Subrecursive Computability in Analysis

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What happens if we restrict the generality of computable processes?
The restriction to the popular classes P, NP, EXP and others is well-studied. But this is not the case with subrecursive hierarchies. The aim of the dissertation is to study computable analysis in the framework of the most popular subrecursive hierarchy - Grzegorczyk's hierarchy of primitive recursive functions.

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We have $\mathcal{M}^2 \subseteq \mathcal{L}^2 \subseteq \mathcal{E}^2$ and whether each of these inclusions is proper is an open question.

Relative computability of real numbers

The triple of functions $\langle f, g, h\rangle$ of type $\mathbb{N} \to \mathbb{N}$ is a *name* of the real number ξ iff for all $n \in \mathbb{N}$,

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Definition

A function $a : \mathbb{N} \to \mathbb{R}$ is $\mathcal{F}$-computable iff there exist binary functions $f, g, h \in \mathcal{F}$, such that for any natural s, the triple $\langle \lambda n.f(s,n), \lambda n.g(n,s), \lambda n.h(n,s) \rangle$ names the real number $a(s)$.

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Theorem

Let $a : \mathbb{N} \to \mathbb{R}$ be an $\mathcal{F}$-computable function, such that the series $\sum_{s=0}^{\infty} a_s$ is convergent and let α be its sum. Let there exist a function $p : \mathbb{N} \to \mathbb{N}$ from the class $\mathcal{F}$, such that $\left|\sum_{s \geq y+1} a_s\right| \leq \frac{1}{n+1}$ for any natural n and $y = p(n)$. Then the number α is $\mathcal{F}$-computable.

Relative computability of famous constants II

This method is not suitable for the class $\mathcal{M}^2$, because it is not known whether this class is closed under bounded summation (that would be the case if and only if $\mathcal{M}^2 = \mathcal{L}^2$). In my master thesis I modify the method above and prove the following

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Using these two methods I proved $\mathcal{L}^2$-computability and $\mathcal{M}^2$-computability of a large number of constants. Among them is the number π, which is $\mathcal{M}^2$-computable and Euler's constant γ, which is $\mathcal{L}^2$-computable.

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$$\xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}},$$

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Examples:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}}}}, \pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}}}}.$$

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An essential part of my dissertation is the study of the converse theorem. It turns out that it is not true.

Counterexample for the converse of the theorem

To prove that I strengtened the theorem to the following

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For $n \geq 2$, if the graph $\{(k, a_k) | k \in \mathbb{N}\}$ of the continued fraction of ξ is an $\mathcal{E}^n$-relation, then ξ is $\mathcal{E}^n$-computable.

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We apply that theorem for

$$\xi_A = A(0,0) + \cfrac{1}{A(1,1) + \cfrac{1}{A(2,2) + \cfrac{1}{\ddots}}},$$

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So the number ξ_A is $\mathcal{E}^2$-computable (and thus $\mathcal{E}^n$-computable for any $n \geq 2$).

Its continued fraction is, however, not primitive recursive and it does not belong to any of the classes $\mathcal{E}^n$.

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It follows that the number π and all real algebraic numbers have continued fraction in $\mathcal{E}^3$ (elementary).

Partial converse of the theorem II

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Theorem

For all $n \geq 2$, if the number ξ is $\mathcal{E}^{n+1}$-computable and there exists a unary function $v \in \mathcal{E}^n$, which majorizes $\lambda k.a_k$ (the terms of the continued fraction of ξ), then ξ has continued fraction in $\mathcal{E}^{n+1}$.

Computing real functions

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Let $k \in \mathbb{N}$ and $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^k$. The triple (F, G, H), where F, G, H are operators of type $\mathcal{T}_1^{3k} \to \mathcal{T}_1$, is called a *computing system* for θ if for all $(\xi_1, \xi_2, \ldots, \xi_k) \in D$ and triples (f_i, g_i, h_i) that name ξ_i for $i = 1, 2, \ldots, k$, the triple

$$(F(f_1, g_1, h_1, f_2, g_2, h_2, \ldots, f_k, g_k, h_k),$$

$$G(f_1, g_1, h_1, f_2, g_2, h_2, \ldots, f_k, g_k, h_k),$$

$$H(f_1, g_1, h_1, f_2, g_2, h_2, \ldots, f_k, g_k, h_k))$$

names the real number $\theta(\xi_1, \xi_2, \ldots, \xi_k)$.

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For example, if $\mathbf{O}$ is the class of all computable operators, then the function θ is $\mathbf{O}$-computable if and only if it is computable in the classical sense of Grzegorczyk.

Computing real functions

Let $\mathbf{O}$ be a class of operators. The function θ is *uniformly* $\mathbf{O}$*-computable*, if there exists a computing system (F, G, H) for θ, such that $F, G, H \in \mathbf{O}$.
For example, if $\mathbf{O}$ is the class of all computable operators, then the function θ is $\mathbf{O}$-computable if and only if it is computable in the classical sense of Grzegorczyk.
There is a more general notion of computability of real functions in which we allow the operators to be partial, that is they might not be defined for some tuples of unary total functions, which are not tuples of names for the arguments of θ.

$\mathcal{F}$-substitutional operators

We fix a class $\mathcal{F}$ of total functions. For any $k, m \in \mathbb{N}$ we define inductively the class of $\mathcal{F}$-*substitutional operators* of type $\mathcal{T}_1^k \to \mathcal{T}_m$ as follows:

- For any m-argument projection h in $\mathbb{N}$, the operator F defined by $F(f_1, \ldots, f_k) = h$ is $\mathcal{F}$-substitutional.

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- For any $i \in \{1, \ldots, k\}$, if F_0 is an $\mathcal{F}$-substitutional operator of type $\mathcal{T}_1^k \to \mathcal{T}_m$, then so is the operator F defined by

$$F(f_1, \ldots, f_k)(n_1, \ldots, n_m) = f_i(F_0(f_1, \ldots, f_k)(n_1, \ldots, n_m)).$$

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- For any $r \in \mathbb{N}$ and $f \in \mathcal{T}_r \cap \mathcal{F}$, if $F_1, \ldots, F_r$ are $\mathcal{F}$-substitutional operators of type $\mathcal{T}_1^k \to \mathcal{T}_m$, then so is the operator F, defined by

$$F(f_1, \ldots, f_k)(n_1, \ldots, n_m) = \\ f(F_1(f_1, \ldots, f_k)(n_1, \ldots, n_m), \ldots, F_r(f_1, \ldots, f_k)(n_1, \ldots, n_m)).$$

Uniformly $\mathcal{F}$-computable real functions

The real function $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^k$ is uniformly $\mathcal{F}$-computable, if there exists a computing system (F, G, H) for θ, such that F, G, H are $\mathcal{F}$-substitutional operators.

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If the class $\mathcal{F}$ contains $\mathcal{M}^2$ and is closed under substitution, it follows easily that the set of all $\mathcal{F}$-computable real numbers is closed under the elementary functions of calculus.

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$$E(f_1, g_1, h_1, \ldots, f_k, g_k, h_k)(s) = 0.$$

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- There exists a natural number s satisfying the equality
$$E(f_1, g_1, h_1, \ldots, f_k, g_k, h_k)(s) = 0.$$
- For any natural number s satisfying the above equality, the triple $(\tilde{f}, \tilde{g}, \tilde{h})$ names the real number $\theta(\xi_1, \ldots, \xi_k)$, where
$$\tilde{f} = \lambda t.F(f_1, g_1, h_1, \ldots, f_k, g_k, h_k)(s, t),$$
$$\tilde{g} = \lambda t.G(f_1, g_1, h_1, \ldots, f_k, g_k, h_k)(s, t),$$
$$\tilde{h} = \lambda t.H(f_1, g_1, h_1, \ldots, f_k, g_k, h_k)(s, t).$$

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- conditional $\mathcal{F}$-computability is preserved by substitution,
- all conditionally $\mathcal{F}$-computable real functions are locally uniformly $\mathcal{F}$-computable,
- the conditionally $\mathcal{F}$-computable real functions with compact domains are uniformly $\mathcal{F}$-computable.

Thank you for your attention!