On the complexity of irrational numbers under different representations

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Acknowledgements

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Our aim is to compare the complexity of these representations. Our framework for complexity is subrecursive: roughly speaking, a computation is *subrecursive* if the number of iterations in any cycle can be computed in advance, just before executing the cycle. Thus unbounded search is not allowed in a subrecursive computation.

Representations of real numbers

Let ξ be an irrational number.

Definition

The function $C : \mathbb{N} \to \mathbb{Q}$ is a *Cauchy sequence* for ξ if and only if for all $n \in \mathbb{N}$

$$|C(n) - \xi| < \frac{1}{2^n}.$$

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The function $T : \mathbb{Q} \to \mathbb{Q}$ is a *trace function* for ξ if and only if for all $q \in \mathbb{Q}$

$$|T(q) - \xi| < |q - \xi|.$$

Subrecursive classes

Let ϕ, ψ be total functions in the natural numbers.
We say that ϕ is *elementary in* ψ ($\phi \leq_E \psi$) if and only if ϕ can be generated from ψ and the initial functions (projections, constants, successor, $\lambda n.2^n$) using elementary operations (composition and bounded primitive recursion).

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We say that ϕ is *primitive recursive in* ψ ($\phi \leq_{PR} \psi$) if and only if ϕ can be generated from ψ and the initial functions using primitive recursive operations (composition and (unbounded) primitive recursion).
The function ϕ is *primitive recursive* if and only if $\phi \leq_{PR} 0$.
We assume some coding of $\mathbb{Z}$ and $\mathbb{Q}$ into the natural numbers. Under this coding all of the usual basic operations will be elementary. Thus we allow $\mathbb{Z}$ or $\mathbb{Q}$ in place of $\mathbb{N}$ for the arguments and/or the result of the functions ϕ, ψ.

Subrecursive classes II

A function $h : \mathbb{N} \to \mathbb{N}$ is *honest* if h is monotonically increasing ($h(n) \leq h(n+1)$), dominates $\lambda n.2^n$ ($h(n) \geq 2^n$) and has elementary graph (the relation $f(x) = y$ is elementary).

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For any subrecursive class $\mathcal{S}$ there exists an honest function f, such that $f \notin \mathcal{S}$.

Roughly speaking, the graph of f is easily computable, but f grows too fast to belong ot $\mathcal{S}$.

Recursive real numbers

Proposition

The following are equivalent for an irrational number ξ:

- *there exists a computable Cauchy sequence for ξ;*
- *the Dedekind cut of ξ is computable;*
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The proof is uniform in ξ, because ξ is assumed irrational.
Without this assumption the uniformity is lost, but the proposition remains true.
From our viewpoint any representation of the rational numbers is considered trivial.

From Dedekind cut to Cauchy sequence

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The remaining digits to the right of the decimal point of ξ can be computed subrecursively.
Finally, by taking q_n to be the rational number, consisting of the whole part of ξ and its first n digits after the decimal point, we obtain a Cauchy sequence for ξ.

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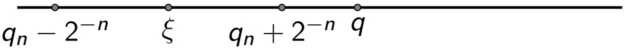

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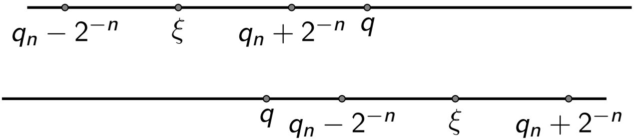

Dedekind cuts and trace functions

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But conversely, given the Dedekind cut of ξ is not possible to obtain a trace function T subrecursively.
Given $q \in \mathbb{Q}$, an unbounded search is needed to find $T(q) \in \mathbb{Q}$, such that $q < T(q) < \xi$ or $\xi < T(q) < q$.

First result

For a class $\mathcal{S}$ of functions, we denote by $\mathcal{S}_T, \mathcal{S}_D, \mathcal{S}_C$ the set of all real numbers, which have a trace function, Dedekind cut or Cauchy sequence in $\mathcal{S}$, respectively.

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Theorem (Kristiansen)

For any subrecursive class $\mathcal{S}$, closed under elementary operations we have

$$\mathcal{S}_T \subset \mathcal{S}_D \subset \mathcal{S}_C.$$

Base-b expansions

From now on we consider only irrational numbers $\xi \in (0, 1)$.

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For any base $b \geq 2$, there exists a unique sequence $D_1 D_2 \dots D_n \dots$ of b-digits, such that

$$0.D_1 D_2 \dots D_n < \xi < 0.D_1 D_2 \dots D_n + \frac{1}{b^n}$$

for any number $n \in \mathbb{N}$.

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Theorem (Mostowski, Kristiansen)

For any subrecursive class $\mathcal{S}$, closed under elementary operations and any two bases a, b we have

$$\mathcal{S}_{bE} \subseteq \mathcal{S}_{aE} \iff \textit{every prime factor of } a \textit{ is a prime factor of } b.$$

Base-b expansions II

It follows trivially that for any base b there exists base a, such that $\mathcal{S}_{aE}$ and $\mathcal{S}_{bE}$ are not comparable with respect to inclusion.

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But from the considerations above we have $\mathcal{S}_D \subseteq \mathcal{S}_{aE} \subseteq \mathcal{S}_C$, therefore

$$\mathcal{S}_T \subset \mathcal{S}_D \subset \mathcal{S}_{bE} \subset \mathcal{S}_C$$

for any subrecursive class $\mathcal{S}$, closed under elementary operations and any base b.

Base-b sum approximations from below

Let us fix some base b. Any irrational number $\xi \in (0,1)$ can be written in the form

$$\xi = \frac{d_1}{b^{k_1}} + \frac{d_2}{b^{k_2}} + \frac{d_3}{b^{k_3}} + \dots,$$

where k_n is a strictly increasing sequence of positive integers and d_n are non-zero base-b digits, $d_n \in \{1, \dots, b-1\}$.

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Definition

The function $\hat{A}_b^{\xi}$, defined by $\hat{A}_b^{\xi}(n) = d_n b^{-k_n}$ for $n > 0$ and $\hat{A}_b^{\xi}(0) = 0$ is called *base-b sum approximation from below* of the number ξ.

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For a class of functions $\mathcal{S}$ we denote by $\mathcal{S}_{b\uparrow}$ the set of all real numbers, which have a base-b sum approximation from below in $\mathcal{S}$, that is

$$\xi \in \mathcal{S}_{b\uparrow} \iff \hat{A}_b^{\xi} \in \mathcal{S}.$$

Base-b sum approximations from above

Moreover, we can write

$$\xi = 1 - \frac{d'_1}{b^{m_1}} - \frac{d'_2}{b^{m_2}} - \frac{d'_3}{b^{m_3}} - \dots,$$

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Results on sum approximations

Theorem (Kristiansen)

Let $\mathcal{S}$ be a subrecursive class, closed under elementary operations. For any base b we have

$$\mathcal{S}_{b\uparrow} \not\subseteq \mathcal{S}_{b\downarrow} \text{ and } \mathcal{S}_{b\downarrow} \not\subseteq \mathcal{S}_{b\uparrow}.$$

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Theorem (Kristiansen)

Let $\mathcal{S}$ be a subrecursive class, closed under primitive recursive operations. For all bases a, b we have

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if and only if every prime factor of a is a prime factor of b.

General sum approximations

Let $\xi \in (0,1)$ be an irrational number and b be any base.

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The function $\hat{G} : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$, such that

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We denote by $\mathcal{S}_{g\uparrow}$ the set of all real numbers, which have a general sum approximation from below in $\mathcal{S}$.

General sum approximations

Let $\xi \in (0,1)$ be an irrational number and b be any base.

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Proposition

For any subrecursive class $\mathcal{S}$, closed under primitive recursive operations we have

$$\mathcal{S}_{T\uparrow} \cap \mathcal{S}_D = \mathcal{S}_{g\uparrow} \text{ and } \mathcal{S}_{T\downarrow} \cap \mathcal{S}_D = \mathcal{S}_{g\downarrow}.$$

Numbers with interesting properties

Let P_n denote the n-th prime ($P_0 = 2, P_1 = 3, \ldots$).
For any honest function f we define the rational number α_n^f and the irrational number α^f by

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For any honest function f we have $f \leq_{PR} \hat{G}$, where $\hat{G}$ is the general sum approximation from below of α^f.

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Thus we have

$$\mathcal{S}_{g\uparrow} \subset \bigcap_{b=2}^{\infty} \mathcal{S}_{b\uparrow}$$

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It follows that α_f has an elementary trace function from above, as well as elementary Dedekind cut.

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A symmetric argument yields $\mathcal{S}_{g\uparrow} \nsubseteq \mathcal{S}_{g\downarrow}$.

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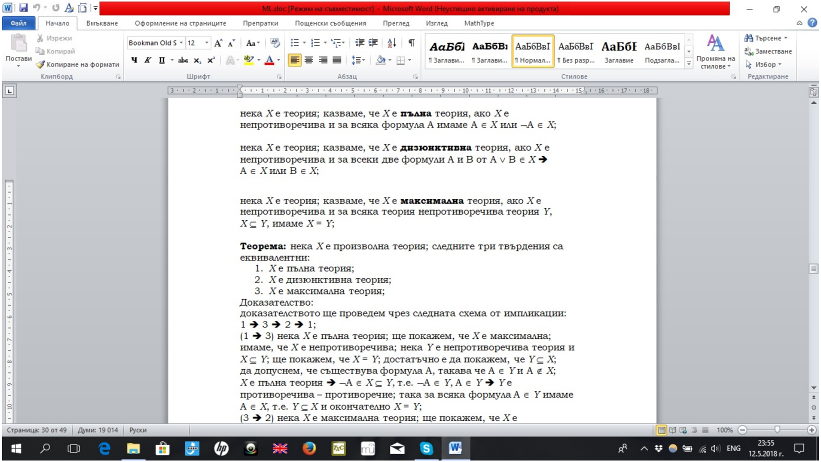
My favourite theorem from logic course



нека X е теория; казваме, че X е **пълна** теория, ако X е непротиворечива и за всяка формула А имаме $A \in X$ или $\neg A \in X$;

нека X е теория; казваме, че X е **дизюнктивна** теория, ако X е непротиворечива и за всеки две формули А и В от $A \vee B \in X \rightarrow A \in X$ или $B \in X$;

нека X е теория; казваме, че X е **максимална** теория, ако X е непротиворечива и за всяка теория непротиворечива теория Y, $X \subseteq Y$, имаме $X = Y$;

Теорема: нека X е произволна теория; следните три твърдения са еквивалентни:

1. X е пълна теория;
2. X е дизюнктивна теория;
3. X е максимална теория;

Доказателство:
доказателството ще проведем чрез следната схема от импликации:
$1 \rightarrow 3 \rightarrow 2 \rightarrow 1$;
($1 \rightarrow 3$) нека X е пълна теория; ще покажем, че X е максимална; имаме, че X е непротиворечива; нека Y е непротиворечива теория и $X \subseteq Y$; ще покажем, че $X = Y$; достатъчно е да покажем, че $Y \subseteq X$; да допуснем, че съществува формула А, такава че $A \in Y$ и $A \notin X$; X е пълна теория $\rightarrow \neg A \in X \subseteq Y$, т.е. $\neg A \in Y$, $A \in Y \rightarrow Y$ е противоречива – противоречие; така за всяка формула $A \in Y$ имаме $A \in X$, т.е. $Y \subseteq X$ и окончателно $X = Y$;
($3 \rightarrow 2$) нека X е максимална теория; ще покажем, че X е

HAPPY 80TH ANNIVERSARY,

PROFESSOR VAKARELOV!

Thank you for your attention!