On representations of irrational numbers in subrecursive context

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Acknowledgements

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Our aim is to compare the complexity of these representations. Our framework for complexity is subrecursive: roughly speaking, a computation is *subrecursive* if the number of iterations in any cycle can be computed in advance, before executing the cycle. Thus unbounded search is not allowed in a subrecursive computation.

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We assume some elementary coding of $\mathbb{Z}$ and $\mathbb{Q}$ into $\mathbb{N}$.

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The function $T : \mathbb{Q} \to \mathbb{Q}$ is a *trace function* for ξ if and only if for all $q \in \mathbb{Q}$

$$|T(q) - \xi| < |q - \xi|.$$

Recursive real numbers

Proposition

The following are equivalent for an irrational number ξ:

- *there exists a computable Cauchy sequence for ξ;*
- *the Dedekind cut of ξ is computable;*
- *there exists a computable trace function for ξ.*

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Finally, by taking

$$q_n = M.D_1 D_2 \ldots D_n$$

we obtain a Cauchy sequence for ξ.

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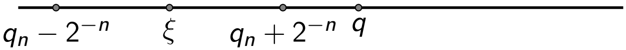

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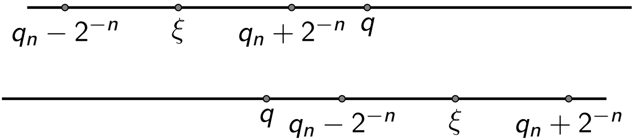

Dedekind cuts and trace functions

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But conversely, given the Dedekind cut of ξ is not possible to obtain a trace function T subrecursively.
Given $q \in \mathbb{Q}$, an unbounded search is needed to find $T(q) \in \mathbb{Q}$, such that $q < T(q) < \xi$ or $\xi < T(q) < q$.

Formal result

For a class $\mathcal{S}$ of functions, we denote by $\mathcal{S}_T, \mathcal{S}_D, \mathcal{S}_C$ the set of all real numbers, which have a trace function, Dedekind cut or Cauchy sequence in $\mathcal{S}$, respectively.

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Theorem (Kristiansen)

For any subrecursive class $\mathcal{S}$, closed under elementary operations we have

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Base-b expansions

From now on we consider only irrational numbers $\xi \in (0, 1)$.

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Theorem (Mostowski, Kristiansen)

For any subrecursive class $\mathcal{S}$, closed under elementary operations and any two bases a, b we have

$$\mathcal{S}_{bE} \subseteq \mathcal{S}_{aE} \iff \textit{every prime factor of } a \textit{ is a prime factor of } b.$$

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$\mathcal{E}^2$-irrationality

Definition

A real number ξ will be called *$\mathcal{E}^2$-irrational* iff there exists a function $v \in \mathcal{E}^2$, such that

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Theorem

For a subrecursive class $\mathcal{S}$, closed under elementary operations and a real number ξ, which is $\mathcal{E}^2$-irrational we have:

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Let R any of the known representations, not equivalent to Cauchy sequences and $\mathcal{S}_R$ be the set of irrational numbers, which possess an R-representation, computable through functions from $\mathcal{S}$.

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and it follows from the last two theorems that $\mathcal{S}_R$ is not closed under addition or multiplication.

Base-b sum approximations from below

Let us fix some base b. Any irrational number $\xi \in (0,1)$ can be written in the form

$$\xi = \frac{d_1}{b^{k_1}} + \frac{d_2}{b^{k_2}} + \frac{d_3}{b^{k_3}} + \dots,$$

where k_n is a strictly increasing sequence of positive integers and d_n are non-zero base-b digits, $d_n \in \{1, \dots, b-1\}$.

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For a class of functions $\mathcal{S}$ we denote by $\mathcal{S}_{b\uparrow}$ the set of all real numbers, which have a base-b sum approximation from below in $\mathcal{S}$, that is

$$\xi \in \mathcal{S}_{b\uparrow} \iff \hat{A}_b^\xi \in \mathcal{S}.$$

Base-b sum approximations from above

Moreover, we can write

$$\xi = 1 - \frac{d'_1}{b^{m_1}} - \frac{d'_2}{b^{m_2}} - \frac{d'_3}{b^{m_3}} - \dots,$$

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$$\hat{A}^{\xi}_{10}(1) = \frac{5}{10^2}, \quad \hat{A}^{\xi}_{10}(2) = \frac{4}{10^3}, \quad \ldots$$

Moreover,

$$\xi = 1 - \frac{9}{10^1} - \frac{4}{10^2} - \frac{5}{10^3} - \frac{6}{10^4} - \frac{9}{10^8} - \frac{9}{10^9} - \frac{9}{10^{10}} - \frac{6}{10^{11}} + \ldots,$$

thus

$$\check{A}^{\xi}_{10}(1) = \frac{9}{10^1}, \quad \check{A}^{\xi}_{10}(2) = \frac{4}{10^2}, \quad \ldots$$

Results on sum approximations

Theorem (Kristiansen)

Let $\mathcal{S}$ be a subrecursive class, closed under elementary operations. For any base b we have

$$\mathcal{S}_{b\uparrow} \not\subseteq \mathcal{S}_{b\downarrow} \text{ and } \mathcal{S}_{b\downarrow} \not\subseteq \mathcal{S}_{b\uparrow}.$$

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It easily follows that

$$\mathcal{S}_{b\uparrow} \subset \mathcal{S}_{bE} \text{ and } \mathcal{S}_{b\downarrow} \subset \mathcal{S}_{bE}.$$

Arbitary base sum approximation

Let us denote $\mathcal{S}_{\uparrow} = \bigcup_{b=2}^{\infty} \mathcal{S}_{b\uparrow}, \;\; \mathcal{S}_{\downarrow} = \bigcup_{b=2}^{\infty} \mathcal{S}_{b\downarrow}$.

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Theorem

For any finite set of primes $\{p_1, \ldots, p_k\}$ and prime p not belonging to the set there exists a real number ξ, such that $\xi \in \mathcal{S}_{p_1\uparrow} \cap \ldots \cap \mathcal{S}_{p_k\uparrow}$ and $\xi \notin \mathcal{S}_{p\uparrow}$.

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But actually the conjecture is false:

Theorem

There exist a real number $\xi \in \mathcal{S}_D$, such that $\xi \notin \mathcal{S}_{\uparrow}$.

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Thank you for your attention!