Transfer of Properties from Uniform to Conditional Computability of Real Functions

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Abstract

For an acceptable pair $(\mathcal{F}, \mathbf{O})$, where $\mathcal{F}$ is a set of total functions in the natural numbers and $\mathbf{O}$ is a class of operators, we consider the uniformly $\mathbf{O}$-computable real functions, which are uniformly realisable by operators from $\mathbf{O}$ with respect to a natural representation of the real numbers. Our most important example is the pair $(\mathcal{M}^2, \mathbf{MSO})$, where $\mathcal{M}^2$ is the set of all Δ_0-definable total functions of polynomial growth and $\mathbf{MSO}$ is the class of all $\mathcal{M}^2$-substitutional operators. All elementary functions of calculus are uniformly $\mathbf{MSO}$-computable, but restricted to compact subsets of their domains. To eliminate this restrictions we make use of a more complex computing procedure and define the broader class of the conditionally $\mathbf{O}$-computable real functions, which are also realisable by operators from $\mathbf{O}$, but with respect to an additional natural parameter, computed by a search operation from the representations of the real arguments. All elementary functions of calculus are conditionally $\mathbf{MSO}$-computable on their whole domains. Our main question in this talk is the following:

Do properties of the uniformly $\mathbf{O}$-computable real functions transfer naturally for the conditionally $\mathbf{O}$-computable real functions?

We give both positive examples like closure under composition and preservation of $\mathcal{F}$-computability as well as negative examples like certain extensions by uniform limits. We also consider the transferability of some results on closure under integration of the uniformly $\mathbf{O}$-computable analytic real functions.

Introduction

The aim of our research is studying the complexity of the representations of objects from mathematical analysis, such as real numbers and real functions. We adopt subrecursive complexity setting, that is we are interested in classes of total functions in $\mathbb{N}$, closed under some natural operations. The narrowest class we are interested in is given by the following

Definition. *The class $\mathcal{M}^2$ is the smallest class of total functions in $\mathbb{N}$, which contains the initial functions $\lambda x_1 \ldots x_n . x_m (1 \leq m \leq n)$, $\lambda x.x+1$, $\lambda xy.max(x-y, 0)$, $\lambda xy.xy$ and is closed under substitution and bounded minimization ($f \mapsto \lambda \vec{x}y.\mu_{z \leq y}[f(\vec{x}, z) = 0]$).*

The class $\mathcal{M}^2$ contains precisely those total functions in $\mathbb{N}$, which are Δ_0-definable and bounded by a polynomial.

The triple of functions (f, g, h) of unary total functions in $\mathbb{N}$ is a name of the real number ξ if for all $n \in \mathbb{N}$,

$$\left| \frac{f(n) - g(n)}{h(n)+1} - \xi \right| < \frac{1}{n+1}.$$

A real number ξ is $\mathcal{M}^2$-computable if there exists a name (f, g, h) of ξ, such that $f, g, h \in \mathcal{M}^2$.

The set of all $\mathcal{M}^2$-computable real numbers is a real-closed field. Many other real constants, such as π and e are also $\mathcal{M}^2$-computable.

Substitutional operators

We denote $\mathcal{T}_m = \{a | a : \mathbb{N}^m \to \mathbb{N}\}$.

Since real functions map tuples of real numbers to real numbers, we need computing procedures of type 2 (operators), which act on names of real numbers.

The mappings $F : \mathcal{T}_1^k \to \mathcal{T}_1$ will be called k-operators.

Definition. *For any $k \in \mathbb{N}$ we define inductively the class $\mathbf{MSO}$ of $\mathcal{M}^2$-substitutional k-operators as follows:*

- *The operator F defined by $F(f_1, \ldots, f_k)(n) = n$ for all $n \in \mathbb{N}$ is $\mathcal{M}^2$-substitutional.*
- *For any $i \in \{1, \ldots, k\}$, if F_0 is an $\mathcal{M}^2$-substitutional k-operator, then so is the operator F defined by*
$$F(f_1, \ldots, f_k)(n) = f_i(F_0(f_1, \ldots, f_k)(n)).$$
- *For any $r \in \mathbb{N}$ and function $f \in \mathcal{T}_r \cap \mathcal{M}^2$, if $F_1, \ldots, F_r$ are $\mathcal{M}^2$-substitutional k-operators, then so is the operator F, defined by*
$$F(f_1, \ldots, f_k)(n) = f(F_1(f_1, \ldots, f_k)(n), \ldots, F_r(f_1, \ldots, f_k)(n)).$$

Uniform computability of real functions

Definition. *Let $k \in \mathbb{N}$ and θ be a real function, $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^k$. The triple (F, G, H), where F, G, H are $3k$-operators will be called a uniform realiser for θ if for all $(\xi_1, \xi_2, \ldots, \xi_k) \in D$ and triples (f_i, g_i, h_i) that name ξ_i for $i = 1, 2, \ldots, k$, the triple*

$$(F(f_1, g_1, h_1, f_2, g_2, h_2, \ldots, f_k, g_k, h_k),$$
$$G(f_1, g_1, h_1, f_2, g_2, h_2, \ldots, f_k, g_k, h_k),$$
$$H(f_1, g_1, h_1, f_2, g_2, h_2, \ldots, f_k, g_k, h_k))$$

names the real number $\theta(\xi_1, \xi_2, \ldots, \xi_k)$.

The function θ will be called uniformly $\mathbf{MSO}$-computable, if there exists a uniform realiser (F, G, H) for θ, such that $F, G, H \in \mathbf{MSO}$.

As shown in [3], all elementary functions of calculus are uniformly $\mathbf{MSO}$-computable on the compact subsets of their domains.

Conditional computability of real functions

Definition. *Let $k \in \mathbb{N}$ and θ be a real function, $\theta : D \to \mathbb{R}, D \subseteq \mathbb{R}^k$. The quadruple (E, F, G, H), where E is a $3k$-operator and F, G, H are $(3k+1)$-operators, will be called a conditional realiser for θ if for all $(\xi_1, \ldots, \xi_k) \in D$ and all triples (f_i, g_i, h_i) that name ξ_i for $i = 1, 2, \ldots, k$, the following two hold:*

- *There exists a natural number s satisfying the equality*
$$E(f_1, g_1, h_1, \ldots, f_k, g_k, h_k)(s) = 0.$$
- *For any natural number s satisfying the above equality, the triple $(\tilde{f}, \tilde{g}, \tilde{h})$ names the real number $\theta(\xi_1, \ldots, \xi_k)$, where*
$$\tilde{f} = F(f_1, g_1, h_1, \ldots, f_k, g_k, h_k, \lambda x.s),$$
$$\tilde{g} = G(f_1, g_1, h_1, \ldots, f_k, g_k, h_k, \lambda x.s),$$
$$\tilde{h} = H(f_1, g_1, h_1, \ldots, f_k, g_k, h_k, \lambda x.s).$$

The function θ will be called conditionally $\mathbf{MSO}$-computable, if there exists a conditional realiser (E, F, G, H) for θ, such that $E, F, G, H \in \mathbf{MSO}$.

Any uniformly $\mathbf{MSO}$-computable real function is also conditionally $\mathbf{MSO}$-computable. All elementary functions of calculus are conditionally $\mathbf{MSO}$-computable on their whole domains, see [2].

Main Properties

The following list of properties is common for both uniform and conditional computability.

1. Any uniformly (conditionally) $\mathbf{MSO}$-computable real function is computable in the usual sense (see [4]).
2. Composition on real functions preserves uniform (conditional) $\mathbf{MSO}$-computability.
3. Every uniformly (conditionally) $\mathbf{MSO}$-computable real function maps tuples of $\mathcal{M}^2$-computable real numbers into an $\mathcal{M}^2$-computable real number.
4. (*gluing property*) For an $\mathcal{M}^2$-computable real number r and a real function $\theta : D \to \mathbb{R}$, $D \subseteq \mathbb{R}$, if the restrictions of θ to $D \cap (-\infty, r]$ and to $D \cap [r, +\infty)$ are uniformly (conditionally) $\mathbf{MSO}$-computable, then θ is uniformly (conditionally) $\mathbf{MSO}$-computable on its whole domain D.

The following two properties distinguish uniform from conditional computability.

1. The absolute value of any uniformly $\mathbf{MSO}$-computable real function is bounded by a polynomial of the absolute values of its arguments.
2. Any uniformly $\mathbf{MSO}$-computable real function is uniformly continuous on the bounded subsets of its domain.

Uniform Limits

Theorem. *Let $l \in \mathbb{N}$, $U \subseteq \mathbb{R}^l$ and $\theta : \mathbb{N} \times U \to \mathbb{R}$ be a real function, which is uniformly $\mathbf{MSO}$-computable, such that the limit $\rho(\vec{\eta}) = \lim_{n \to \infty} \theta(n, \vec{\eta})$ exists for any $\vec{\eta} \in U$. Let there also exist a $3l$-operator $R \in \mathbf{MSO}$, such that for any $\vec{\eta} \in U$ and any triples (f_i, g_i, h_i) naming η_i for $i = 1, \ldots, l$, we have the inequality*

$$|\rho(\vec{\eta}) - \theta(n, \vec{\eta})| \leq \frac{1}{t+1}$$

for all $t \in \mathbb{N}$ and $n = R(f_1, g_1, h_1, \ldots, f_l, g_l, h_l)(t)$. Then the real function $\rho : U \to \mathbb{R}$ is uniformly $\mathbf{MSO}$-computable.

This theorem is not true when we substitute conditional for uniform computability.

Let $\chi : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be the real function with value 1 for the positive and value 0 for the negative real numbers. Let $D = \mathbb{R} \setminus \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ and the real function $\theta : D \to \mathbb{R}$ be defined by

$$\theta(\xi) = \sum_{k=1}^{\infty} \frac{1}{2^k} \chi \left(\xi - \frac{1}{k} \right).$$

Then θ is not conditionally $\mathbf{MSO}$-computable, but it can be shown that θ is the uniform limit of a conditionally $\mathbf{MSO}$-computable sequence.

Question. Does there exist a real function, which is computable in the usual sense, but which is not the uniform limit of a conditionally $\mathbf{MSO}$-computable sequence?

Integration

The following theorem is proven in [1].

Theorem. *Let α, β be $\mathcal{M}^2$-computable real numbers, $D \subseteq \mathbb{R}^l$ be a set for some $l \in \mathbb{N}$ and $\theta : [\alpha, \beta] \times U \to \mathbb{R}$ be a real function, which is uniformly $\mathbf{MSO}$-computable. Let there exist $A \in \mathbb{R}$, $A > 0$, such that for every fixed $\vec{\eta} \in U$, $\theta(x, \vec{\eta})$ (as a real function of x) has an analytic continuation defined in the set $[\alpha, \beta] \times [-A, A] \subseteq \mathbb{C}$. Let there also exist a polynomial P, such that $|\theta(x + Bi, \vec{\eta})| \leq P(|\vec{\eta}|)$ for all $\vec{\eta} \in U, x \in [\alpha, \beta], B \in [-A, A]$. Then the real function $I : U \to \mathbb{R}$ defined by*

$$I(\vec{\eta}) = \int_{\alpha}^{\beta} \theta(x, \vec{\eta}) \, dx$$

is uniformly $\mathbf{MSO}$-computable.

For conditional computability we have the following result: retaining all other assumptions if θ is conditionally $\mathbf{MSO}$-computable, then I is the uniform limit of a conditionally $\mathbf{MSO}$-computable sequence.

Question. Can we generalise these results for uniformly or conditionally $\mathbf{MSO}$-computable real functions, which are not analytic?

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