Minimal ω-Turing Degrees*

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Abstract

In the present paper we characterize the minimal degrees in the structure of the ω-Turing degrees. Namely, we show that all minimal degrees in $\mathcal{D}_{T,\omega}$ are inherited from the Turing degrees and bounded by the first jump of the least ω-Turing degree.

1 The ω-Turing degrees

The ω-Turing reducibility $\leq_{T,\omega}$ arises as a formal way to compare the information content of sequences of sets of natural numbers. In this computational framework the information content of a sequence is uniquely determined by the collection of the Turing degrees of the sets that *code* the sequence. We say that a set codes a sequence iff *uniformly* in k, it can compute the k-th element of the considered sequence in its k-th Turing jump:

$$X \subseteq \omega \text{ codes } \{A_k\}_{k<\omega} \iff A_k \leq_T X^{(k)} \text{ uniformly in } k.$$

Having this, we shall say that the sequence $\mathcal{A}$ is ω-*Turing reducible* to the sequence $\mathcal{B}$ iff each set that codes $\mathcal{B}$ also codes $\mathcal{A}$:

$$\mathcal{A} \leq_{T,\omega} \mathcal{B} \iff (\forall X \subseteq \omega)[X \text{ codes } \mathcal{B} \Rightarrow X \text{ codes } \mathcal{A}].$$

This reducibility is introduced in [3], where its basic properties are explored. The relation $\leq_{T,\omega}$ is a preorder on the set of the sequences of sets of natural numbers and in the standard way induces a degree structure – the upper semi-lattice $\mathcal{D}_{T,\omega}$ of the ω-Turing degrees. The least element of $\mathcal{D}_{T,\omega}$ is the degree $\mathbf{0}_{T,\omega}$ of the sequence $\{\emptyset\}_{k<\omega}$. The degree of the sequence $\{A_k \oplus B_k\}_{k<\omega}$ is the least upper bound of the degrees of the sequences $\{A_k\}_{k<\omega}$ and $\{B_k\}_{k<\omega}$.

Although we refer to $\leq_{T,\omega}$ as a reducibility relation between sequences, its definition does not give us an immediate way for computing, say $\mathcal{A}$ from $\mathcal{B}$, given that $\mathcal{A} \leq_{T,\omega} \mathcal{B}$. In order to characterize the ω-Turing reducibility in more approachable way we need the following notion. Given a sequence $\mathcal{A} = \{A_k\}_{k<\omega}$ we define its *jump-sequence* $\mathcal{P}(\mathcal{A}) = \{P_k(\mathcal{A})\}_{k<\omega}$ as the sequence:

$$P_0(\mathcal{A}) = A_0 \text{ and for each } k,\ P_{k+1}(\mathcal{A}) = P_k(\mathcal{A})' \oplus A_{k+1}.$$

Now, according to [3], the ω-Turing reducibility is characterized as:

$$\mathcal{A} \leq_{T,\omega} \mathcal{B} \iff A_n \leq_T P_n(\mathcal{B}) \text{ uniformly in } n. \qquad (1)$$

From here, one can show that each sequence is ω-Turing equivalent with its jump-sequence, i.e. for all $\mathcal{A}$, $\mathcal{A} \equiv_{T,\omega} \mathcal{P}(\mathcal{A})$.

Again in [3], is defined a jump operation on sequences, which induces a corresponding jump operation in the degree structure. Namely the jump $\mathcal{A}'$ of the sequence $\mathcal{A}$ is defined in such a way that:

$$X \text{ codes } \mathcal{A}' \iff (\exists Y)[X \equiv_T Y' \ \&\ Y \text{ codes } \mathcal{A}]. \qquad (2)$$

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Although for each $\mathcal{A}$ there are exactly continuum many sequences satisfying (2), all of them are ω-Turing equivalent. Following the lines of [3], as a canonical representative of the jump of $\mathcal{A} = \{A_k\}_{k<\omega}$ we take the sequence:

$$\mathcal{A}' = (P_1(\mathcal{A}), A_2, A_3, \ldots, A_k, \ldots).$$

Note, that for each k, $P_k(\mathcal{A}') = P_{1+k}(\mathcal{A})$, so $\mathcal{A}' \equiv_{T,\omega} \{P_{k+1}(\mathcal{A})\}_{k<\omega}$. The jump operator is strictly expanding and monotone, i.e. $\mathcal{A} \lneq_{T,\omega} \mathcal{A}'$ and $\mathcal{A} \leq_{T,\omega} \mathcal{B} \Rightarrow \mathcal{A}' \leq_{T,\omega} \mathcal{B}'$. This allows to define a jump operation on the ω-Turing degrees by setting $\mathbf{a}' = \deg_{T,\omega}(\mathcal{A}')$, where $\mathcal{A} \in \mathbf{a}$ is an arbitrary. Clearly $\mathbf{a} < \mathbf{a}'$ and $\mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{a}' \leq \mathbf{b}'$. Let us note that:

$$\mathcal{A}^{(n)} = (P_n(\mathcal{A}), A_{n+1}, A_{n+2}, \ldots) \equiv_{T,\omega} \{P_{n+k}(\mathcal{A})\}_{k<\omega}.$$

How $\mathcal{D}_{T,\omega}$ can be seen as an extension of the structure $\mathcal{D}_T$ of the Turing degrees? By the uniform properties of the Turing jump, it is well known that for all $A, X \subseteq \omega$:

$$A \leq_T X \iff A^{(k)} \leq_T X^{(k)} \text{ uniformly in } k.$$

Thus, the information content of the set A, described in the Turing universe by the set of the degrees of the sets that decides A, is the same as the content of the sequence $\{A^{(k)}\}_{k<\omega}$ in the context of the ω-Turing reducibility. This observation allows us to define a very natural embedding of the Turing degrees into the ω-Turing:

$$\kappa : \deg_T(A) \longmapsto \deg_{T,\omega}(\{A^{(k)}\}_{k<\omega}).$$

This embedding preserves the order, the least upper bound operation and even the jump. In this way we may assume the Turing degrees as a proper substructure of $\mathcal{D}_{T,\omega}$. But there are much more strong connections between the both structures. In [3] it is shown that $\mathcal{D}_T$ is definable in $\mathcal{D}_{T,\omega}$ by a first-order formula in the language of the structure order and the jump operation. Also it is proved that the group $\mathrm{Aut}(\mathcal{D}_T)$ of the automorphisms of the Turing degrees is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(\mathcal{D}_{T,\omega})$ of $\mathcal{D}_{T,\omega}$ – namely to the subgroup $\mathrm{Aut}(\mathcal{D}'_{T,\omega})$ of the jump preserving automorphisms of the ω-Turing degrees.

2 Almost zero degrees

In this section we shall introduce the special class of degrees, which differ from the least ω-Turing degree $\mathbf{0}_{T,\omega}$ only by the lack of uniformity. Namely, we call a degree *almost zero* (or simply, a.z.) if it contains sequence $\mathcal{A}$ such that:

$$P_k(\mathcal{A}) \equiv_T \emptyset^{(k)} \text{ for every natural } k, \tag{3}$$

In particular, the least degree $\mathbf{0}_{T,\omega}$ is a.z. Note that if the sequence $\mathcal{A} = \{A_k\}_{k<\omega}$ belongs to the a.z. degree $\mathbf{a}$, then for each k, $A_k \leq_T P_k(\{\emptyset\}_{n<\omega}) \equiv_T \emptyset^{(k)}$. Since this reduction is not necessarily uniform, it does not imply that $\mathbf{a}$ is equal to $\mathbf{0}_{T,\omega}$. It is easy to see that there are continuum many sequences satisfying (3). For example a sequence consisting only of finite sets, or more generally one consisting only of computable sets, satisfies (3). Thus there are continuum many nonzero a.z. degrees.

We finish this section concluding that the a.z. degrees are downwards dense. This shall be used later in the characterization of the minimal degrees.

Theorem 1. *For every nonzero a.z. degree* $\mathbf{a}$ *there is a nonzero a.z. degree* $\mathbf{b} < \mathbf{a}$.

Proof. Let $\mathbf{a}$ be a nonzero a.z. degree and let $\mathcal{A} \in \mathbf{a}$ satisfy (3). We shall construct a sequence $\mathcal{B}$, such that $\{\emptyset\}_{t<\omega} \lneq_{T,\omega} \mathcal{B} \lneq_{T,\omega} \mathcal{A}$. Let us fix a computable in $\emptyset''$ enumeration $f_0, f_1, \ldots, f_n, \ldots$ of all total computable functions. In order to build $\mathcal{B}$ as desired, it suffices to ensure that $\mathcal{B} \leq_{T,\omega} \mathcal{A}$ and to meet the following requirements[1]:

$$R_{2e} : \exists k \left(A_k \neq \varphi_{f_e(k)}^{P_k(\mathcal{B})} \right),$$

$$R_{2e+1} : \exists k \left(B_k \neq \varphi_{f_e(k)}^{\emptyset^{(k)}} \right).$$

The construction of $\mathcal{B} = \{B_k\}_{k<\omega}$ shall use an induction on k. For every k we shall set either $B_k = \emptyset$ or $B_k = A_k$. Note that this gives us automatically, that $\mathcal{B}$ satisfies (3).

The construction: During the construction we shall use a global variable ρ, which shall indicate the least requirement that is (possibly) not yet satisfied. Set $\rho = 0$ and $B_0 = B_1 = \emptyset$. Suppose that $k \geq 2$ and that B_s is defined for $s \leq k$. Note that our assumption yields that for $s \leq k$, $P_s(\mathcal{B})$ is defined as well.

- *Case 1:* $\rho = 2e$. If $A_{k-2} \neq \varphi_{f_e(k-2)}^{P_{k-2}(\mathcal{B})}$, set $B_k = A_k$ and augment ρ by 1. Otherwise set $B_k = \emptyset$ and keep ρ the same.
- *Case 2:* $\rho = 2e + 1$. If $B_{k-2} \neq \varphi_{f_e(k-2)}^{\emptyset^{(k-2)}}$, set $B_k = \emptyset$ and augment ρ by 1. Otherwise set $B_k = A_k$ and keep ρ the same.

End of construction.

First of all let us note that, according to the definition of the jump sequence $\mathcal{P}(\mathcal{A})$, $\emptyset''$ is uniformly computable in $P_k(\mathcal{A})$ for $k \geq 2$. Hence for $k \geq 2$, $P_k(\mathcal{A})$ can uniformly compute our fixed enumeration $f_0, f_1, f_2, \ldots$ of the total computable functions. Also, for $k \geq 2$, $P_{k-2}(\mathcal{A})'' \leq_T P_k(\mathcal{A})$ uniformly in k. Using an induction on $k \geq 2$, one can easily see that $P_k(\mathcal{A})$ uniformly compute $P_{k-2}(\mathcal{B})$, as well as the outcomes of the questions $A_{k-2} \neq \varphi_{f_e(k-2)}^{P_{k-2}(\mathcal{B})}$ and $B_{k-2} \neq \varphi_{f_e(k-2)}^{\emptyset^{(k-2)}}$. In particular $P_k(\mathcal{A})$ can compute uniformly the value of ρ at stage k and hence it can compute uniformly B_k. Therefore $\mathcal{B} \leq_{T,\omega} \mathcal{A}$.

Note that all the requirements are met. Towards a contradiction assume that n is the least index of requirement which is not met. Note that the construction yields that at some stage s, the global variable ρ has been set to be equal to n, and from then on ρ has never changed its value. First let us suppose that $n = 2e$ for some $e \in \omega$. Then for every $k > s$, $A_{k-2} = \varphi_{f_e(k-2)}^{P_{k-2}(\mathcal{B})}$, so that $B_k = \emptyset$ for $k \geq s$ and $A_k \leq_T P_k(\mathcal{B})$ uniformly in $k > s$. On the other hand for $0 \leq k \leq s$,

$$B_k \leq_T P_k(\mathcal{A}) \leq_T \emptyset^{(k)},$$

which together with our previous observation yields $\mathcal{B} \leq_{T,\omega} \{\emptyset\}_{t<\omega}$ and $\mathcal{A} \leq_{T,\omega} \mathcal{B}$. Thus $\mathcal{A} \leq_{T,\omega} \{\emptyset\}_{t<\omega}$, contradicting the choice of $\mathcal{A}$.

In the case when $n = 2e + 1$, we obtain in a similar way $\mathcal{A} \leq_{T,\omega} \{\emptyset\}_{t<\omega}$, contradicting once again the choice of $\mathcal{A}$. Therefore our assumption, that some of the requirements are not met, is incorrect and hence $\{\emptyset\}_{t<\omega} \lneq_{T,\omega} \mathcal{B} \lneq_{T,\omega} \mathcal{A}$. ☐

[1] in this proof, the e-th partial recursive function with oracle $X \subseteq \omega$ is denoted by φ_e^X

3 Minimal degrees

Given a degree structure $\mathcal{D} = (\mathbf{D}, \leq, \mathbf{0})$, we shall call an element $\mathbf{m} \in \mathbf{D}$ *minimal* if it is nonzero and if it strictly bounds only the least element $\mathbf{0}$:

$$\mathbf{0} < \mathbf{m} \ \& \ (\forall \mathbf{a})[\mathbf{a} \leq \mathbf{m} \rightarrow (\mathbf{a} = \mathbf{m} \ \vee \ \mathbf{a} = \mathbf{0})].$$

In other words, a degree is minimal if it describes a specialized problem, i.e. a problem that can solve (in the context of the reducibility $\leq$) exept the problems that are equivalent to it only trivial ones (the trivial problems modulo $\leq$ are exactly the elements of $\mathbf{0}$).

We shall use a generalization of the notion of minimal degree. Namely, we shall call an element $\mathbf{m} \in \mathbf{D}$ *minimal cover* of the element $\mathbf{x} \in \mathbf{D}$ if $\mathbf{m}$ strictly bounds $\mathbf{x}$ and between them there are no other degrees:

$$\mathbf{x} < \mathbf{m} \ \& \ \mathbf{D}(\mathbf{x}, \mathbf{m}) = \emptyset.$$

The minimal degrees in the Turing degree structure are well studied. Spector first prove the existence of a minimal Turing degree. Later Sacks [2] prove the existence of a minimal degree below the degree of the halting problem $\mathbf{0}'_T$. Cooper [1] improve this result by showing that there is a low minimal degree, i.e. such minimal degree $\mathbf{m} < \mathbf{0}'_T$ that $\mathbf{m}' = \mathbf{0}'_T$. The relativization of the Cooper's result over $\mathbf{0}_T^{(n)}$ gives us for each n a degree $\mathbf{m}$ that is low over $\mathbf{0}_T^{(n)}$ minimal cover of $\mathbf{0}_T^{(n)}$:

$$\mathbf{0}_T^{(n)} <_T \mathbf{m} <_T \mathbf{0}_T^{(n+1)}, \ \mathbf{m}' = \mathbf{0}_T^{(n+1)} \text{ and } \mathbf{D}_T(\mathbf{0}_T^{(n)}, \mathbf{m}) = \emptyset.$$

Note that if M has a degree that is a low over $\mathbf{0}_T^{(n)}$ (i.e. $\emptyset^{(n)} <_T M <_T M' \equiv_T \emptyset^{(n+1)}$) minimal cover of $\mathbf{0}_T^{(n)}$ (i.e. there are no A such that $\emptyset^{(n)} <_T A <_T M$), then the sequence

$$\mathcal{M} = (\underbrace{\emptyset, \emptyset, \ldots, \emptyset}_{n}, M, \emptyset, \ldots, \emptyset, \ldots), \tag{4}$$

is a minimal ω-Turing degree.

Indeed $M \not\leq_T \emptyset^{(n)} \equiv_T P_n(\{\emptyset\}_{t<\omega})$, so that $\mathcal{M} \not\leq_{T,\omega} \{\emptyset\}_{t<\omega}$. In particular the ω-Turing degree, $\mathbf{m}$, of $\mathcal{M}$ is nonzero.

Now let us suppose that $\mathcal{A} \leq_{T,\omega} \mathcal{M}$. Note that $M' \equiv_T \emptyset^{(n+1)}$ implies that $P_k(\mathcal{M}) \equiv_T \emptyset^{(k)}$ uniformly in $k \neq n$ and hence for $k \neq n$, $A_k \leq_T \emptyset^{(k)}$ uniformly in k. On the other hand $A_n \leq_T P_n(\mathcal{M})$ yields $A_n \leq_T M$, so that either $A_n \leq_T \emptyset^{(n)}$, or $A_n \equiv_T M$. In the first case we obtain $\mathcal{A} \leq_{T,\omega} \{\emptyset\}_{t<\omega}$ and in the second $\mathcal{A} \equiv_{T,\omega} \mathcal{M}$. Hence $\mathbf{m}$ is minimal.

In the rest of this paper we shall show that the possessing of a sequence such that (4) is also a necessary condition to be a degree minimal. For the purpose, suppose that $\mathbf{m}$ is a minimal and choose a $\mathcal{M} \in \mathbf{m}$. By Theorem 1, $\mathbf{m}$ is not a.z. degree, so for a least one n, $P_n(\mathcal{M}) \not\equiv_T \emptyset^{(n)}$. Note that there must be a unique such n. Indeed, assume that there are at least two natural numbers, say $n < m$, such that $P_n(\mathcal{M}) \not\equiv_T \emptyset^{(n)}$ and $P_m(\mathcal{M}) \not\equiv_T \emptyset^{(m)}$. But then

$$\{\emptyset\}_{t<\omega} \lneq_{T,\omega} (\underbrace{\emptyset, \emptyset, \ldots, \emptyset}_{m}, P_m(\mathcal{M}), \emptyset, \ldots, \emptyset, \ldots) \lneq_{T,\omega} \mathcal{M},$$

and hence $\mathbf{m}$ is not minimal. A contradiction.

Hence there is a unique n such that $P_n(\mathcal{M}) \not\equiv_T \emptyset^{(n)}$. Note that the Turing degree of $P_n(\mathcal{M})$ must be a minimal cover of $\emptyset^{(n)}$, for otherwise (i.e. if there is a set $\emptyset^{(n)} \lneq_T A \lneq_T P_n(\mathcal{M})$)

$$\{\emptyset\}_{t<\omega} \lneq_{T,\omega} (\underbrace{\emptyset, \emptyset, \ldots, \emptyset}_{n}, A, \emptyset, \ldots) \lneq_{T,\omega} \mathcal{M},$$

which contradicts with the minimality of $\mathbf{m}$.

Further, $P_n(\mathcal{M})'$ is equivalent to $\emptyset^{(n+1)}$, since clearly $\emptyset^{(n)} \leq_T P_n(\mathcal{M})$ and $P_{n+1}(\mathcal{M}) = P_n(\mathcal{M})' \oplus M_{n+1} \equiv_T \emptyset^{(n+1)}$.

Finally let us consider the sequence $\mathcal{M}^-$ obtained from $\mathcal{P}(\mathcal{M})$ by replacing its first $n+1$ elements by the empty set, i.e.

$$\mathcal{M}^- = (\underbrace{\emptyset, \ldots, \emptyset}_{n+1}, P_{n+1}(\mathcal{M}), P_{n+2}(\mathcal{M}), \ldots).$$

Clearly $\mathcal{M}^- \lneq_{T,\omega} \mathcal{M}$ and hence $\mathcal{M}^- \equiv_{T,\omega} \{\emptyset\}_{t<\omega}$. In particular $P_k(\mathcal{M}) \leq_T \emptyset^{(k)}$ uniformly in $k > n$.

Therefore

$$\mathcal{M} \equiv_{T,\omega} (\underbrace{\emptyset, \emptyset, \ldots, \emptyset}_{n}, M, \emptyset, \ldots, \emptyset, \ldots),$$

where M is such that, $M' \equiv_T \emptyset^{n+1}$ and the Turing degree of M is a minimal cover of $\mathbf{0}_T^{(n)}$.

Thus we have proven the following theorem

Theorem 2. *An ω-Turing degree $\mathbf{m}$ is minimal, if and only if it contains a sequence of the form*

$$(\underbrace{\emptyset, \emptyset, \ldots, \emptyset}_{n}, M, \emptyset, \ldots, \emptyset, \ldots),$$

where the Turing degree of M is a low over $\mathbf{0}_T^{(n)}$ minimal cover of $\mathbf{0}_T^{(n)}$.

Note that if $\mathbf{m}$ is a minimal degree with witness the sequence

$$\mathcal{M} = (\underbrace{\emptyset, \emptyset, \ldots, \emptyset}_{n}, M, \emptyset, \ldots, \emptyset, \ldots),$$

then $P_k(\mathcal{M}) \leq_T \emptyset^{(k)}$ uniformly in k, so $\mathcal{M} \leq_{T,\omega} \{\emptyset^{(t+1)}\}_{t<\omega} \equiv_{T,\omega} \{\emptyset\}'_{t<\omega}$. Thus $\mathbf{m} \leq_{T,\omega} \mathbf{0}_{T,\omega}{}'$. Hence the first jump $\mathbf{0}_{T,\omega}{}'$ of the least ω-Turing degree bounds all minimal ω-Turing degrees. Still remains unsolved the problem if $\mathbf{0}_{T,\omega}{}'$ is the least degree which bounds every minimal degree.

References

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