On subrecursive complexity of integration

Ivan Georgiev

Prof. d-r Asen Zlatarov University, Faculty of Natural Sciences,
1 Prof. Yakimov Str., 8010 Burgas, Bulgaria

Abstract

We consider the complexity of the integration operator on real functions with respect to the subrecursive class $\mathcal{M}^2$. We prove that the definite integral of a uniformly $\mathcal{M}^2$-computable analytic real function with $\mathcal{M}^2$-computable limits is itself $\mathcal{M}^2$-computable real number. We generalise this result to integrals with parameters and with varying limits. As an application, we show that the Euler-Mascheroni constant is $\mathcal{M}^2$-computable.

Keywords: computable real function, relative computability, the subrecursive class $\mathcal{M}^2$, integration, Euler-Mascheroni constant
2010 MSC: 03F60, 03D78, 03D15, 03D20

1. Introduction

This paper is about relative computability of real numbers and real functions. Our aim is to study the complexity of integration. The motivating question is:

Given real numbers α,β and a real function $\theta : [\alpha,\beta] \to \mathbb{R}$, which are efficiently computable, is it true that the real number

$$\int_{\alpha}^{\beta} \theta(x)\, dx$$

is also efficiently computable?

To evaluate the complexity of real numbers, we introduce a naming system based on Cauchy sequences. To represent a real function, we define a computing system of type-2 operators, which transform arbitrary names of the arguments into a name of the value of the real function. Thus the complexity of the real function can be defined in terms of the complexity of the corresponding type-2 operators.

In the framework of discrete complexity theory the question is studied in [1, 5, 6] and more systematically in Section 5.4 in [4]. In fact, it is shown in [6] that the definite integral of an analytic polynomial-time computable real function is itself polynomial-time computable. Our aim is to prove a similar result, but our framework for complexity is subrecursive, that is we are interested in inductively defined classes of total functions in the natural numbers, contained in the low levels of Grzegorczyk's hierarchy of the primitive recursive functions. The tool that we will use to prove the result is the trapezoidal rule for numerical integration, combined with a suitable change of variables, as described in [12] and in more detail in [13].

Email address: `ivandg@yahoo.com` (Ivan Georgiev)

2. The classes $\mathcal{M}^2, \mathcal{L}^2, \mathcal{E}^2$

We denote $\mathcal{T}_m = \{a | a : \mathbb{N}^m \to \mathbb{N}\}$ and $\mathcal{T} = \bigcup_m \mathcal{T}_m$. Unless otherwise specified, a *function* means a function from $\mathcal{T}$. We will use vector notation $\vec{x}, \vec{y}, \vec{s}$ for tuples of natural numbers and $\vec{f}, \vec{g}, \vec{h}$ for tuples of unary functions. The size will be clear from the context.

For functions $f, g \in \mathcal{T}_n$, we say that g *majorises* f (or f *is majorised by* g), if $f(\vec{x}) \leq g(\vec{x})$ for all $\vec{x} \in \mathbb{N}^n$.

The projection functions $\lambda x_1 \ldots x_n.x_m (1 \leq m \leq n)$, the successor function $\lambda x.x + 1$, the modified subtraction function $\lambda xy.x \dot{-} y = \lambda xy. \max(x - y, 0)$ and the product function $\lambda xy.xy$, belonging to $\mathcal{T}$, will be called the *initial functions*.

Definition 2.1. *The class $\mathcal{M}^2$ is the smallest subclass of $\mathcal{T}$, which contains the initial functions and is closed under substitution and bounded minimisation*

$$\left(f \mapsto \lambda \vec{x} y. \mu_{z \leq y}[f(\vec{x}, z) = 0]\right).$$

For any $\vec{x}, y$, the natural number $\mu_{z \leq y}[f(\vec{x}, z) = 0]$ is the least $z \leq y$, such that $f(\vec{x}, z) = 0$, if such z exists, and $y + 1$, otherwise.

If we replace bounded minimisation with bounded summation in the definition, we obtain the class $\mathcal{L}^2$ of the lower elementary functions.

By using limited primitive recursion in place of bounded minimisation in the definition, we obtain the third level $\mathcal{E}^2$ of Grzegorczyk's hierarchy. Limited primitive recursion is the same as primitive recursion, but the resulting function must be bounded by a given function.

It is known that $\mathcal{M}^2 \subseteq \mathcal{L}^2 \subseteq \mathcal{E}^2$, but whether each of these inclusions is proper is an open question.

A function is Δ_0-definable, if its graph is definable in the standard model of Peano arithmetic with a formula, containing only bounded quantifiers. The class $\mathcal{M}^2$ contains precisely those functions from $\mathcal{T}$, which are Δ_0-definable and majorised by a polynomial.

Remark 2.2. *It is well-known that the relation $z = 2^y$ is Δ_0-definable. It follows that the function L defined by*

$$L(y) = \lfloor \log_2(y + 1) \rfloor,$$

belongs to the class $\mathcal{M}^2$, since $L(y) \leq y + 1$ and

$$z = L(y) \Leftrightarrow \exists u \leq y + 1 (u = 2^z \;\; \& \;\; 2u > y + 1)$$

for all $z, y \in \mathbb{N}$.

The classes $\mathcal{L}^2$ and $\mathcal{E}^2$ are closed under bounded summation, but it is not known whether the same is true for $\mathcal{M}^2$. Nevertheless, we have the following:

Theorem 2.3 ([7])**.** *For any $k, m \in \mathbb{N}$ and any function $f \in \mathcal{T}_{m+1} \cap \mathcal{M}^2$, the function $g \in \mathcal{T}_{m+1}$ defined by*

$$g(\vec{x}, y) = \sum_{z \leq L(y)^k} f(\vec{x}, z)$$

also belongs to $\mathcal{M}^2$.

3. Subrecursive classes of operators

As already noted, real functions are computed by type-2 operators. In this section we introduce three important subrecursive classes of such operators. Their definitions resemble the definitions for classes of functions from the previous section. Namely, a set of initial operators is closed under a set of operations.

For $k, m \in \mathbb{N}$, a (k, m)-*operator* F is a total mapping $F : \mathcal{T}_1^k \to \mathcal{T}_m$. An *operator* is a (k, m)-operator for some $k, m \in \mathbb{N}$.

For a class of operators $\mathbf{O}$, we denote by $\mathbf{O}_1$ the class of all $(k, 1)$-operators in $\mathbf{O}$ (for some $k \in \mathbb{N}$).

Definition 3.1. *The class* **RO** *of* rudimentary operators *is the smallest class of operators, such that:*

1. *For any n, m and m-argument initial function a, the (n, m)-operator F defined by $F(\vec{f})(\vec{x}) = a(\vec{x})$ belongs to* **RO**.
2. *For all n, k with $1 \leq k \leq n$, the $(n, 1)$-operator F defined by $F(f_1, \ldots, f_n)(x) = f_k(x)$ belongs to* **RO**.
3. *For all n, m, k, if F_0 is an (n, k)-operator and $F_1, \ldots, F_k$ are (n, m)-operators all belonging to* **RO**, *then the (n, m)-operator F defined by*
$$F(\vec{f})(\vec{x}) = F_0(\vec{f})(F_1(\vec{f})(\vec{x}), \ldots, F_k(\vec{f})(\vec{x}))$$
also belongs to **RO**.
4. *For all m, n, if F_0 is an $(n, m + 1)$-operator which belongs to* **RO**, *then so is the operator F defined by*
$$F(\vec{f})(\vec{x}, y) = \mu_{z \leq y}[F_0(\vec{f})(\vec{x}, z) = 0].$$

The definition of the class **logRO** of *log-rudimentary* operators contains the same clauses as the definition for **RO** and also the following clause:

5. For all m, n, k, if F_0 is an $(n, m + 1)$-operator which belongs to **logRO**, *then so is the operator F defined by*
$$F(\vec{f})(\vec{x}, y) = \sum_{z \leq L(y)^k} [F_0(\vec{f})(\vec{x}, z) = 0].$$

Of course, **RO** $\subseteq$ **logRO**. Moreover, if there is a uniform definition of log-bounded summation for the class $\mathcal{M}^2$ (that is, if Theorem 2.3 has a uniform proof), then the same definition, easily modified for operators, will show that **RO** = **logRO**. As we shall see in the next section whether this equality holds is immaterial for our considerations.

The next definition is a slightly generalised version of Definition 6 from Section 2.2 in [11] with $\mathcal{F} = \mathcal{M}^2$.

Definition 3.2. *The class* **MSO** *of* $\mathcal{M}^2$-substitutional operators *is the smallest class of operators, such that:*

1. *For all m, n, i with $1 \leq i \leq m$, the (n, m)-operator F defined by $F(\vec{f})(\vec{x}) = x_i$ belongs to* **MSO**.
2. *For any m, n and $k \in \{1, \ldots, n\}$, if F_0 is an (n, m)-operator which belongs to* **MSO**, *then the (n, m)-operator F defined by*
$$F(\vec{f})(\vec{x}) = f_k(F_0(\vec{f})(\vec{x}))$$

also belongs to **MSO**.

3. *For any m, n, k and $a \in \mathcal{T}_k \cap \mathcal{M}^2$, if $F_1, \ldots, F_k$ are (n, m)-operators which belong to* **MSO**, *then so is the operator F defined by*

$$F(\vec{f})(\vec{x}) = a(F_1(\vec{f})(\vec{x}), \ldots, F_k(\vec{f})(\vec{x})).$$

The following propositions list the most important properties of the three classes of operators. The proofs of them are straight-forward inductions.

Proposition 3.3. *For any natural numbers m, n and function $a \in \mathcal{M}^2 \cap \mathcal{T}_m$, the (n, m)-operator F defined by $F(\vec{f})(\vec{x}) = a(\vec{x})$ belongs to the class* **RO**.

Corollary 3.4. **MSO** $\subseteq$ **RO**.

In fact **MSO** is a proper subclass of **RO**. Clause 4 of Definition 3.1 cannot be expressed by operators from **MSO**. The full proof can be found in [2].

Proposition 3.5. *Let* **O** $\in$ {**RO**, **logRO**, **MSO**}. *For natural numbers l, m, n, p, let F be an (l, m)-operator and $G_1, \ldots, G_l$ be $(n, p+1)$-operators, all belonging to* **O**. *Then the $(n, m+p)$-operator H defined by the equality*

$$H(\vec{f})(\vec{x}, \vec{y}) = F(\lambda t.G_1(\vec{f})(t, \vec{y}), \ldots, \lambda t.G_l(\vec{f})(t, \vec{y}))(\vec{x})$$

for $\vec{f} \in \mathcal{T}_1^n, \vec{x} \in \mathbb{N}^m, \vec{y} \in \mathbb{N}^p$, also belongs to the class **O**.

Proposition 3.6. *Let l, m, n be natural numbers and $a_1, \ldots, a_n \in \mathcal{M}^2 \cap \mathcal{T}_{l+1}$ be functions. For any (n, m)-operator $F \in$* **logRO**, *the function $b \in \mathcal{T}_{l+m}$ defined by*

$$b(\vec{s}, \vec{x}) = F(\lambda t.a_1(\vec{s}, t), \ldots, \lambda t.a_n(\vec{s}, t))(\vec{x})$$

also belongs to the class $\mathcal{M}^2$.

Of course, the last proposition also holds for **MSO** and **RO**, since **MSO** $\subseteq$ **RO** $\subseteq$ **logRO**.

An (n, m)-operator F will be called *monotonically increasing*, if $F(\vec{f})(\vec{x}) \leq F(\vec{g})(\vec{y})$ for all $\vec{f}, \vec{g} \in \mathcal{T}_1^n$ and $\vec{x}, \vec{y} \in \mathbb{N}^m$, such that g_l majorises f_l for $l \in \{1, \ldots, n\}$ and $x_k \leq y_k$ for $k \in \{1, \ldots, m\}$.

The proofs of the last two propositions can be found in [2] for the class **RO** and they can be adapted almost immediately for the class **logRO**.

Proposition 3.7. *Let* **O** $\in$ {**RO**, **logRO**}. *For natural numbers m, n and any (n, m)-operator F belonging to* **O**, *there exists a monotonically increasing $(1, m)$-operator G, also belonging to* **O**, *such that $F(\vec{f})(\vec{x}) \leq G(f)(\vec{x})$ whenever $\vec{f} \in \mathcal{T}_1^n, f \in \mathcal{T}_1, \vec{x} \in \mathbb{N}^m$ and f majorises $f_1, \ldots, f_n$.*

Proposition 3.8 (Uniformity Theorem). *Let* **O** $\in$ {**RO**, **logRO**}. *For natural numbers m, n and any (n, m)-operator $F \in$* **O** *there exists a $(1, m)$-operator $H \in$* **O**, *such that the following holds: for any $\vec{x} \in \mathbb{N}^m$ and $f \in \mathcal{T}_1$, if the unary functions $g_1, \ldots, g_n, h_1, \ldots, h_n$ are majorised by f and $g_1(t) = h_1(t), \ldots, g_n(t) = h_n(t)$ for all $t \leq H(f)(\vec{x})$, then $F(\vec{g})(\vec{x}) = F(\vec{h})(\vec{x})$.*

Proposition 3.9. *For any* **O** $\in$ {**RO**, **logRO**, **MSO**}, *the pair $(\mathcal{M}^2, \mathbf{O}_1)$ is acceptable in the sense of Definition 4 in [9].*

Proof. The pair $(\mathcal{M}^2, \mathbf{MSO}_1)$ is acceptable by Theorem 1 in [9] and the fact that any function in $\mathcal{M}^2$ is majorised by a polynomial (the class $\mathbf{MSO}_1$ is denoted by $\mathbf{O}_{\mathcal{M}^2}$ in [9]). The acceptability of the other two pairs follows from the propositions above. □

4. Relative computability of real numbers and real functions

The next definition introduces the naming system for the real numbers, which we use to define relative computability.

Definition 4.1. *The triple of functions* $(f, g, h) \in \mathcal{T}_1^3$ *is a* name *of the real number* ξ *iff for all* $n \in \mathbb{N}$*,*

$$\left| \frac{f(n) - g(n)}{h(n) + 1} - \xi \right| < \frac{1}{n + 1}.$$

For a class $\mathcal{F}$ *of functions* ($\mathcal{F} \subseteq \mathcal{T}$)*, a real number* ξ *is* $\mathcal{F}$-computable *iff there exists a triple* $(f, g, h) \in \mathcal{F}^3$ *which is a name of* ξ*.*

It is important to note that we use Cauchy sequences with linear convergence rate. The usual definition for computable real number here uses 2^n in place of $n + 1$, but this is not suitable for classes of polynomially bounded functions.

It is proven in [8] that for $\mathcal{F} \in \{\mathcal{M}^2, \mathcal{L}^2, \mathcal{E}^2\}$ the set of all $\mathcal{F}$-computable real numbers is a real-closed field. Therefore all real algebraic numbers are $\mathcal{M}^2$-computable. Examples from [11] show that the numbers π and e are also $\mathcal{M}^2$-computable.

Definition 4.2. *Let* $k \in \mathbb{N}$ *and* θ *be a real function,* $\theta : D \to \mathbb{R}$*, where* $D \subseteq \mathbb{R}^k$*. The triple* (F, G, H)*, where* F, G, H *are* $(3k, 1)$*-operators, is called a* computing system *for* θ *if for all* $(\xi_1, \xi_2, \ldots, \xi_k) \in D$ *and triples* (f_i, g_i, h_i) *that name* ξ_i *for* $i = 1, 2, \ldots, k$*, the triple*

$$(F(f_1, g_1, h_1, f_2, g_2, h_2, \ldots, f_k, g_k, h_k),$$
$$G(f_1, g_1, h_1, f_2, g_2, h_2, \ldots, f_k, g_k, h_k),$$
$$H(f_1, g_1, h_1, f_2, g_2, h_2, \ldots, f_k, g_k, h_k))$$

names the real number $\theta(\xi_1, \xi_2, \ldots, \xi_k)$*.*

For a class $\mathbf{O}$ of operators, the real function θ is *uniformly* $\mathbf{O}$*-computable*, if there exists a computing system (F, G, H) for θ, such that $F, G, H \in \mathbf{O}$. Of course, since F, G, H produce unary function, we can replace the last $\mathbf{O}$ by $\mathbf{O}_1$ in this definition.

By Theorem 2 of Skordev in [9] and Proposition 3.9, the following three conditions are equivalent for a real function θ:

- θ is uniformly **MSO**-computable;
- θ is uniformly **RO**-computable;
- θ is uniformly **LogRO**-computable.

So the three classes of operators, which might be all different, have exactly the same computing power with respect to real functions.

Remark 4.3. *For any* $k \in \mathbb{N}$ *we have that* $(\lambda x.k, \lambda x.0, \lambda x.0)$ *is a name of* k*. Conversely, if* (f, g, h) *is a name of a natural number* k*, then*

$$k = \left\lfloor \frac{|f(1) - g(1)|}{h(1) + 1} + \frac{1}{2} \right\rfloor.$$

The remark can be used for transferring functional arguments of an operator into additional natural arguments of the functional value of the operator and vice versa. This is particularly useful for computing systems of real functions, which have natural-valued arguments. More concretely, for $\mathbf{O} \in \{\mathbf{MSO}, \mathbf{RO}, \mathbf{logRO}\}$, a real function $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{N}^k$ is uniformly $\mathbf{O}$-computable if and only if there exist $f, g, h \in \mathcal{T}_{k+1} \cap \mathcal{M}^2$, such that for all $\vec{s} \in D$

$$(\lambda n.f(\vec{s}, n), \lambda n.g(\vec{s}, n), \lambda n.h(\vec{s}, n))$$

is a name for the real number $\theta(\vec{s})$. Of course, we can generalise this to real functions $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{N}^k \times \mathbb{R}^l$. Any computing system $(F, G, H) \in \mathbf{O}^3$ for θ can be identified with a triple $(F', G', H') \in \mathbf{O}^3$ of $(3l, k + 1)$-operators, such that for any $(\vec{s}, \xi_1, \ldots, \xi_l) \in D$ and any tuple of names for $\xi_1, \ldots, \xi_l$, the operators F', G', H' transform this tuple into three functions $f, g, h \in \mathcal{T}_{k+1}$, such that $(\lambda n.f(\vec{s}, n), \lambda n.g(\vec{s}, n), \lambda n.h(\vec{s}, n))$ is a name for $\theta(\vec{s}, \xi_1, \ldots, \xi_l)$.

It is easy to see that the class of uniformly **MSO**-computable real functions is closed under substitution and under restrictions of the domain.

Results from [11] show that all elementary functions of calculus, restricted to compact subsets of their domains, are uniformly **MSO**-computable. More concretely:

- addition, subtraction and multiplication are uniformly **MSO**-computable on the whole $\mathbb{R}^2$;
- the absolute value real function is uniformly **MSO**-computable on the whole $\mathbb{R}$, hence the binary max and min real functions are uniformly **MSO**-computable on $\mathbb{R}^2$;
- the restriction of the reciprocal real function to any set of the form $(-\infty, -r) \cup (r, +\infty)$ for $r > 0$ is uniformly **MSO**-computable (Corollary 6 in [11]);
- the restriction of the logarithmic real function to any interval of the form $(r, +\infty)$ for $r > 0$ is uniformly **MSO**-computable (Corollary 9 in [11]);
- the restriction of the exponential real function to any interval of the form $(-\infty, r)$ is uniformly **MSO**-computable (Corollary 10 in [11]).

The reciprocal, the logarithmic and the exponential real functions are not uniformly **MSO**-computable on their whole domains, since the absolute value of any uniformly **MSO**-computable real function is bounded by some polynomial (Section 2.2 in [11]).

Lemma 4.4. *Let $\theta : \mathbb{N} \times D \to \mathbb{R}$, $D \subseteq \mathbb{R}$ be a real function, which is uniformly* **MSO***-computable. For any fixed natural number k, the real function $\theta^{\Sigma} : \mathbb{N} \times D \to \mathbb{R}$ defined by the equality*

$$\theta^{\Sigma}(y, \xi) = \sum_{z \le L(y)^k} \theta(z, \xi)$$

for $y \in \mathbb{N}, \xi \in D$, is also uniformly **MSO***-computable.*

Proof. Let (F, G, H) be a computing system for θ, where F, G and H are $(6, 1)$-operators, belonging to **MSO**. By applying the operator K from Section 1.3 in [11], we can assume that $H(f_1, g_1, h_1, f, g, h)(t) = t$ for all $t \in \mathbb{N}$.

We define $(3, 2)$-operators F' and G' by

$$F'(f, g, h)(z, t) = F(\lambda x.z, \lambda x.0, \lambda x.0, f, g, h)(t),$$

$$G'(f,g,h)(z,t) = G(\lambda x.z, \lambda x.0, \lambda x.0, f, g, h)(t).$$

Using Proposition 3.5 it is easy to see that F' and G' belong to **MSO**.

For all $z, t \in \mathbb{N}$ and $\xi \in D$, if (f,g,h) is a name of ξ, then by Remark 4.3 we have the inequality

$$\left| \frac{F'(f,g,h)(z,t) - G'(f,g,h)(z,t)}{t+1} - \theta(z,\xi) \right| < \frac{1}{t+1}.$$

Let us define the $(3,2)$-operators F_1^Σ and G_1^Σ by

$$F_1^\Sigma(f,g,h)(y,n) = \sum_{z \le L(y)^k} F'(f,g,h)(z, L(y)^k n + n + L(y)^k),$$

$$G_1^\Sigma(f,g,h)(y,n) = \sum_{z \le L(y)^k} G'(f,g,h)(z, L(y)^k n + n + L(y)^k),$$

$$H_1^\Sigma(f,g,h)(y,n) = L(y)^k n + n + L(y)^k.$$

It is clear that F_1^Σ, G_1^Σ and H_1^Σ are log-rudimentary (since $L \in \mathcal{M}^2$ and k is fixed).

For all $y, n \in \mathbb{N}, \xi \in D$ and a name (f,g,h) of ξ we have

$$\left| \frac{F_1^\Sigma(f,g,h)(y,n) - G_1^\Sigma(f,g,h)(y,n)}{H_1^\Sigma(f,g,h)(y,n)+1} - \theta^\Sigma(y,\xi) \right|$$

$$\le \sum_{z \le L(y)^k} \left| \frac{F'(f,g,h)(z, L(y)^k n + n + L(y)^k) - G'(f,g,h)(z, L(y)^k n + n + L(y)^k)}{(L(y)^k+1)(n+1)} - \theta(z,\xi) \right|$$

$$< \sum_{z \le L(y)^k} \frac{1}{(L(y)^k+1)(n+1)} = \frac{1}{n+1}.$$

It other words, the triple

$$(\lambda n.F_1^\Sigma(f,g,h)(y,n), \lambda n.G_1^\Sigma(f,g,h)(y,n), \lambda n.H_1^\Sigma(f,g,h)(y,n))$$

is a name of $\theta^\Sigma(y,\xi)$. Therefore, by defining the $(6,1)$-operators $F^\Sigma, G^\Sigma, H^\Sigma$ with the equalities

$$F^\Sigma(f_1,g_1,h_1,f,g,h)(n) = F_1^\Sigma(f,g,h)\left(\left\lfloor \frac{|f_1(1) - g_1(1)|}{h_1(1)+1} + \frac{1}{2} \right\rfloor, n\right),$$

$$G^\Sigma(f_1,g_1,h_1,f,g,h)(n) = G_1^\Sigma(f,g,h)\left(\left\lfloor \frac{|f_1(1) - g_1(1)|}{h_1(1)+1} + \frac{1}{2} \right\rfloor, n\right),$$

$$H^\Sigma(f_1,g_1,h_1,f,g,h)(n) = H_1^\Sigma(f,g,h)\left(\left\lfloor \frac{|f_1(1) - g_1(1)|}{h_1(1)+1} + \frac{1}{2} \right\rfloor, n\right),$$

and using Remark 4.3, we obtain that $(F^\Sigma, G^\Sigma, H^\Sigma)$ is a computing system for θ^Σ, which consists of log-rudimentary operators. It follows that θ^Σ is uniformly **LogRO**-computable, hence uniformly **MSO**-computable (as noted above, by Skordev's theorem). □

Lemma 4.4 can immediately be generalised for real functions $\theta : \mathbb{N} \times D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^l$ with $l > 1$.

5. First theorem on integration

Lemma 5.1. *The real function* tanh *and its derivative*

$$\tanh'(t) = \frac{1}{\cosh^2(t)}$$

are uniformly **MSO***-computable on* $(-\infty, \infty)$.

Proof. For all $t \in \mathbb{R}$ we have the equality

$$\frac{1}{\cosh^2(t)} = 1 - \tanh^2(t).$$

Thus it is enough to consider the real function tanh only. We have

$$\tanh(t) = \frac{\sinh(t)}{\cosh(t)} = \frac{e^t - e^{-t}}{e^t + e^{-t}} = \frac{e^{2t} - 1}{e^{2t} + 1} = 1 - \frac{2}{e^{2t} + 1}.$$

Results from [11] show that the restriction of e^t to $(-\infty, 0]$ and the restriction of the reciprocal function to $(1, +\infty)$ are uniformly **MSO**-computable. Therefore, the restriction of tanh to $(-\infty, 0]$ is uniformly **MSO**-computable. Using the equality

$$\tanh(t) = \tanh(\min(t, 0)) - \tanh(-\max(t, 0))$$

we obtain that tanh is uniformly **MSO**-computable on its whole domain $(-\infty, \infty)$. □

Lemma 5.2. *For any real number* $A > 0$ *there exists a real number* $a \in \left(0, \frac{\pi}{2}\right)$*, such that for all* z *with* $|\operatorname{Im}(z)| < a$ *we have*

$$|\operatorname{Re}(\tanh(z))| \leq 1, \quad |\operatorname{Im}(\tanh(z))| \leq A, \quad |\cosh(z)|^2 \geq \cosh^2(\operatorname{Re}(z)) - \frac{1}{2}.$$

Proof. Let $z = x + ib$ for $x, b \in \mathbb{R}$ and $|b| < \frac{\pi}{2}$. We have

$$\tanh(z) = \tanh(x + ib) = \frac{\tanh(x) + \tanh(ib)}{1 + \tanh(x).\tanh(ib)} = \frac{\tanh(x) + i.\tan(b)}{1 + i.\tanh(x).\tan(b)}$$
$$= \frac{(\tanh(x) + i.\tan(b))\,(1 - i.\tanh(x).\tan(b))}{1 + \tanh^2(x).\tan^2(b)}.$$

Therefore,

$$\operatorname{Re}(\tanh(z)) = \frac{\tanh(x) + \tanh(x).\tan^2(b)}{1 + \tanh^2(x).\tan^2(b)} = \frac{\tanh(x)}{\cos^2(b) + \tanh^2(x).\sin^2(b)}$$

and

$$\operatorname{Im}(\tanh(z)) = \frac{\tan(b) - \tanh^2(x).\tan(b)}{1 + \tanh^2(x).\tan^2(b)} = \frac{\frac{1}{\cosh^2(x)}.\tan(b)}{1 + \tanh^2(x).\tan^2(b)}.$$

Clearly, since $\cosh(x) \geq 1$, $|\operatorname{Im}(\tanh(z))| \leq |\tan(b)|$.

Let $|\tanh(x)| = T$, $\cos^2(b) = \alpha$, $\sin^2(b) = \beta$.

We have $0 \le T < 1,\ \alpha > 0,\ \beta \ge 0,\ \alpha + \beta = 1$. The inequality

$$|\operatorname{Re}(\tanh(z))| \le 1$$

is equivalent to

$$\frac{T}{\alpha + T^2\beta} \le 1 \Longleftrightarrow T \le \alpha + T^2\beta$$

$$\Longleftrightarrow T \le 1 - \beta + T^2\beta \Longleftrightarrow T - 1 \le \beta(T^2 - 1) \Longleftrightarrow \beta(T + 1) \le 1.$$

If $\beta = 0$ (that is $b = 0$), then the last inequality is obviously true. For $\beta > 0$ it is equivalent to

$$T \le \frac{1}{\beta} - 1 \Longleftrightarrow T \le \frac{\alpha}{\beta} = \cot^2(b).$$

Hence $|\operatorname{Re}(\tanh(z))| \le 1$ will certainly be true, if $\cot^2(b) > 1$.

We also have

$$|\cosh(z)|^2 = |\cosh(x + ib)|^2 = |\cosh(x).\cosh(ib) + \sinh(x).\sinh(ib)|^2$$

$$= |\cosh(x).\cos(b) + i.\sinh(x).\sin(b)|^2$$

$$= \cosh^2(x).\cos^2(b) + \sinh^2(x).\sin^2(b) = \cosh^2(x) - \sin^2(b).$$

Therefore, if $\sin^2(b) < \frac{1}{2}$ then

$$|\cosh(z)|^2 \ge \cosh^2(x) - \frac{1}{2} = \cosh^2(\operatorname{Re}(z)) - \frac{1}{2}.$$

Let us fix $A > 0$. By using the limits

$$\lim_{b\to 0} \tan(b) = 0,\quad \lim_{b\to 0} \cot^2(b) = +\infty,\quad \lim_{b\to 0} \sin^2(b) = 0$$

we can choose a real number $a \in \left(0, \frac{\pi}{2}\right)$, such that

$$|\tan(b)| \le A,\quad \cot^2(b) > 1,\quad \sin^2(b) < \frac{1}{2}$$

for all $b \in (-a, a) \setminus \{0\}$. For this choice of a and for all z in the strip $|\operatorname{Im}(z)| < a$ we have

$$|\operatorname{Im}(\tanh(z))| \le A,\quad |\operatorname{Re}(\tanh(z))| \le 1,\quad |\cosh(z)|^2 \ge \cosh^2(\operatorname{Re}(z)) - \frac{1}{2}.$$

□

Theorem 5.3. *Let α,β be $\mathcal{M}^2$-computable real numbers and $\theta : [\alpha,\beta] \to \mathbb{R}$ be uniformly* **MSO**-*computable and analytic real function. Then the definite integral*

$$\int_\alpha^\beta \theta(x)\, dx$$

is an $\mathcal{M}^2$-computable real number.

Proof. We begin by applying the linear change of variables

$$x = \frac{\beta - \alpha}{2}.u + \frac{\beta + \alpha}{2}$$

to the given integral and we obtain

$$\int_{\alpha}^{\beta} \theta(x)\, dx = \frac{\beta - \alpha}{2} \int_{-1}^{1} \theta_1(u)\, du = \frac{\beta - \alpha}{2}.I,$$

where

$$\theta_1(u) = \theta\Big(\frac{\beta - \alpha}{2}.u + \frac{\beta + \alpha}{2}\Big).$$

Of course, since α and β are $\mathcal{M}^2$-computable, θ_1 is uniformly **MSO**-computable and also analytic in $[-1, 1]$. It suffices to prove that the new integral I is $\mathcal{M}^2$-computable.

We apply another change of variables $u = \tanh(t)$ (the so-called *tanh-rule*):

$$I = \int_{-1}^{1} \theta_1(u)\, du = \int_{-\infty}^{\infty} \theta_1(\tanh(t)) \frac{1}{\cosh^2(t)}\, dt.$$

As in Section 5 of [13] we approximate the last integral with the infinite sum

$$I_h = h \sum_{k=-\infty}^{+\infty} \theta_1(\tanh(kh)) \frac{1}{\cosh^2(kh)}.$$

We will choose the step h below. The error $|I_h - I|$ of the approximation is called *discretisation error*. We will apply Theorem 5.1 in [13] to estimate this error. Since θ_1 is analytic in $[-1, 1]$, it has an analytic continuation defined in $[-1, 1] \times [-A, A] \subseteq \mathbb{C}$ for some positive real number A. Let M be an upper bound of $|\theta_1(u)|$ for (complex) $u \in [-1, 1] \times [-A, A]$.

Let us choose a from Lemma 5.2 (corresponding to the choice of A). Then the integrand

$$\theta_1(\tanh(z)) \frac{1}{\cosh^2(z)}$$

is analytic in the strip $|\operatorname{Im}(z)| < a$, because $\tanh(z) \in [-1, 1] \times [-A, A]$ and $\cosh(z) \neq 0$ for z in this strip. Moreover,

$$\left| \theta_1(\tanh(z)) \frac{1}{\cosh^2(z)} \right| \leq \frac{M}{|\cosh(z)|^2} \leq \frac{M}{\cosh^2(\operatorname{Re}(z)) - \frac{1}{2}}$$

for any z, such that $|\operatorname{Im}(z)| < a$. Therefore, the integrand converges to 0 uniformly as $|z| \to \infty$ in the strip $|\operatorname{Im}(z)| < a$. For all $b \in (-a, a)$ we have

$$\int_{-\infty}^{\infty} \left| \theta_1(\tanh(x + ib)) \frac{1}{\cosh^2(x + ib)} \right| dx \leq \int_{-\infty}^{\infty} \frac{M}{|\cosh(x + ib)|^2}\, dx$$

$$\leq M \int_{-\infty}^{\infty} \frac{dx}{\cosh^2(x) - \frac{1}{2}} = 2M \int_{-\infty}^{\infty} \frac{dx}{\cosh(2x)} = M \int_{-\infty}^{\infty} \frac{dx}{\cosh(x)}$$

$$= M \int_{-\infty}^{\infty} \frac{2e^x}{e^{2x} + 1}\, dx = 2M. \arctan(e^x)|_{-\infty}^{\infty} = M\pi.$$

Using Theorem 5.1 in [13] we obtain

$$|I_h - I| \leq \frac{2M\pi}{e^{2\pi a/h} - 1}$$

(for any $h > 0$).

The second step of the approximation of I is to truncate the infinite sum I_h to

$$I_h^{[n]} = h \sum_{k=-n}^{n} \theta_1(\tanh(kh)) \frac{1}{\cosh^2(kh)}$$

for a large enough n. The error $|I_h^{[n]} - I_h|$ is called *truncation error*. We can estimate it as follows:

$$|I_h^{[n]} - I_h| = \left| h \sum_{|k|>n} \theta_1(\tanh(kh)) \frac{1}{\cosh^2(kh)} \right|$$

$$\leq h \sum_{|k|>n} \frac{M}{\cosh^2(kh)} = 4hM. \sum_{|k|>n} \frac{1}{(e^{kh} + e^{-kh})^2}$$

$$= 8hM. \sum_{k>n} \frac{1}{(e^{kh} + e^{-kh})^2} \leq 8hM. \sum_{k>n} \frac{1}{e^{2kh}}$$

$$= 8hM. \frac{1}{e^{2(n+1)h}} . \frac{1}{1 - \frac{1}{e^{2h}}} = \frac{8hM}{(e^{2h} - 1).e^{2nh}} \leq \frac{4M}{e^{2nh}},$$

because $e^{2h} \geq 2h + 1$ for $h > 0$.

Since

$$|I_h^{[n]} - I| \leq |I_h - I| + |I_h^{[n]} - I_h|$$

we must balance the discretisation and the truncation error by choosing a suitable h, depending on n.

For any $n > 0$, let us take $h = \frac{1}{\sqrt{n}}$. Then

$$|I_h - I| \leq \frac{2M\pi}{e^{2\pi a \sqrt{n}} - 1}$$

and

$$|I_h^{[n]} - I_h| \leq \frac{4M}{e^{2\sqrt{n}}}.$$

Therefore

$$|I - I_h^{[n]}| \leq \frac{2M\pi}{e^{2\pi a \sqrt{n}} - 1} + \frac{4M}{e^{2\sqrt{n}}}.$$

We choose $C, E \in \mathbb{N}, C > 0$, such that $\frac{1}{C} < 2\pi a$ and $E > 2M(\pi + 2)$. Then it is easy to see that

$$|I - I_h^{[n]}| \leq \frac{E}{e^{\frac{1}{C}\sqrt{n}} - 1}$$

for all $n > 0$. This convergence is fast enough to be suitable for introducing a log-bounded sum. We replace n with $L(n)^2$ and h with $\dfrac{1}{L(n)}$, accordingly. Since $e^{L(n)} \geq \frac{n+1}{2}$, we obtain that

$$|I - I_h^{[L(n)^2]}| \leq \frac{E}{\frac{n+1}{2}^{\frac{1}{C}} - 1}$$

for all $n > 0$, hence the inequality

$$|I - I_h^{[L(n)^2]}| \leq \frac{1}{t+1}$$

holds for all $t \in \mathbb{N}$ and $n = p(t) = (2Et + 2E + 1)^C$, where $p \in \mathcal{M}^2$.

It remains to extract a name of I from $I_h^{[L(n)^2]}$. This can be done in the following way: for any real number $h > 0$

$$I_h^{[L(n)^2]} = h \sum_{k=-L(n)^2}^{L(n)^2} \theta_1(\tanh(kh)) \frac{1}{\cosh^2(kh)} = h \sum_{k=0}^{L(n)^2} \theta_1(\tanh(kh)) \frac{1}{\cosh^2(kh)} +$$

$$+h \sum_{k=-L(n)^2}^{0} \theta_1(\tanh(kh)) \frac{1}{\cosh^2(kh)} - h\theta_1(\tanh(kh)) \frac{1}{\cosh^2(kh)}\bigg|_{k=0}$$

$$= h \sum_{k=0}^{L(n)^2} \theta_1(\tanh(kh)) \frac{1}{\cosh^2(kh)} + h \sum_{k=0}^{L(n)^2} \theta_1(-\tanh(kh)) \frac{1}{\cosh^2(kh)} - h\theta_1(0).$$

In both sums, the summands are uniformly **MSO**-computable real functions of $k \in \mathbb{N}$ and $h \in (0, +\infty)$. This easily follows from Lemma 5.1 and from the fact that θ_1 is uniformly **MSO**-computable in $[-1, 1]$. Using Lemma 4.4 (with $k = 2$), we obtain that these sums are uniformly **MSO**-computable in $n \in \mathbb{N}$ and $h > 0$. Therefore by substituting $h = \dfrac{1}{L(n)}$ ($L \in \mathcal{M}^2$) we obtain that $I_h^{[L(n)^2]}$ is uniformly **MSO**-computable in $n \in \mathbb{N}, n > 0$.

Using a computing system for $I_h^{[L(n)^2]}$, consisting of $(3, 1)$-operators from **MSO**, and by applying Remark 4.3 we can choose binary functions $a, b, c \in \mathcal{M}^2$, such that for all natural numbers $n > 0$ and t,

$$\left| \frac{a(n,t) - b(n,t)}{c(n,t) + 1} - I_h^{[L(n)^2]} \right| < \frac{1}{t+1}.$$

Let us define the functions $f, g, h \in \mathcal{T}_1$ by the equalities

$$f(t) = a(p(2t+1), 2t+1), \quad g(t) = b(p(2t+1), 2t+1), \quad h(t) = c(p(2t+1), 2t+1).$$

Of course, f, g and h belong to $\mathcal{M}^2$. For all $t \in \mathbb{N}$ and $n = p(2t+1) > 0$ we have

$$\left| I - \frac{f(t) - g(t)}{h(t) + 1} \right| \leq \left| I - I_h^{[L(n)^2]} \right| + \left| I_h^{[L(n)^2]} - \frac{a(n, 2t+1) - b(n, 2t+1)}{c(n, 2t+1) + 1} \right|$$

$$< \frac{1}{2t+2} + \frac{1}{2t+2} = \frac{1}{t+1}.$$

Therefore, (f, g, h) is a name of I and I is $\mathcal{M}^2$-computable. □

6. Second theorem on integrals with parameters

Theorem 6.1. *Let α, β be $\mathcal{M}^2$-computable real numbers, $D \subseteq \mathbb{R}$ be a set and $\theta : [\alpha, \beta] \times D \to \mathbb{R}$ be a real function, which is uniformly* **MSO***-computable. Let there exist $A \in \mathbb{R}$, $A > 0$, such that for every fixed $\xi \in D$, θ has an analytic continuation defined in $[\alpha, \beta] \times [-A, A] \subseteq \mathbb{C}$. Let there also exist a polynomial P with natural coefficients, such that $|\theta(x + Bi, \xi)| \le P(|\xi|)$ for all $\xi \in D, x \in [\alpha, \beta], B \in [-A, A]$. Then the real function $I : D \to \mathbb{R}$ defined by*

$$I(\xi) = \int_\alpha^\beta \theta(x, \xi)\, dx$$

is uniformly **MSO***-computable.*

Proof. The proof follows the same argument as in Theorem 5.3. Roughly speaking, we just add a real parameter ξ to all formulas. There is a subtle difference: the A that we choose in the proof of Theorem 5.3 might be different for different values of $\xi \in D$ and this is why we assume there exists one A, which works for all $\xi \in D$.

After the linear change of variables

$$x = \frac{\beta - \alpha}{2}.u + \frac{\beta + \alpha}{2}$$

we obtain a real function $\theta_1 : [-1, 1] \times D \to \mathbb{R}$,

$$\theta_1(u, \xi) = \theta\Big(\frac{\beta - \alpha}{2}.u + \frac{\beta + \alpha}{2}, \xi\Big),$$

which is uniformly **MSO**-computable and for any fixed $\xi \in D$, θ_1 is analytic in $[-1, 1] \times [-A', A'] \subseteq \mathbb{C}$ for $A' = \dfrac{2A}{\beta - \alpha}$. For all (complex) $u \in [-1, 1] \times [-A', A']$ we have $\theta_1(u, \xi) = \theta(x + iB, \xi)$ for some $x \in [\alpha, \beta], B \in [-A, A]$, therefore

$$|\theta_1(u, \xi)| \le P(|\xi|),$$

where P is the polynomial from the statement of the theorem.

Since

$$I(\xi) = \frac{\beta - \alpha}{2} \int_{-1}^1 \theta_1(u, \xi)\, du$$

and α, β are $\mathcal{M}^2$-computable, it suffices to consider the integral

$$J(\xi) = \int_{-1}^1 \theta_1(u, \xi)\, du$$

and prove that $J : D \to \mathbb{R}$ is uniformly **MSO**-computable.

Let us fix $\xi \in D$ and an arbitrary name $(f', g', h') \in \mathcal{T}_1^3$ of ξ. As in the proof of Theorem 5.3 we apply the *tanh*-rule

$$J(\xi) = \int_{-1}^1 \theta_1(u, \xi)\, du = \int_{-\infty}^{+\infty} \theta_1(\tanh(t), \xi) \frac{1}{\cosh^2(t)}\, dt$$

and then discretise this integral to

$$J_h(\xi) = h \sum_{k=-\infty}^{+\infty} \theta_1(\tanh(kh), \xi) \frac{1}{\cosh^2(kh)}.$$

To estimate the discretisation error we use Lemma 5.2 and choose a, corresponding to A' (and therefore not depending on ξ). Since $P(|\xi|)$ is an upper bound of $|\theta_1(u, \xi)|$ for any (complex) $u \in [-1, 1] \times [-A', A']$, we obtain in exactly the same way

$$|J_h(\xi) - J(\xi)| \leq \frac{2P(|\xi|)\pi}{e^{2\pi a/h} - 1}$$

(for any $h > 0$).

Next we truncate the infinite sum to

$$J_h^{[n]}(\xi) = h \sum_{k=-n}^{n} \theta_1(\tanh(kh), \xi) \frac{1}{\cosh^2(kh)}$$

for a large enough n. The estimate of the truncation error is

$$\left| J_h^{[n]}(\xi) - J_h(\xi) \right| \leq \frac{4P(|\xi|)}{e^{2nh}}.$$

To balance the two errors we put $h = \dfrac{1}{\sqrt{n}}$ for $n > 0$ and obtain

$$|J(\xi) - J_h^{[n]}(\xi)| \leq \frac{2P(|\xi|)\pi}{e^{2\pi a\sqrt{n}} - 1} + \frac{4P(|\xi|)}{e^{2\sqrt{n}}}.$$

We can again choose a non-zero natural number C, such that $\dfrac{1}{C} < 2\pi a$ (not depending on ξ). But the choice of E will depend on ξ. Since (f', g', h') is a name of ξ we have

$$|\xi| < |f'(0) - g'(0)| + 1$$

and we can choose

$$E = 11P(|f'(0) - g'(0)| + 1).$$

For this choice of E we obviously have $E > 2P(|\xi|)(\pi + 2)$. Moreover, E is obtained from f', g', h' using an operator from **MSO**.

We obtain the inequality

$$|J(\xi) - J_h^{[L(n)^2]}(\xi)| \leq \frac{1}{t+1}$$

for all $t \in \mathbb{N}$ and $n = U(f', g', h')(t) = (2Et + 2E + 1)^C$, where U is a $(3, 1)$-operator from **MSO** (and $h = \dfrac{1}{L(n)}$).

Using the fact that $\theta_1 : [-1, 1] \times D \to \mathbb{R}$ is uniformly **MSO**-computable and by applying Lemma 4.4 (for $k = 2$ and two parameters h, ξ), we obtain that $J_h^{[L(n)^2]}$ is uniformly **MSO**-computable in $n \in \mathbb{N}, n > 0$ and $\xi \in D$ (after substituting $h = \dfrac{1}{L(n)}$).

It remains to extract a computing system for J from $J_h^{[L(n)^2]}$. Using Remark 4.3 we can choose $(3,2)$-operators $F_1, G_1, H_1 \in \mathbf{MSO}$, such that for all natural numbers $n > 0$ and t, any $\xi \in D$ and any name (f', g', h') of ξ we have

$$\left|\frac{F_1(f',g',h')(n,t) - G_1(f',g',h')(n,t)}{H_1(f',g',h')(n,t)+1} - J_h^{[L(n)^2]}(\xi)\right| < \frac{1}{t+1}.$$

Let us define the $(3,1)$-operators F, G, H by the equalities

$$F(f',g',h')(t) = F_1(f',g',h')(U(f',g',h')(2t+1), 2t+1),$$

$$G(f',g',h')(t) = G_1(f',g',h')(U(f',g',h')(2t+1), 2t+1),$$

$$H(f',g',h')(t) = H_1(f',g',h')(U(f',g',h')(2t+1), 2t+1).$$

Obviously, F, G and H belong to $\mathbf{MSO}$. For any $\xi \in D$, any name (f', g', h') of ξ and natural numbers t and $n = U(f', g', h')(2t+1) > 0$ we have

$$\left|J(\xi) - \frac{F(f',g',h')(t) - G(f',g',h')(t)}{H(f',g',h')(t)+1}\right|$$

$$\leq \left|J(\xi) - J_h^{[L(n)^2]}(\xi)\right| + \left|J_h^{[L(n)^2]}(\xi) - \frac{F_1(f',g',h')(n,2t+1) - G_1(f',g',h')(n,2t+1)}{H_1(f',g',h')(n,2t+1)+1}\right|$$

$$< \frac{1}{2t+2} + \frac{1}{2t+2} = \frac{1}{t+1}.$$

Therefore, (F, G, H) is a computing system of $J : D \to \mathbb{R}$ and J is uniformly $\mathbf{MSO}$-computable. □

The proof of Theorem 6.1 easily extends to the case of a definite integral with more than one real parameter ξ, that is for real functions $\theta : [\alpha, \beta] \times D \to \mathbb{R}$, such that $D \subseteq \mathbb{R}^l$ for $l > 1$. Of course, the polynomial P will have l variables in this case.

7. Third theorem on integrals with varying limits

Theorem 7.1. *Let α be an $\mathcal{M}^2$-computable real number, D be an interval of the form $[\alpha, \beta)$, $[\alpha, \beta]$ or $[\alpha, +\infty)$ and $\theta : D \to \mathbb{R}$ be a real function, which is uniformly* **MSO***-computable. Let there exist a real number $A > 0$, such that for any fixed $\xi \in D$, θ has an analytic continuation to the set $D_\xi = [\alpha, \xi] \times [A(\alpha - \xi), A(\xi - \alpha)] \subseteq \mathbb{C}$. Let there also exist a polynomial P with natural coefficients, such that $|\theta(x + Bi)| \leq P(|\xi|)$ for all $\xi \in D$ and $(x, B) \in D_\xi$. Then the real function $I : D \to \mathbb{R}$ defined by*

$$I(\xi) = \int_\alpha^\xi \theta(x)\, dx,$$

is uniformly **MSO***-computable.*

Proof. For any fixed $\xi \in D$ we apply the linear change of variables

$$x = \frac{\xi - \alpha}{2}.u + \frac{\xi + \alpha}{2}$$

to the given integral and we obtain

$$\int_\alpha^\xi \theta(x)\,dx = \frac{\xi-\alpha}{2}\int_{-1}^1 \theta_1(u,\xi)\,du,$$

where

$$\theta_1(u,\xi) = \theta\Big(\frac{\xi-\alpha}{2}.u + \frac{\xi+\alpha}{2}\Big).$$

We will prove that the new integral is uniformly **MSO**-computable as a function of $\xi \in D$. From this it follows easily that I is uniformly **MSO**-computable.

Of course, since α is $\mathcal{M}^2$-computable and θ is uniformly **MSO**-computable in D, θ_1 is uniformly **MSO**-computable in $u \in [-1,1]$ and $\xi \in D$.

For any $u \in [-1,1]\times[-2A,2A] \subseteq \mathbb{C}$ we have that $\theta_1(u,\xi) = \theta(x+Bi)$ for some $x \in [\alpha,\xi]$ and $B = \dfrac{\xi-\alpha}{2}.\,\mathrm{Im}(u)$, $|B| \le A(\xi-\alpha)$, that is for some $(x,B) \in D_\xi$. Therefore, for any fixed $\xi \in D$, θ_1 is analytic in $[-1,1]\times[-2A,2A] \subseteq \mathbb{C}$. Moreover,

$$|\theta_1(u,\xi)| = |\theta(x+Bi)| \le P(|\xi|)$$

for any $u \in [-1,1]\times[-2A,2A] \subseteq \mathbb{C}$ (again since $(x,B) \in D_\xi$). It remains to apply Theorem 6.1 for the real function θ_1. □

It is clear that Theorem 7.1 is also true for an interval D of the form $(\beta,\alpha]$, $[\beta,\alpha]$ or $(-\infty,\alpha]$. Of course, $D_\xi = [\xi,\alpha]\times[A(\xi-\alpha),A(\alpha-\xi)]$ in this case.

Using Theorem 7.1 and the fact that

$$\arctan'(x) = \frac{1}{1+x^2},$$

a much simpler proof for the uniform **MSO**-computability of the arctan real function can be given than the proof in [11].

Corollary 7.2. *In the assumptions of Theorem 7.1 with $D = [\alpha,+\infty)$, let the improper integral*

$$I = \int_\alpha^\infty \theta(x)\,dx$$

be convergent. Moreover, let there exist a function $r \in \mathcal{M}^2$, such that

$$\left|\int_{n+\alpha}^\infty \theta(x)\,dx\right| \le \frac{1}{t+1}$$

for all $t \in \mathbb{N}$ and $n = r(t)$. Then I is $\mathcal{M}^2$-computable.

Proof. We define the sequence I_n by

$$I_n = \int_\alpha^{n+\alpha} \theta(x)\,dx.$$

According to Theorem 7.1, this sequence is uniformly $\mathcal{M}^2$-computable in $n \in \mathbb{N}$ (since α is $\mathcal{M}^2$-computable). Therefore, by Remark 4.3 we can choose functions $a,b,c \in \mathcal{T}_2 \cap \mathcal{M}^2$, such that for all $n,t \in \mathbb{N}$ we have

$$\left|\frac{a(n,t)-b(n,t)}{c(n,t)+1} - I_n\right| < \frac{1}{t+1}.$$

We also have

$$|I - I_n| = \left|\int_{n+\alpha}^{\infty} \theta(x)\, dx\right| \le \frac{1}{2t+2}$$

for all $t \in \mathbb{N}$ and $n = r(2t+1)$.

Let us define the functions $f, g, h \in \mathcal{T}_1$ by the equalities

$$f(t) = a(r(2t+1), 2t+1), \quad g(t) = b(r(2t+1), 2t+1), \quad h(t) = c(r(2t+1), 2t+1).$$

Of course, f, g and h belong to $\mathcal{M}^2$. For all natural numbers t and $n = r(2t+1)$ we have

$$\left|I - \frac{f(t) - g(t)}{h(t)+1}\right| \le |I - I_n| + \left|I_n - \frac{a(n, 2t+1) - b(n, 2t+1)}{c(n, 2t+1) + 1}\right| < \frac{1}{2t+2} + \frac{1}{2t+2} = \frac{1}{t+1}.$$

Thus (f, g, h) is a name of I, hence the real number I is $\mathcal{M}^2$-computable. □

8. $\mathcal{M}^2$-computability of the Euler-Mascheroni constant γ

We will apply Corollary 7.2 to answer positively an open question from [11], regarding the Euler-Mascheroni constant γ.

The following representation is well-known

$$-\gamma = \int_0^{\infty} e^{-x} \ln x\, dx.$$

Theorem 8.1. *The constant γ is $\mathcal{M}^2$-computable.*

Proof. We have

$$\int_0^{\infty} e^{-x} \ln x\, dx = I_1 + I_2,$$

where

$$I_1 = \int_0^1 e^{-x} \ln x\, dx, \quad I_2 = \int_1^{\infty} e^{-x} \ln x\, dx.$$

It suffices to show that I_1 and I_2 are $\mathcal{M}^2$-computable.

By the change of variables $x = \frac{1}{t}$, the integral I_1 transforms to

$$I_1 = -\int_1^{\infty} e^{-\frac{1}{t}} \ln t \frac{1}{t^2}\, dt.$$

Results from [11] show that the restrictions of $\ln t$ and $\frac{1}{t^2}$ to the interval $[1, +\infty)$, as well as the restriction of e^x to $[-1, 0)$, are uniformly **MSO**-computable. Therefore, the integrand is uniformly **MSO**-computable in $[1, +\infty)$. Moreover, it has an analytic continuation $\theta(z)$, defined in the half-plane $\mathrm{Re}(z) > 0$ (assuming the principal value of the logarithm with branch cut the non-negative real numbers). Let us choose $A = 1$ (in fact any choice of $A > 0$ will do). Let $\xi \ge 1$ and $z = x + Bi$, where $1 \le x \le \xi$ and $|B| \le A(\xi - 1) = \xi - 1$. We have

$$|\theta(z)| = \left|e^{-\frac{1}{z}}\right| . |\ln z| . \left|\frac{1}{z^2}\right| .$$

Since

$$|e^{-\frac{1}{z}}| = e^{\mathrm{Re}(-\frac{1}{z})} = e^{-\frac{x}{x^2+B^2}} < 1,$$

$$\left|\frac{1}{z^2}\right| = \frac{1}{|z|^2} = \frac{1}{x^2+B^2} \leq \frac{1}{1+B^2} \leq 1$$

and (due to the inequality $\ln r < r$ for all real numbers $r \geq 1$)

$$|\ln z| = \sqrt{\ln^2 |z| + \mathrm{Arg}^2\, z} \leq |\ln |z|| + |\,\mathrm{Arg}\, z|$$

$$\leq \frac{1}{2}\ln(x^2+B^2) + \pi < \frac{1}{2}\ln(2\xi^2) + 4 = \frac{1}{2}\ln 2 + \ln\xi + 4 < \ln\xi + 5 < \xi + 5.$$

So

$$|\theta(z)| \leq 1.(\xi+5).1 = \xi + 5$$

and we can take $P(\xi) = \xi + 5$. The assumptions of Theorem 7.1 are satisfied and to apply Corollary 7.2 we need to estimate the remainder of I_1. Let us choose a non-zero natural number C, such that

$$\left|e^{-\frac{1}{t}} \ln t \frac{1}{t^2}\right| \leq \frac{C}{t\sqrt{t}}$$

for all real numbers $t \geq 1$. We have

$$\left|\int_{n+1}^{\infty} e^{-\frac{1}{t}} \ln t \frac{1}{t^2}\, dt\right| \leq C\int_{n+1}^{\infty} \frac{dt}{t\sqrt{t}} = \frac{2C}{\sqrt{n+1}} = \frac{1}{t+1}$$

for all $t \in \mathbb{N}$ and $n = (2Ct + 2C)^2 - 1 = r_1(t)$, where $r_1 \in \mathcal{M}^2$. So the integral is $\mathcal{M}^2$-computable and therefore the same is true for I_1.

Results from [11] show that the restriction of $\ln x$ to $[1, +\infty)$ and the restriction of e^x to $(-\infty, -1]$ are uniformly **MSO**-computable. Therefore, the integrand of I_2 is uniformly **MSO**-computable in $[1, +\infty)$. Moreover, this integrand has an analytic continuation $\theta(z)$, defined in the half-plane $\mathrm{Re}(z) > 0$. As for I_1, we can take any value for A, for example $A = 1$. Let $\xi \geq 1$ and $z = x + Bi$, where $1 \leq x \leq \xi$ and $|B| \leq A(\xi - 1) = \xi - 1$. We have

$$|\theta(z)| = \left|e^{-z}\right|.|\ln z|.$$

Since

$$|e^{-z}| = e^{\mathrm{Re}(-z)} = e^{-x} < 1,$$

and (exactly as above)

$$|\ln z| < \xi + 5,$$

we obtain

$$|\theta(z)| \leq 1.(\xi+5) = \xi + 5$$

and we can take $P(\xi) = \xi + 5$. In order to apply Corollary 7.2 we estimate the remainder of I_2. Let us choose a non-zero natural number D, such that

$$\left|e^{-x}\ln x\right| \leq \frac{D}{x^2}$$

for all real numbers $x \geq 1$. We have

$$\left|\int_{n+1}^{\infty} e^{-x}\ln x\, dx\right| \leq D\int_{n+1}^{\infty} \frac{dx}{x^2} = \frac{D}{n+1} = \frac{1}{t+1}$$

for all $t \in \mathbb{N}$ and $n = Dt + D - 1 = r_2(t)$, where $r_2 \in \mathcal{M}^2$. So I_2 is $\mathcal{M}^2$-computable. □

9. Conclusion

By a more careful estimation of the error of approximation, an actual sequence can be extracted from the proof in the previous section, which converges to γ with subexponential rate. Of course, there are much faster and more effective methods for the computation of (the digits of) γ, but none of them appears to be appropriate in the subrecursive setting.

In order to compute the elementary functions of calculus on their whole domains, a more general non-uniform notion for computability of real functions is studied in [3, 10], called *conditional computability* of a real function (with respect to a class of operators). In a future research we plan to extend the results on integration from the paper to this broader class of real functions.

The study of subrecursive computability in analysis is still near its beginning. Much more progress has been made on computational complexity in analysis with respect to the discrete complexity classes $P, NP, \ldots$. Any question on complexity in analysis can be asked and studied with respect to the subrecursive classes. The connection between the two approaches for estimating complexity is not at all clear. So far it appears that the topics in the subrecursive setting require separate study, usually using methods that are quite independent from the methods in the discrete complexity setting. This is convincing evidence that the area is fruitful and it should be studied on a larger scale.

Acknowledgments

This paper is supported by the Bulgarian National Science Fund through the project "Models of computability", DN-02-16/19.12.2016.

References

[1] H. Friedman, On the computational complexity of maximization and integration. Adv. Math. 53 (1984) 80–98.
[2] I. Georgiev, Computing real functions with rudimentary operators, Ann. "Prof. d-r Asen Zlatarov" Univ. 44 (1) (2015) 9–14.
[3] I. Georgiev, D. Skordev, Conditional computability of real functions with respect to a class of operators, Ann. of Pure and Appl. Logic 164 (2013) 550–565.
[4] K. Ko, Complexity Theory of Real Functions, Birhäuser-Boston, Cambridge, MA, 1991.
[5] K. Ko, H. Friedman, Computational complexity of real functions, Theoret. Comput. Sci. 20 (1982) 323–352.
[6] K. Ko, H. Friedman, Computing power series in polynomial time, Adv. Appl. Math. 9 (1988) 40–50.
[7] J. Paris, A. Wilkie, A. Woods, Provability of the pigeonhole principle and the existence of infinitely many primes, J. Symb. Logic 53 (4) (1998) 1235–1244.
[8] D. Skordev, Computability of real numbers by using a given class of functions in the set of the natural numbers, Math. Logic Quart. 48 (1) (2002) 91–106.
[9] D. Skordev, On some computability notions for real functions, Computability 2 (2013) 67–73.
[10] D. Skordev, I. Georgiev, On a relative computability notion for real functions, Lecture Notes in Comput. Sci. 6735 (2011) 270–279.
[11] D. Skordev, A. Weiermann, I. Georgiev, $\mathcal{M}^2$-computable real numbers, J. Logic Comp. 22 (4) (2012) 899–925.
[12] H. Takahasi, M. Mori, Quadrature formulas obtained by variable transformation, Numer. Math. 21 (1973) 206–219.
[13] L. Trefethen, J. Weideman, The exponentially convergent trapezoidal rule, Soc. Ind. Appl. Math. 56 (3) (2014) 385–458.