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FAST CONVERGING SEQUENCE TO EULER-MASCHERONI CONSTANT

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The aim of the paper is to apply an exponential trapezoidal quadrature rule to an integral representation of the Euler-Mascheroni constant. The resulting sequence has subexponential convergence rate and is particularly useful for estimating the subrecursive complexity of the constant.

Keywords: computable real number, subrecursive complexity, Euler-Mascheroni constant, exponential trapezoidal rule.
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1. INTRODUCTION

The Euler-Mascheroni constant is usually denoted by γ and is defined by the equality

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right).$$

Since this sequence converges to γ very slowly, it is not suitable for the effective computation of γ. This is why other much faster methods are invented, like the method of Karatsuba in [2], which is suitable to prove polynomial-time computability of γ.

Our aim is to study the complexity of γ in another context, namely the subrecursive class $\mathcal{M}^2$, contained in the third level $\mathcal{E}^2$ of Grzegorczyk's hierarchy. It appears that the known methods for effective computation of γ are not suitable in this context.

The author has proven in [1] that γ is $\mathcal{M}^2$-computable, as a consequence of some results on $\mathcal{M}^2$-computability of integration. The aim of the present paper is to extract an actual sequence from this proof, which converges to γ with subexponential convergence rate. This will be done by a careful estimation of the error of approximation.

The starting point is the following well-known representation

$$-\gamma = \int_0^\infty e^{-x} \ln x \, dx.$$

Let us define

$$I_1 = \int_0^1 e^{-x} \ln x \; dx = -\int_1^\infty t^{-2} \, e^{-\frac{1}{t}} \ln t \, dt,$$

$$I_2 = \int_1^\infty e^{-x} \ln x \, dx,$$

so that $\gamma = -I_1 - I_2$.

2. EXPONENTIAL TRAPEZOIDAL RULE

The trapezoidal quadrature rule is a famous method for numerical integration. It approximates the definite integral of a real function over an interval with the area of a trapezoid. In practice, the initial interval is split into sufficiently many subintervals of equal length, the rule is applied to each one of them and the obtained results are summed. It is known that in certain situations the result of this method approaches the exact answer very quickly when increasing the number of subintervals. This phenomenon is described in great detail in the paper [3]. The results from Section 1.5 in [3] can be used to estimate the subrecursive complexity of the integration operation. More concretely, the author has proved in [1] that the definite integral of an analytic real function, belonging to a certain low subrecursive class of computable real functions, is itself computable in this class. The steps in this proof proceed as follows:

1. We start with a function θ, which is analytic on an open set of the complex plane containing the interval $[\alpha, \beta]$ and we wish to approximate $I = \int_\alpha^\beta \theta(x) \; dx$.
2. By a linear change of variables we may assume $[\alpha, \beta] = [-1, 1]$.
3. We apply the transformation $x = \tanh(t)$ and thus we obtain the integral

   $$I = \int_{-\infty}^{+\infty} \theta(\tanh(t)) \frac{1}{\cosh^2(t)} \, dt.$$

   (This is the so-called *tanh-rule*.)

4. We discretise the integral by the trapezoidal quadrature rule with step h and then truncate the obtained infinite series to its (two-way) n-th partial sum. Finally, we put $h = \dfrac{1}{\sqrt{n}}$ and the result is

$$I_h^{[n]} = h \sum_{k=-n}^{n} \theta(\tanh(kh)) \frac{1}{\cosh^2(kh)} = \frac{1}{\sqrt{n}} \sum_{k=-n}^{n} \theta\left(\tanh(\frac{k}{\sqrt{n}})\right) \frac{1}{\cosh^2(\frac{k}{\sqrt{n}})}$$

for every positive integer number n.

5. The error of the approximation is

$$|I - I_h^{[n]}| \leq \frac{M(2\pi + 4)}{e^{\frac{1}{C}\sqrt{n}} - 1},$$

where C, M depend on θ, α, β only. (See the proof of Theorem 5.3 in [1].)

3. APPROXIMATION OF I_1

Let $\phi : [1, \infty) \to \mathbb{R}$ be defined by

$$\phi(t) = t^{-2} e^{-\frac{1}{t}} \ln t.$$

Lemma 1. *For any fixed $\xi > 0$, the function ϕ has an analytic continuation to an open set of the complex plane $\mathbb{C}$ containing the set*

$$D_\xi = \left\{ z \in \mathbb{C} \,\middle|\, \operatorname{Re} z \in [1, \xi + 1],\ \operatorname{Im} z \in \left[-\frac{\xi}{2}, \frac{\xi}{2}\right] \right\}. \tag{1}$$

Moreover, $|\phi(z)| \leq \ln(\xi + 1) + \frac{7}{2}$ *for* $z \in D_\xi$.

Proof. The first claim is obviously true, assuming the principal value of the logarithmic function with branch cut the non-positive real numbers. In fact, ϕ has an analytic continuation defined on the whole halfplane $\operatorname{Re} z > 0$.

Now we estimate $|\phi(z)|$, $z \in D_\xi$. Let $\xi > 0$ and $z \in D_\xi$, where $z = x + Bi$ with $1 \leq x \leq \xi + 1$ and $|B| \leq \frac{\xi}{2}$. We have

$$|\phi(z)| = \frac{1}{|z|^2} |e^{-\frac{1}{z}}|.| \ln z|.$$

For the first two factors we have

$$\frac{1}{|z|^2} = \frac{1}{x^2 + B^2} \leq \frac{1}{1 + B^2} \leq 1,$$

$$|e^{-\frac{1}{z}}| = e^{\operatorname{Re}(-\frac{1}{z})} = e^{-\frac{x}{x^2+B^2}} < 1.$$

The third factor is estimated as follows:

$$
\begin{aligned}
|\ln z| = \sqrt{\ln^2|z| + \operatorname{Arg}^2 z} &\le |\ln|z|| + |\operatorname{Arg} z| \\
&\le \frac{1}{2}\ln(x^2+B^2) + \pi \le \frac{1}{2}\ln((\xi+1)^2 + \frac{\xi^2}{4}) + \pi \\
&< \frac{1}{2}\ln(\frac{5}{4}(\xi+1)^2) + \pi < \ln(\xi+1) + \frac{7}{2}.
\end{aligned}
$$

The result follows trivially. □

We replace the integral $-I_1 = \int_1^\infty \phi(t)\,dt$ by

$$
J_\xi^1 = \int_1^{\xi+1} \phi(t)\,dt,
$$

where $\xi > 0$ will be specified later. For the truncation error $e_1(\xi)$ we have (using that $e^{-x} \le 1$ for any $x \ge 0$)

$$
\begin{aligned}
e_1(\xi) = \int_{\xi+1}^\infty t^{-2}\, e^{-\frac{1}{t}} \ln t\, dt &= \int_0^{\frac{1}{\xi+1}} (-\ln x) e^{-x}\, dx \\
&\le \int_0^{\frac{1}{\xi+1}} (-\ln x)\, dx = \frac{\ln(\xi+1)+1}{\xi+1}.
\end{aligned}
$$

Following the steps from the previous section we have

$$
J_\xi^1 = \frac{\xi}{2}\int_{-1}^1 \phi_1(u,\xi)\,du,
$$

where

$$
\phi_1(u,\xi) = \phi\left(\frac{\xi}{2}.u + \frac{\xi+2}{2}\right).
$$

Then we approximate J_ξ^1 by

$$
J_{\xi,n}^1 = \frac{\xi}{2}\frac{1}{\sqrt{n}}\sum_{k=-n}^{n} \phi_1\left(\tanh(\frac{k}{\sqrt{n}}), \xi\right)\frac{1}{\cosh^2(\frac{k}{\sqrt{n}})}.
$$

In the proof of Theorem 5.3 in [1] we can arrange $A' = 1, a = \dfrac{\pi}{4}, C = 1$ and by Lemma 1 we obtain

$$
\left|J_\xi^1 - J_{\xi,n}^1\right| \le \frac{\xi}{2}\frac{(\ln(\xi+1)+\frac{7}{2})(2\pi+4)}{e^{\sqrt{n}}-1}
$$

for any $\xi > 0$ and positive integer number n.

4. APPROXIMATION OF I_2

Let $\psi : [1, \infty) \to \mathbb{R}$ be defined by

$$\psi(x) = e^{-x} \ln x.$$

Lemma 2. *For any fixed $\xi > 0$, the function ψ has an analytic continuation to an open set in $\mathbb{C}$ containing the set D_ξ defined in* (1).

Moreover, $|\psi(z)| \le \ln(\xi + 1) + \frac{7}{2}$ *for* $z \in D_\xi$.

Proof. Analogous to the proof of Lemma 1, this time using that

$$|e^{-z}| = e^{\mathrm{Re}(-z)} = e^{-\mathrm{Re}(z)} < 1,$$

for any complex number z with $\mathrm{Re}\, z > 0$. $\square$

We replace I_2 by

$$J_\xi^2 = \int_1^{\xi+1} \psi(x)\, dx,$$

where $\xi > 0$. For the error $e_2(\xi)$ after this replacement we have (using $\ln x \le x$ for any real number $x \ge 1$)

$$e_2(\xi) = \int_{\xi+1}^{\infty} e^{-x} \ln x\, dt \le \int_{\xi+1}^{\infty} e^{-x} x\, dx = \frac{\xi + 2}{e^{\xi+1}}.$$

Following the steps from Section 2 we have

$$J_\xi^2 = \frac{\xi}{2} \int_{-1}^{1} \psi_1(u, \xi)\, du,$$

where

$$\psi_1(u, \xi) = \psi\left(\frac{\xi}{2}.u + \frac{\xi + 2}{2}\right).$$

Then we approximate J_ξ^2 by

$$J_{\xi,n}^2 = \frac{\xi}{2} \frac{1}{\sqrt{n}} \sum_{k=-n}^{n} \psi_1\left(\tanh(\frac{k}{\sqrt{n}}), \xi\right) \frac{1}{\cosh^2(\frac{k}{\sqrt{n}})}.$$

Again we can arrange $C = 1$ in the proof of Theorem 5.3 in [1] and similarly obtain

$$\left|J_\xi^2 - J_{\xi,n}^2\right| \le \frac{\xi}{2} \frac{(\ln(\xi + 1) + \frac{7}{2})(2\pi + 4)}{e^{\sqrt{n}} - 1}$$

for any $\xi > 0$ and positive integer number n.

5. MAIN RESULT

After approximating $-I_1$ by $J^1_{\xi,n}$ and I_2 by $J^2_{\xi,n}$, we are ready to approximate $\gamma = -I_1 - I_2$ and estimate the error of the approximation by choosing a suitable ξ depending on n.

Let $p(x) = \dfrac{\ln x + 1}{x}$. It is easy to see that p is decreasing in the interval $[1, +\infty)$. Therefore,

$$e_2(\xi) = p(e^{\xi+1}) \leq p(\xi + 1) = e_1(\xi),$$

since $e^{\xi+1} > \xi + 1$ for $\xi > 0$.

Now the approximation of γ by $J^1_{\xi,n} - J^2_{\xi,n}$ leads to an error, which is bounded above by

$$e_1(\xi) + e_2(\xi) + 2\frac{\xi}{2}\frac{(\ln(\xi+1) + \frac{7}{2})(2\pi + 4)}{e^{\sqrt{n}} - 1} \leq 2e_1(\xi) + \frac{\xi(\ln(\xi+1) + \frac{7}{2})(2\pi + 4)}{e^{\sqrt{n}} - 1}$$

for any $\xi > 0$ and any positive integer number n. To produce the desired sequence A, we choose $\xi = \sqrt{e}^{\sqrt{n}} - 1$ to obtain

$$A(n) = J^1_{\xi,n} - J^2_{\xi,n} = \frac{\sqrt{e}^{\sqrt{n}} - 1}{2\sqrt{n}} \sum_{k=-n}^{n} \theta\left(\tanh(\frac{k}{\sqrt{n}}), \sqrt{e}^{\sqrt{n}} - 1\right) \frac{1}{\cosh^2(\frac{k}{\sqrt{n}})},$$

where $\theta(u, \xi) = \phi_1(u, \xi) - \psi_1(u, \xi)$.

For any positive integer n, the error of approximation of γ by $A(n)$ satisfies

$$\begin{aligned}|A(n) - \gamma| &\leq 2e_1(\sqrt{e}^{\sqrt{n}} - 1) + \frac{(\sqrt{e}^{\sqrt{n}} - 1)(\frac{1}{2}\sqrt{n} + \frac{7}{2})(2\pi + 4)}{e^{\sqrt{n}} - 1} \\ &= \frac{2(\frac{1}{2}\sqrt{n} + 1)}{\sqrt{e}^{\sqrt{n}}} + \frac{(\sqrt{n} + 7)(\pi + 2)}{\sqrt{e}^{\sqrt{n}} + 1} \leq \frac{(\pi + 3)\sqrt{n} + 7\pi + 16}{\sqrt{e}^{\sqrt{n}}}.\end{aligned}$$

6. CONCLUSION

The sequence A is suitable for proving $\mathcal{M}^2$-computability of γ. It turns out that the sequence B, defined by

$$B(m) = A(\lfloor \log_2(m+1) \rfloor^2),$$

is $\mathcal{M}^2$-computable and has polynomial convergence rate.

Unfortunately, the expression for the general term of A is too complex to be used in practice for computation of many decimal digits of γ. Simple numerical experiments with Simulink/Matlab using high precision calculations give 18 correct decimal digits for $n = 10000$ and 30 correct decimal digits for $n = 25000$.

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