Minimal ω-Turing degrees

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Degree structures

- set of objects Ω;
  - Turing and enumeration degrees: sets of natural numbers;
  - Muchnik and Medvedev degrees: sets of total functions;
  - ω-Turing degrees: sequences of sets of natural numbers;
- reflexive and transitive reducibility $\leq$ that compares the information content of the objects from Ω;
  - $A \leq_T B \iff$ there is an algorithm s.t. on any input n, in finitely many steps and using finitely many membership queries to B, determines the membership of n in A;
- $A \in \Omega$, degree of A: $\deg(A) = \{B \in \Omega \mid A \leq B \text{ and } B \leq A\}$;
- the set of all degrees: $\mathbf{D} = \{\deg(A) \mid A \in \Omega\}$;
- induced order in $\mathbf{D}$: $\deg(A) \leq \deg(B) \iff A \leq B$;
- $\mathcal{D} = (\mathbf{D}, \vee, \mathbf{0}, \leq)$ upper semi-lattice with least element $\mathbf{0}$;

The least degree and Minimal degrees

- $\mathbf{0}$: the degree of the trivial with respect to $\leq$ objects,

$$A \in \mathbf{0} \iff (\forall B \in \Omega)[A \leq B];$$

- $\mathbf{m} \in \mathbf{D}$ is minimal iff there is no $\mathbf{a}$ s.t. $\mathbf{0} < \mathbf{a} < \mathbf{m}$;
  - there are continuum many minimal Turing degrees;
  - there are no minimal enumeration degrees;
  - there are countably many minimal ω-Turing degrees;

ω-Turing reducibility

- objects: sequences of sets of natural numbers;
- the informational content of sequence is uniquely determined by the set of the Turing degrees of all sets that *code* the sequence;
- $X \subseteq \omega$ *codes* $\{A_k\}_{k<\omega}$ iff $A_k \leq_T X^{(k)}$ uniformly in k.
- ω-Turing reducibility:

$$\mathcal{A} \leq_\omega \mathcal{B} \iff (\forall X \subseteq \omega)[X \text{ codes } \mathcal{B} \Rightarrow X \text{ codes } \mathcal{A}];$$

- $\mathcal{A} \equiv_\omega \mathcal{B} \iff \mathcal{A} \leq_\omega \mathcal{B}$ and $\mathcal{B} \leq_\omega \mathcal{A}$.

ω-Turing degrees

- ω-Turing degree of the sequence of sets of natural numbers $\mathcal{A}$:

$$\deg_\omega(\mathcal{A}) = \{\mathcal{B} : \mathcal{B} \equiv_\omega \mathcal{A}\};$$

- partial order:

$$\deg_\omega(\mathcal{A}) \leq \deg_\omega(\mathcal{B}) \iff \mathcal{A} \leq_\omega \mathcal{B}.$$

- Denote by $\mathcal{D}_\omega$ the partial ordering of the ω-Turing degrees.
- $\mathcal{D}_\omega$ is an upper semi-lattice:
  - least element: $\mathbf{0}_\omega = \deg_\omega(\{\emptyset\}_{k<\omega})$;
  - l.u.b.: $\deg_\omega(\{A_k\}_{k<\omega}) \vee \deg_\omega(\{B_k\}_{k<\omega}) = \deg_\omega(\{A_k \oplus B_k\}_{k<\omega})$.

Polynomial sequence

- Let $\mathcal{A} = \{A_k\}_{k<\omega}$ be a sequence of sets of natural numbers. Define its polynomial sequence $\mathcal{P}(\mathcal{A}) = \{P_k(\mathcal{A})\}_{k<\omega}$ by induction:
  - $P_0(\mathcal{A}) = A_0$;
  - $P_{k+1}(\mathcal{A}) = (P_k(\mathcal{A}))' \oplus A_{k+1}$.
- example: $P_k(\{\emptyset\}_{t<\omega}) \equiv_T \emptyset^{(k)}$ uniformly in k; $P_k(A, \emptyset, \dots, \emptyset, \dots) \equiv_T A^{(k)}$ uniformly in k;
- canonical representative:

$$\mathcal{A} \equiv_\omega \mathcal{P}(\mathcal{A}).$$

- characterization of $\leq_\omega$:

$$\mathcal{A} \leq_\omega \mathcal{B} \iff A_k \leq_T P_k(\mathcal{B}) \text{ uniformly in } k.$$

- each lower cone $[\mathbf{0}_\omega, \mathbf{a}]$ is at most countable;

jump operation

- X codes $\mathcal{A}' \iff (\exists Y)[X \equiv_T Y' \text{ and } Y \text{ codes } \mathcal{A}]$
- jump of a sequence: $\mathcal{A}' = \{P_{k+1}(\mathcal{A})\}_{k<\omega}$.
  - strictly expansive: $\mathcal{A} <_\omega \mathcal{A}'$;
  - monotone: $\mathcal{A} \leq_\omega \mathcal{B} \Rightarrow \mathcal{A}' \leq_\omega \mathcal{B}'$;
- jump operation in the degrees: $\mathrm{deg}_\omega(\mathcal{A})' = \mathrm{deg}_\omega(\mathcal{A}')$;
- example: $\{\emptyset^{(k+1)}\}_{k<\omega} \in \mathbf{0}'_\omega$.
- $\mathbf{0}'_\omega$ is first-order definable if and only if the jump operation is first-order definable;

Embedding of the Turing degrees

- $A \leq_T X \iff A^{(k)} \leq_T X^{(k)}$ uniformly in $k \iff$
  $\iff X$ codes $\{A^{(k)}\}_{k<\omega} \iff X$ codes $(A, \emptyset, \ldots, \emptyset, \ldots)$;
- $\kappa : \mathcal{D}_T \to \mathcal{D}_\omega$ defined by

$$\deg_T(A) \stackrel{\kappa}{\longmapsto} \deg_\omega(A, \emptyset, \ldots, \emptyset, \ldots)$$

  is an embedding which preserves the order, l.u.b. operation and the jump;
- $\kappa[\mathcal{D}_T]$ is definable in $\mathcal{D}_\omega$ by a first-order formula in the language $\mathcal{L}(\leq, ')$;
- $\mathrm{Aut}(\mathcal{D}_T) \cong \mathrm{Aut}(\mathcal{D}'_\omega)$;

Minimal degrees in $\mathcal{D}_\omega$

- Characterisation of the minimal degrees: The degree $\mathbf{m}$ is minimal iff there exist $M \subseteq \omega$ and $n < \omega$ such that:
  - $\emptyset^{(n)} <_T M \leq_T \emptyset^{(n+1)}$ and $M' \equiv_T \emptyset^{(n+1)}$;
  - $\deg_T(M)$ is a minimal cover of $\mathbf{0}_T^{(n)}$;
  - $(\underbrace{\emptyset, \emptyset, \ldots, \emptyset}_{n}, M, \emptyset, \ldots, \emptyset, \ldots) \in \mathbf{m}$.
- all minimal ω-Turing degrees are bounded by $\mathbf{0}'_\omega$;
- at most countably many minimal degrees;

Minimal degrees in $\mathcal{D}_\omega$

- if M has a minimal Turing degree and it is low ($M' \equiv_T \emptyset'$) then:

$$(M, \emptyset, \dots, \emptyset, \dots)$$

  has a minimal ω-Turing degree;
- Since there are countably many low minimal Turing degrees, then there are countably many minimal ω-Turing degrees;
- Open question: Can $\mathbf{0}'_\omega$ be defined as the least degree, which bounds all the minimal degrees?

Thank You!