$\kappa(\deg_e(X)) = \deg_\omega(X \uparrow \omega),$$

is an embedding of $\mathcal{D}_e$ into $\mathcal{D}_\omega$. Further, the so defined embedding κ preserves the order, the least element and the binary least upper bound operation.

We shall refer to κ as *the natural embedding* of the enumeration degrees into the ω-enumeration degrees. The range of κ shall be denoted by $\mathbf{D}_1$ and shall be called *the natural copy* of the enumeration degrees.

As shown in [17], each ω-enumeration degree is uniquely determined by the set of the degrees in $\mathbf{D}_1$, which bound it,

$$\mathbf{a} \leq_\omega \mathbf{b} \iff (\forall \mathbf{x} \in \mathbf{D}_e)[\mathbf{b} \leq_\omega \kappa(\mathbf{x}) \to \mathbf{a} \leq_\omega \kappa(\mathbf{x})]. \tag{4}$$

From here, one can easily show that $\mathbf{D}_1$ is an automorphism base of $\mathcal{D}_\omega$.

2.4 The jump operator

Following the lines of Soskov and Ganchev [17], the ω-enumeration jump $\mathcal{A}'$ of $\mathcal{A} \in \mathcal{S}_\omega$ is defined as the jump sequence of $\mathcal{A}$ with the first element deleted:

$$\mathcal{A}' = \{P_{k+1}(\mathcal{A})\}_{k<\omega}.$$

Note, that $\mathcal{A}' \equiv_\omega (P_1(\mathcal{A}), A_2, A_3, \dots, A_k, \dots) = \mathcal{B}$, because for each k, $P_k(\mathcal{B}) = P_{1+k}(\mathcal{A})$.

The jump operator is strictly monotone, i.e. $\mathcal{A} \lneq_\omega \mathcal{A}'$ and $\mathcal{A} \leq_\omega \mathcal{B} \Rightarrow \mathcal{A}' \leq_\omega \mathcal{B}'$. This allows to define a jump operation on the ω-enumeration degrees by setting

$$\deg_\omega(\mathcal{A})' = \deg_\omega(\mathcal{A}').$$

Clearly for all $\mathbf{a}, \mathbf{b} \in \mathbf{D}_\omega$, $\mathbf{a} <_\omega \mathbf{a}'$ and $\mathbf{a} \leq_\omega \mathbf{b} \Rightarrow \mathbf{a}' \leq_\omega \mathbf{b}'$.

Also the jump operation on ω-enumeration degrees agrees with the jump operation on the enumeration degrees, i.e. we have

$$\kappa(\mathbf{x}') = \kappa(\mathbf{x})', \text{ for all } \mathbf{x} \in \mathbf{D}_e.$$

We shall denote by $\mathcal{A}^{(n)}$ the n-the iteration of the jump operator on $\mathcal{A}$. Let us note that

$$\mathcal{A}^{(n)} = \{P_{n+k}(\mathcal{A})\}_{k<\omega} \equiv_\omega (P_n(\mathcal{A}), A_{n+1}, A_{n+2}, \ldots). \tag{5}$$

It is clear that if $\mathcal{A} \in \mathbf{a}$, then $\mathcal{A}^{(n)} \in \mathbf{a}^{(n)}$, where $\mathbf{a}^{(n)}$ denotes the n-th iteration of the jump operation on the ω-enumeration degree $\mathbf{a}$.

The first author [4] proved that the jump operator on $\mathcal{D}_\omega$ preserves the greatest lower bound, i.e. for each $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{D}_\omega$,

$$\mathbf{x} \wedge \mathbf{y} = \mathbf{z} \Rightarrow \mathbf{x}' \wedge \mathbf{y}' = \mathbf{z}'. \tag{6}$$

On the other hand it is not difficult to see that the jump operator does not always preserve least upper bounds. In [6] one can find a sufficient condition for the preservation of the least upper bound.

In [16] it is proved that the range of the jump operator is exactly the upper cone over the first jump $\mathbf{0}'_\omega$ of the least element. Soskov and Ganchev [17] showed an even stronger jump inversion property, which does not hold either in the Turing degrees, or in the enumeration degrees. Namely, for each natural number n if $\mathbf{b}$ is above $\mathbf{a}^{(n)}$, then there is a least ω-enumeration degree $\mathbf{x}$ above $\mathbf{a}$ with $\mathbf{x}^{(n)} = \mathbf{b}$. We shall denote this degree by $\mathbf{I}^n_{\mathbf{a}}(\mathbf{b})$. An explicit representative of $\mathbf{I}^n_{\mathbf{a}}(\mathbf{b})$ can be given by setting

$$I^n_{\mathcal{A}}(\mathcal{B}) = (A_0, A_1, \ldots, A_{n-1}, B_0, B_1, \ldots, B_k, \ldots), \tag{7}$$

where $\mathcal{A} \in \mathbf{a}$ and $\mathcal{B} \in \mathbf{b}$ are arbitrary.

3 A property of the least enumeration degree

The aim of this section is to provide a characterizing property of the least enumeration degree $\mathbf{0}_e$ which we shall use later. Namely, we shall show that $\mathbf{0}_e$ is the only degree $\mathbf{x}$ in $\mathcal{D}_e$ such that for each enumeration degree $\mathbf{b}$, if $\mathbf{x} \vee \mathbf{b} \geq_e \mathbf{0}'_e$ then necessarily $\mathbf{b} \geq_e \mathbf{0}'_e$. In order to do so, we need the following notion of $\mathcal{K}$-pair.

Definition 1 Let A and B be sets of natural numbers. The pair $\{A, B\}$ is a Kalimullin pair ($\mathcal{K}$-pair) if there is a c.e. set W, such that:

$$A \times B \subseteq W \text{ and } \overline{A} \times \overline{B} \subseteq \overline{W}.$$

Note that if U is c.e. then for any set A of natural numbers, U and A form a trivial example of a $\mathcal{K}$-pair via the c.e. set $U \times \omega$. Further, we shall call a $\mathcal{K}$-pair $\{A, B\}$ nontrivial if both A and B are not c.e. The existence of nontrivial $\mathcal{K}$-pairs follows from the fact that there are *semi-recursive* non c.e. sets.

Definition 2 A set of natural numbers A is semi-recursive if there is a total computable selector function $s_A : \omega \times \omega \to \omega$, such that for any $x, y \in \omega$,

$$s_A(x, y) \in \{x, y\} \text{ and whenever } \{x, y\} \cap A \neq \emptyset, s_A(x, y) \in A.$$

By Jockusch [7], every nonzero Turing degree contains a semi-recursive set A such that both A and $\overline{A}$ are not c.e. It remains to note that if A is a semi-recursive then $\{A, \overline{A}\}$ is a $\mathcal{K}$-pair. Indeed, if A is semi-recursive set then $A \times \overline{A} \subseteq W$ and $\overline{A} \times A \subseteq \overline{W}$, where W is the c.e. set $\{(x, y) \mid s_A(x, y) = x\}$.

We list some additional properties of $\mathcal{K}$-pairs, which we shall use later. A proof of all of them can be found in [8].

Proposition 1 *Let A and B be a nontrivial $\mathcal{K}$-pair. Then:*

1. $A \leq_e \overline{B}$ *and* $\overline{A} \leq_e B \oplus J_e(\emptyset)$*;*
2. *The enumeration degrees $deg_e(A)$ and $deg_e(B)$ are incomparable and quasi-minimal;*
3. *The set $\{\mathbf{x} \in \mathbf{D}_e \mid \{deg_e(A), \mathbf{x}\}$ is a $\mathcal{K}$-pair$\}$ is an ideal.*

$\mathcal{K}$-pairs and the definability of the jump in $\mathcal{D}_e$ are closely connected. $\mathcal{K}$-pairs were introduced by Kalimullin in [8], where he proves the first-order definability of the enumeration jump operator. Later, Ganchev and M. Soskova [6] found a much simpler definition of the jump.

Namely for every nonzero enumeration degree $\mathbf{u} \in \mathbf{D}_e$, $\mathbf{u}'$ is greatest among the all the least upper bounds $\mathbf{a} \vee \mathbf{b}$ of nontrivial $\mathcal{K}$-pairs $\{\mathbf{a}, \mathbf{b}\}$, such that $\mathbf{a} \leq_e \mathbf{u}$.

From here one can easily derive that if $\mathbf{x}$ be nonzero enumeration degree, then there is a degree $\mathbf{y}$ such that $\mathbf{x} \vee \mathbf{y} \geq_e \mathbf{0}_e'$ but $\mathbf{y}$ is not above $\mathbf{0}_e'$. Indeed, let $\mathbf{x} \in \mathbf{D}_e$ be a nonzero. Let $\{\mathbf{a}, \mathbf{b}\}$ be a nontrivial $\mathcal{K}$-pair, such that $\mathbf{a} \leq_e \mathbf{x}$, which realizes $\mathbf{x}'$, i. e. $\mathbf{a} \vee \mathbf{b} = \mathbf{x}'$. Since $\mathbf{a} \leq_e \mathbf{x}$, we have that $\mathbf{x} \vee \mathbf{b} \geq_e \mathbf{a} \vee \mathbf{b} = \mathbf{x}' \geq_e \mathbf{0}_e'$. Suppose now that $\mathbf{0}_e' \leq_e \mathbf{b}$. Then by the third property in Proposition 1, $\{\mathbf{a}, \mathbf{0}_e'\}$ must be a nontrivial $\mathcal{K}$-pair. If $A \in \mathbf{a}$, then $A \leq_e \overline{J_e(\emptyset)} \equiv_e J_e(\emptyset)$ by the first property in Proposition 1 and the fact that $J_e(\emptyset)$ is a total set. Hence $\mathbf{a} \leq_e \mathbf{0}_e' \leq_e \mathbf{b}$, which contradicts with the second property in Proposition 1. Thus $\mathbf{0}_e' \not\leq_e \mathbf{b}$.

Note also, that the formula: $\varphi(\mathbf{x}) \rightleftharpoons (\forall \mathbf{y})[\mathbf{x} \vee \mathbf{y} \geq_e \mathbf{0}_e' \to \mathbf{y} \geq_e \mathbf{0}_e']$ is satisfied by the least enumeration degree $\mathbf{0}_e$. Thus, we have proven the following proposition.

Lemma 1 *The least element $\mathbf{0}_e$ is the only enumeration degree $\mathbf{x}$ such that*

$$(\forall \mathbf{y})[\mathbf{x} \vee \mathbf{y} \geq_e \mathbf{0}_e' \to \mathbf{y} \geq_e \mathbf{0}_e'].$$

4 Definability of the enumeration degrees

In [17] Soskov and Ganchev showed the first-order definability of the isomorphic copy $\mathbf{D}_1$ under the embedding κ of the enumeration degrees in $\mathcal{D}_\omega$ in the terms of the structure order and the jump operation. In this section we shall improve this result by finding a first-order formula only in the language of structure order which defines $\mathbf{D}_1$ in $\mathcal{D}_\omega$.

For this purpose, let us first consider the set of the ω-enumeration degrees, defined by the formula φ from the previous section. Let $\mathcal{X} = \{X_k\}_{k<\omega}$ be a sequence such that $\mathcal{D}_\omega \models \varphi(\deg_\omega(\mathcal{X}))$. In other words, $\mathcal{X}$ is such that for each sequence $\mathcal{Y} = \{Y_k\}_{k<\omega}$ if ${\emptyset_\omega}' \leq_\omega \mathcal{X} \oplus \mathcal{Y}$ then ${\emptyset_\omega}' \leq_\omega \mathcal{Y}$. Noting that for each sequence $\mathcal{A} = \{A_k\}_{k<\omega}$, ${\emptyset_\omega}' \leq_\omega \mathcal{A}$ is equivalent to $\emptyset' \leq_e A_0$, and then using Lemma 1, we conclude that $X_0 \equiv_e \emptyset$.

Now, let $\mathcal{X} = \{X_k\}_{k<\omega}$ be such that $X_0 \equiv_e \emptyset$ and the sequence $\mathcal{Y} = \{Y_k\}_{k<\omega}$ be such that ${\emptyset_\omega}' \leq_\omega \mathcal{X} \oplus \mathcal{Y}$. Then we have that $\emptyset' \leq_e X_0 \oplus Y_0 \equiv_e Y_0$ and hence ${\emptyset_\omega}' \leq_\omega \mathcal{Y}$.

Thus, the degrees in $\mathcal{D}_\omega$, which satisfy the formula φ, are exactly those that contain a sequence whose zeroth element is the empty set. We shall denote the set of all these degrees by $\widetilde{\mathbf{D}_1}$,

$$\widetilde{\mathbf{D}_1} = \{\mathbf{x} \in \mathbf{D}_\omega \mid (\exists \{A_k\}_{k<\omega} \in \mathbf{x})[\ A_0 = \emptyset\ ]\}.$$

Here is the moment when we use the first-order definability of $\mathbf{0}'_\omega$, proved in [5]. By this result, we now conclude the first-order definability of the set $\widetilde{\mathbf{D}_1}$.

Using the set $\widetilde{\mathbf{D}_1}$ a simple definition of $\mathbf{D}_1$ can be derived. Indeed, for each $\mathbf{a} \in \mathbf{D}_\omega$, denote by $\mu(\mathbf{a})$ the least (ω-enumeration) degree $\mathbf{x}$ for which there exists degree $\mathbf{y} \in \widetilde{\mathbf{D}_1}$ such that $\mathbf{x} \vee \mathbf{y} = \mathbf{a}$. It is not difficult to see that the operation μ is correctly defined. Moreover, for each $\mathbf{a}$, if $\{A_k\}_{k<\omega} \in \mathbf{a}$ then $\mu(\mathbf{a})$ contains the sequence $(A_0, \emptyset, \ldots, \emptyset, \ldots)$. Hence, the range of μ is exactly the copy $\mathbf{D}_1$ of the enumeration degrees under the embedding κ:

$$\mathbf{D}_1 = \{\mu(\mathbf{a}) \mid \mathbf{a} \in \mathbf{D}_\omega\}.$$

Combining all these facts, we conclude that the enumeration degrees are first order definable in the structure $\mathcal{D}_\omega$ of the ω-enumeration degrees.

Lemma 2 *The copy $\mathbf{D}_1$ of the enumeration degrees under the embedding κ is first-order definable in $\mathcal{D}_\omega$.*

5 Definability of the jump

As we noted at the beginning of the previous section, by [17] $\mathbf{D}_1$ is first order definable in the language of the structure order an the jump operator. So obviously the definability of the jump operator implies the definability of the isomorphic copy of the enumeration degrees. The aim of this section is to show the reverse implication, and by Lemma 2 to conclude the first-order definability of the jump operator.

We start with a structural property of $\mathcal{D}_\omega$. As Ganchev showed in [4], for all ω-enumeration degrees $\mathbf{a}$ and $\mathbf{g}$, if $\mathbf{a} \leq_\omega \mathbf{g}$ then $\mathbf{a}$ is the greatest lower bound of $\mathbf{g}$ and some degree $\mathbf{f}$ in $\mathbf{D}_1$. From here one can easily derive that each degree $\mathbf{a} \in \mathbf{D}_\omega$ is the greatest lower bound of two degrees in $\mathbf{D}_1$. Indeed, it is sufficient to show that each degree $\mathbf{a} \in \mathbf{D}_\omega$ is bounded by a degree in $\mathbf{D}_1$. In order to show this let us recall that $\mathbf{a} \lneq_\omega \mathbf{a}'$. Then by (4), there is $\mathbf{x} \in \mathbf{D}_e$, such that $\mathbf{a} \leq_\omega \kappa(\mathbf{x})$ and $\mathbf{a}' \nleq_\omega \kappa(\mathbf{x})$. So $\kappa(\mathbf{x})$ is a degree from $\mathbf{D}_1$ which bounds $\mathbf{a}$.

Again in [4] it was shown that the jump operator on $\mathbf{D}_\omega$ preserves the greatest lower bound. Hence for each ω-enumeration degree $\mathbf{a}$ there are ω-enumeration degrees $\mathbf{g}, \mathbf{f} \in \mathbf{D}_1$, such that

$$\mathbf{a} = \mathbf{g} \wedge \mathbf{f} \quad \text{and} \quad \mathbf{a}' = \mathbf{g}' \wedge \mathbf{f}', \tag{8}$$

and if there is another pair of degrees, whose greatest lower bound exists and is equal to $\mathbf{a}$, then the greatest lower bound of their jumps also exists and is equal exactly to $\mathbf{a}'$.

As we stated in the preliminaries, $\mathbf{D}_1$ is closed under the jump, and the ω-enumeration jump agrees with the enumeration jump. Also, by Kalimullin [8], the jump operator is definable in the structure $\mathcal{D}_e$ of the enumeration degrees. Hence the restriction of the ω-enumeration jump operator over $\mathbf{D}_1$ is definable in the structure $(\mathbf{D}_1, \leq_\omega, \vee)$. In this way, by (8), we conclude that the definability of $\mathbf{D}_1$ implies that of the jump. Thus, by Lemma 2, we have the definability of the jump operation.

Theorem 1 *The jump operator is first-order definable in the structure $\mathcal{D}_\omega$ of the ω-enumeration degrees.*

From here we directly have that each automorphism of $\mathcal{D}_\omega$ is jump preserving[2], i.e. $\mathrm{Aut}(\mathcal{D}_\omega')$=$\mathrm{Aut}(\mathcal{D}_\omega)$. Now, using the previously mentioned result by Soskov and Ganchev [17] stating the isomorphicity of the groups of the automorphisms of the enumeration degrees and of the jump preserving automorphisms of the ω-enumeration degrees, we conclude the following:

Theorem 2 *The groups $Aut(\mathcal{D}_e)$ and $Aut(\mathcal{D}_\omega)$ are isomorphic.*

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[2] A mapping $\pi : \mathbf{D}_\omega \to \mathbf{D}_\omega$ is said to be *jump preserving*, if for each degree $\mathbf{a} \in \mathbf{D}_\omega$, $\pi(\mathbf{a}') = \pi(\mathbf{a})'$.

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