Jump inversion of structures

Computability Seminar at Notre Dame University

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The plan

- Review some results of Ash and Knight about how to **strongly** code a set by a sequence of structures.
- Show some variants of their work - how to **weakly** code a set by a sequence of structures.
- Some applications of these ideas - new proofs of old results.
- Connection with Alexandra's talk.

Introduction

The idea of coding a set by a sequence of structures is an old one. It is studied thoroughly by Ash and Knight (1990). Here I will give a few applications connected by the theme of "jump inversion" of structures.

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The idea of coding a set by a sequence of structures is an old one. It is studied thoroughly by Ash and Knight (1990). Here I will give a few applications connected by the theme of "jump inversion" of structures.
Recall the following classical result - a jump inversion of Turing degrees.

Theorem (Friedberg, 1957)

For every natural number n and Turing degree $\mathbf{a}$, there exists a Turing degree $\mathbf{b}$ such that

$$\mathbf{b^{(n)}} = \mathbf{a} \vee \mathbf{0^{(n)}}.$$

If $\mathbf{a} \geq \mathbf{0}^{(n)}$, then

$$\mathbf{b^{(n)}} = \mathbf{a}.$$

Later generalized to any computable ordinal by MacIntyre (1977).

Associate a Turing degree to a structure

- We consider countable structures whose domains are $\mathbb{N}$ or a computable subset of $\mathbb{N}$.
- We say that $\mathcal{B}$ is a copy of $\mathcal{A}$ if $\mathcal{B} \cong \mathcal{A}$.
- Usually we identify the copy $\mathcal{B}$ by its atomic diagram, which is a set of natural numbers (under some effective coding of formulas).
- Associate the Turing degree **b** with the copy $\mathcal{B}$ of $\mathcal{A}$ if $deg_T(\mathcal{D}(\mathcal{B})) \in \mathbf{b}$. We say that $\mathcal{B}$ is a computable structure if $\mathcal{D}(\mathcal{B})$ is a computable set of natural numbers.

Spectra of structures

- **The Turing degree spectrum** of $\mathcal{A}$ is the set

$$Spec(\mathcal{A}) = \{d_T(\mathcal{D}(\mathcal{B})) \mid \mathcal{B} \text{ is a copy of } \mathcal{A}\}.$$

- The n-th jump Turing spectrum of $\mathcal{A}$ is the set $Spec_n(\mathcal{A}) = \{\mathbf{a}^{(n)} \mid \mathbf{a} \in Spec(\mathcal{A})\}$.
- In all non-trivial cases, $Spec(\mathcal{A})$ is **closed upwards** relative to $\leq_T$.
- One way to compare the structures $\mathcal{A}$ and $\mathcal{B}$ is by comparing their Turing degree spectra. For example, a question in the style of "jump inversion" is:

$$(\forall \alpha < \omega)(\forall \mathcal{A})(\exists \mathcal{B})[\ Spec(\mathcal{A}) = Spec_n(\mathcal{B})\ ]?$$

Computable infinitary formulas

The computable infinitary formulas are infinitary formulas, in which the conjunctions and disjunctions are over c.e. sets.

1) the Σ_0^c and Π_0^c formulas are the finitary quantifier-free formulas.
2) for every computable ordinal $\alpha > 0$,
    a) $\varphi(\bar{x})$ is a Σ_α^c formula if $\varphi(\bar{x}) = \bigvee_{i \in W_e} \exists \bar{y}_i \psi_i(\bar{x}, \bar{y}_i)$, where each $\psi_i(\bar{x}, \bar{y}_i)$ is $\Pi_{\beta_i}^c$ for some $\beta_i < \alpha$.
    b) $\varphi(\bar{x})$ is a Π_α^c formula if $\varphi(\bar{x}) = \bigwedge_{i \in W_e} \forall \bar{y}_i \psi_i(\bar{x}, \bar{y}_i)$, where each $\psi_i(\bar{x}, \bar{y}_i)$ is $\Sigma_{\beta_i}^c$ for some $\beta_i < \alpha$.

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We can code the compuable infinitary formulas into the natural numbers.

Theorem (ĐŘsh)

If $\varphi(\bar{x})$ is a Σ_α^c or Π_α^c formula, then the relation $\varphi^{\mathcal{A}} = \{\bar{a} \in A^r \mid \mathcal{A} \models \varphi(\bar{a})\}$ is $\Sigma_\alpha^0(\mathcal{D}(\mathcal{A}))$ or $\Pi_\alpha^0(\mathcal{D}(\mathcal{A}))$. We can do this uniformly, i.e. for a fixed notation a for α, by the code of $\varphi(\bar{x})$ we can effectively find the code of $\varphi^{\mathcal{A}}$.

Relatively intrinsically Σ^0_α relations

Definition

We say that the relation R over A is relatively intrinsically Σ^0_α in $\mathcal{A}$, if for every isomorphism f of $\mathcal{A}$, $f^{-1}(R)$ is $\Sigma^0_\alpha(f^{-1}(\mathcal{A}))$. Similarly, we can define relatively intrinsically Π^0_α relations.

Theorem (Ash-Knight-Manasse-Slaman, Chisholm)

For a given relation R over $\mathcal{A}$. The following are equivalent:

1) R is relatively intrinsically Σ^0_α;
2) there exists a Σ^c_α formula $\phi(\bar{x}, \bar{y})$ and parameters $\bar{b} \in A$, for which $(\forall \bar{a} \in A)[\bar{a} \in R \leftrightarrow \mathcal{A} \models \phi(\bar{a}, \bar{b})]$, usually denoted $R \in \Sigma^c_\alpha(\mathcal{A})$.

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For $\alpha < \omega_1^{CK}$, and any structure $\mathcal{A}$, can we find a structure $\mathcal{B}$ with the following "jump inversion" property:

$$(\forall R \subseteq A)[R \in \Sigma^c_1(\mathcal{A}) \leftrightarrow R \in \Sigma^c_\alpha(\mathcal{B})]?$$

Jump structures

It is natural to ask what would be the jump $\mathcal{A}'$ of the structure $\mathcal{A}$. We will define $\mathcal{A}'$ so that we have the following property:

$$Spec_1(\mathcal{A}) = Spec(\mathcal{A}').$$

Jump structures

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$$Spec_1(\mathcal{A}) = Spec(\mathcal{A}').$$

Moreover, we want the following: a relation R is r.i.c.e. on $\mathcal{A}'$ iff R is relatively intrinsically Σ_2^0 on $\mathcal{A}$. Probably the most straightforward definition is the one given by Antonio Montalbán:

$$\mathcal{A}' = (\mathcal{A}, \{R_i\}_{i<\omega}),$$

where R_i is an effective enumeration of all r.i.c.e. relations in $\mathcal{A}$. Actually, we use the effective listing of all Σ_1^c formulas.

It is easy to see that any copy of the structure $\mathcal{A}'$ computes the halting set.
Obviously, for any copy $\mathcal{B}'$ of $\mathcal{A}'$,

$$\mathcal{D}(\mathcal{B}') \leq_T \mathcal{D}(\mathcal{B})'.$$

Thus, $Spec(\mathcal{A}') \subseteq Spec_1(\mathcal{A})$.
For any i, consider the Σ_1^c sentence

$$\phi_i \equiv \bigvee_{i \in W_i} \exists x (x = x).$$

Clearly, $i \in \emptyset'$ iff $\mathcal{A} \models \phi_i$.

- $\mathcal{B}$ is a strong jump invert of $\mathcal{A}$ if

$$\mathbf{a}' \in Spec(\mathcal{B}) \leftrightarrow \mathbf{a} \in Spec(\mathcal{A}).$$

- $\mathcal{B}$ is a weak jump invert of $\mathcal{A}$ if

$$Spec(\mathcal{B}) = Spec_1(\mathcal{A}).$$

- Strong jump inversion implies weak jump inversion.
- Why is this weaker? If $\mathbf{a}' \in Spec(\mathcal{A}')$, then $\mathbf{a}' \in Spec_1(\mathcal{A})$, then there is $\mathbf{b}$ such that $\mathbf{b}' = \mathbf{a}'$ and $\mathbf{b} \in Spec(\mathcal{A})$. $\mathbf{a}$ and $\mathbf{b}$ may be incomparable and we cannot be sure that $\mathbf{a}$ is in $Spec(\mathcal{A})$.

Recall that for any structure $\mathcal{A}$, $\mathcal{A}'$ is the weak jump invert of $\mathcal{A}$. This is not the case for strong jump inversion.
For any Boolean algebra $\mathcal{B}$, $\mathcal{B}'$ is a strong jump invert of $\mathcal{B}$, i.e.

$$\mathbf{a}' \in \mathit{Spec}(\mathcal{B}') \leftrightarrow \mathbf{a} \in \mathit{Spec}(\mathcal{B}).$$

- The direction $\rightarrow$ is obvious since $\mathit{Spec}(\mathcal{B}') = \mathit{Spec}_1(\mathcal{B})$.
- The direction $\leftarrow$ is a Theorem by Downey-Jockusch (1994).

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Not every linear ordering Ł has the strong jump inversion property.
Example by Downey-Knight, There is a linear ordering Ł such that $\mathbf{0} \notin Spec(Ł)$, but $\mathbf{0}' \in Spec_1(Ł)$.

Let's look at the Boolean algebra strong jump inversion more closely. It say that if $\Delta^0_2(X)$ computes $\mathcal{B}'$, then there is $\mathcal{A} \cong \mathcal{B}$ such that X computes $\mathcal{A}$. What is the complexity of f, where $\mathcal{A} \cong_f \mathcal{B}$. By the original Downey-Jockusch theorem, f is Δ^0_4. By an unpublished result of Frolov, this is sharp.

Let's look at the Boolean algebra strong jump inversion more closely. It say that if $\Delta^0_2(X)$ computes $\mathcal{B}'$, then there is $\mathcal{A} \cong \mathcal{B}$ such that X computes $\mathcal{A}$. What is the complexity of f, where $\mathcal{A} \cong_f \mathcal{B}$. By the original Downey-Jockusch theorem, f is Δ^0_4. By an unpublished result of Frolov, this is sharp.

Theorem

Our theorem where f is a Δ^0_3 isomorphism.

We produce a computable Boolean algebra by a finite injury priority construction effective relative to Δ^0_2.

- Given a structure $\mathcal{A}$, is there a structure $\mathcal{B}$ such that $\mathcal{A}$ is a strong jump invert of $\mathcal{B}$, i.e.

$$\mathbf{a}' \in Spec(\mathcal{A}) \;\leftrightarrow\; \mathbf{a} \in Spec(\mathcal{B}).$$

- Given a structure $\mathcal{A}$, is there a structure $\mathcal{B}$ such that $\mathcal{A}$ is a weak jump invert of $\mathcal{B}$, i.e.

$$Spec(\mathcal{A}) = Spec_1(\mathcal{B}).$$

Marker's extensions
Soskova-Soskova, using Goncharov-Khoussainov,
the structure $\mathcal{A}$ is a strong jump inversion of $\mathcal{M}^{\exists\forall}$.

Strongly coding a set by a sequence of structures

Let S be a set of natural numbers and $\mathcal{B}_0$, $\mathcal{B}_1$ are structures in the same language. We say that the sequence of structures $\mathscr{C} = \{\mathcal{C}_n\}_{n<\omega}$ *code* the set S if

$$\mathcal{C}_n \cong \begin{cases} \mathcal{B}_1, & \text{if } n \in S \\ \mathcal{B}_0, & \text{if } n \notin S. \end{cases}$$

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The sequence $\mathscr{C} = \{\mathcal{C}_n\}_{n<\omega}$ is **uniformly computable**, if it consists of computable copies of $\mathcal{B}_0, \mathcal{B}_1$ and for each n we can effectively find a computable index for $\mathcal{C}_n$, although we do not know whether this index corresponds to $\mathcal{B}_0$ or $\mathcal{B}_1$.
If $\mathscr{C}$ is a uniformly commputable sequence, then we say that $\mathscr{C}$ **strongly codes** the set S.

Strongly coding a set by a sequence of structures

Example

The following are equivalent:

1) $\mathscr{C} = \{\mathcal{C}_n\}$ strongly codes the set S, where

$$\mathcal{C}_n \cong \begin{cases} \omega, & \text{if } n \in S \\ \omega^\star, & \text{if } n \notin S, \end{cases}$$

2) S is a Δ^0_2 set.

The question what sets we can strongly coded by what kind of structures was studied by Ash and Knight (1990).

Pairs of Structures (Ash & Knight)

Fix two structures $\mathcal{A}$ and $\mathcal{B}$ and a countable ordinal $\beta \geq 1$. For all tuples $\bar{a} \in \mathcal{A}$ and $\bar{b} \in \mathcal{B}$ with the same length, define $\bar{a} \leq_\beta \bar{b}$ iff the infinitary Π_β formulas true of $\bar{a}$ in $\mathcal{A}$ are true of $\bar{b}$ in $\mathcal{B}$. These are called the *standard back-and-forth* relations. A pair of structures $\{\mathcal{A}, \mathcal{B}\}$ is called α-**friendly** if $\mathcal{A}, \mathcal{B}$ are computable structures and for all $\beta < \alpha$ the relations $\leq_\beta$ are c.e. uniformly in β.

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Theorem (Ash-Knight, 1990)

Let $\mathcal{B}_0, \mathcal{B}_1$ be structures, α be a computable successor ordinal and

1) $\mathcal{B}_0$ and $\mathcal{B}_1$ are computable structures in the same language $\mathscr{L}$,
2) $\{\mathcal{B}_0, \mathcal{B}_1\}$ is α-friendly;
3) $\mathcal{B}_0$ and $\mathcal{B}_1$ satisfy the same $\Sigma^{\text{inf}}_\beta$ sentences for all $\beta < \alpha$.

Then for any Δ^0_α set S there is a sequence $\mathscr{C}$, consisting of copies of $\mathcal{B}_0, \mathcal{B}_1$, which strongly codes S.

Strong jit structure $\mathcal{N}$

Let $\mathcal{A} = (A; R_0, R_1, \ldots, R_{s-1})$ be a structure. Consider R_i in place of the set S above. Suppose we have the sequences $\mathscr{C}_i$, which code the relations R_i. Then we can build a new structure $\mathcal{N}$, which is, roughly speaking, the join of all the structures in $\mathscr{C}_i$, for $i < s$.

Theorem (Goncharov-Harizanov-Knight-McCoy-Miller-Solomon)

Fix a computable succ. ordinal $\alpha \geq 2$ and a structure $\mathcal{A}$. Let $\mathcal{B}_0, \mathcal{B}_1$ be such that:

1) $\mathcal{B}_0$ and $\mathcal{B}_1$ are computable structures in the same language $\mathscr{L}$,
2) $\{\mathcal{B}_0, \mathcal{B}_1\}$ is α-friendly,
3) $\mathcal{B}_0, \mathcal{B}_1$ satisfy the same $\Sigma^{\text{inf}}_\beta$ sentences for all $\beta < \alpha$,
4) **each $\mathcal{B}_i$ satisfies some Σ^c_α sentence that is not true in the other.**

Let $\mathcal{N}$ be the structure built from the sequences $\mathscr{C}_i$ which strongly encode R_i. Then $\mathcal{A}$ has a $\Delta^0_\alpha(X)$-computable copy iff $\mathcal{N}$ has an X-computable copy.

Coding $\{\mathcal{C}_n\}$ into a structure

Let us consider the structure $\mathcal{A} = (A, R)$, where R is unary, and a pair of structures $\mathcal{B}_0$, $\mathcal{B}_1$ for the same relational language, let $\mathcal{N} = (A \cup U, A, U, Q, \dots)$, where

1) $A \cap U = \emptyset$;
2) Q assigns to each element a in A an infinite set U_a, where $x \in U_a$ iff $\mathcal{N} \models Q(a, x)$;
3) The sets U_a form a partition of U;
4) each of the other relations of $\mathcal{N}$ (in $\dots$) correspond to some symbol in the language of $\mathcal{B}_0$, $\mathcal{B}_1$, and is the union of its restrictions to the sets $\mathcal{A}_a$;
5) For each element a in A, if $\mathcal{U}_a = (U_a, \dots)$, then

$$\mathcal{U}_a \cong \begin{cases} \mathcal{B}_0, & \text{if } \mathcal{A} \models R(a) \\ \mathcal{B}_1, & \text{if } \mathcal{A} \models \neg R(a) \end{cases}$$

Such pairs $\{\mathcal{B}_0, \mathcal{B}_1\}$ exist

Denote $\xi_\beta = \sum_{\gamma<\beta} \mathbb{Z}^\gamma \cdot \omega$. Then for ordinals α, where

- $\alpha = 2\beta + 1$,

$$\mathcal{B}_0 \cong \xi_\beta \oplus (\xi_\beta + \mathbb{Z}^\beta);$$
$$\mathcal{B}_1 \cong (\xi_\beta + \mathbb{Z}^\beta) \oplus \xi_\beta;$$

- $\alpha = 2\beta + 2$,

$$\mathcal{B}_0 \cong \mathbb{Z}^\beta \cdot \omega;$$
$$\mathcal{B}_1 \cong \mathbb{Z}^\beta \cdot \omega^\star;$$

This is from the GHKMMS paper.

Weakly coding of a set

Question

Let α be a computable *successor* ordinal, $\mathcal{B}_0$, $\mathcal{B}_1$ are computable structures in the same language. Determine conditions for $\mathcal{B}_0$, $\mathcal{B}_1$, and a set S, for which there exists a (may not be computable) sequence $\mathscr{C}$ of copies of $\mathcal{B}_0$ and $\mathcal{B}_1$, which codes S, and

$$\Delta^0_\alpha(\bigoplus_n \mathcal{C}_n) \leq_T S.$$

In this case we say that $\mathscr{C}$ **weakly codes** the set S.

Weakly coding of a set

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Theorem (Vatev, 2013)

Let $\mathcal{B}_0, \mathcal{B}_1$ be computable structures, α be a computable successor ordinal and $\mathcal{B}_0$ and $\mathcal{B}_1$ satisfy the same Σ^c_β sentences for all $\beta < \alpha$. Then for any Δ^0_α set S there is a sequence $\mathscr{C}$, consisting of copies of $\mathcal{B}_0, \mathcal{B}_1$, which weakly codes S.

Weak JIT structure $\mathcal{N}$

The requirement for α-friendliness is removed.

Theorem (Vatev 2013)

Fix a computable *successor* ordinal $\alpha \geq 2$. Let $\mathcal{A}$ be a countable structure such that **every copy of $\mathcal{A}$ is above Δ^0_α**. Let $\mathcal{B}_0, \mathcal{B}_1$, satisfy the following properties:

a) $\mathcal{B}_0$ and $\mathcal{B}_1$ are computable structures in the same language $\mathscr{L}$;

b) $\mathcal{B}_0, \mathcal{B}_1$ satisfy the same Σ^c_β sentences for every $\beta < \alpha$,

c) each $\mathcal{B}_i$ satisfies some Σ^c_α sentence, which is not true in the other structure.

Let $\mathcal{N}$ be the structure built from the sequences $\mathscr{C}_i$ which **weakly code** the relations R_i in $\mathcal{A}$. Then:

1) $Spec_{\alpha^-}(\mathcal{N}) = Spec(\mathcal{A})$, and

2) $(\forall X \subseteq A)[X \in \Sigma^c_\alpha(\mathcal{N}) \leftrightarrow X \in \Sigma^c_1(\mathcal{A})]$.

Here $\alpha^- = \alpha - 1$, if $\alpha < \omega$ and $\alpha^- = \alpha$, otherwise.

Some details

The proof is by forcing similar to [AKMS, C]. Here the forcing conditions are finite sequences of finite mappings, called *partial conditions*, and have the form $\mathscr{C} = (\tau_0, \tau_1, \ldots, \tau_{k-1})$.
We define the *diagram* of $\mathscr{C}$ *with respect to* $X \in 2^\omega$ as

$$D_X(\mathscr{C}) = \bigoplus_{j < len(\mathscr{C})} \tau_j^{-1}(\mathcal{B}_{X(j)}).$$

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We define the *diagram* of $\mathscr{C}$ *with respect to* $X \in 2^\omega$ as

$$D_X(\mathscr{C}) = \bigoplus_{j < len(\mathscr{C})} \tau_j^{-1}(\mathcal{B}_{X(j)}).$$

Total conditions are infinite sequences of bijections

$$\mathbf{C} = (f_0, f_1, f_2, \ldots, f_i, \ldots).$$

We define the *diagram* of the total condition $\mathbf{C}$ *with respect to* $X \in 2^\omega$ as

$$D_X(\mathbf{C}) = \bigoplus_{j<\omega} f_j^{-1}(\mathcal{B}_{X(j)}).$$

The forcing relation

It models the definition of the Turing jump.

(i) $\mathscr{C} \Vdash_1^X F_e(x) \;\leftrightarrow\; x \in W_e^{D_X(\mathscr{C})}$.

(ii) Let $\alpha = \beta + 1$. Then

$$\begin{aligned}\mathscr{C} \Vdash_{\beta+1}^X F_e(x) \;\leftrightarrow\; & (\exists \delta \in 2^{<\omega})[x \in W_e^\delta \;\&\; (\forall z \in Dom(\delta))[\\ & (\delta(z) = 1 \;\&\; \mathscr{C} \Vdash_\beta^X F_z(z)) \vee \\ & (\delta(z) = 0 \;\&\; \mathscr{C} \Vdash_\beta^X \neg F_z(z))]].\end{aligned}$$

(iii) Let $\alpha = \lim \alpha(p)$. Then

$$\begin{aligned}\mathscr{C} \Vdash_\alpha^X F_e(x) \;\leftrightarrow\; & (\exists \delta \in 2^{<\omega})[x \in W_e^\delta \;\&\; (\forall z \in Dom(\delta))[z = \langle x_z, p_z \rangle \\ & ((\delta(z) = 1 \;\&\; \mathscr{C} \Vdash_{\alpha(p_z)}^X F_{x_z}(x_z)) \vee \\ & (\delta(z) = 0 \;\&\; \mathscr{C} \Vdash_{\alpha(p_z)}^X \neg F_{x_z}(x_z)))]].\end{aligned}$$

(iv) $\mathscr{C} \Vdash_\alpha^X \neg F_e(x) \;\leftrightarrow\; (\forall \mathscr{D})[\mathscr{C} \subseteq \mathscr{D} \;\rightarrow\; \mathscr{D} \nVdash_\alpha^X F_e(x)]$.

Properties of the forcing relation

Let us denote

$$\mathscr{C} \approx_k \mathscr{D} \;\leftrightarrow\; \bigwedge_{i \neq k} (\tau_i^{\mathscr{C}} = \tau_i^{\mathscr{D}}),$$

i.e. the partial conditions $\mathscr{C}$ and $\mathscr{D}$ might differ only in the k-th coordinate.

Lemma

Let $\mathcal{B}_0$ and $\mathcal{B}_1$ be **computable** *structures, $X \in 2^\omega$ is* **computable**, *$\mathscr{C}$ be a partial condition. Then for all natural numbers e, z, there is a Σ^c_α sentence $\Phi^\alpha_{\mathscr{C},e,z}$ such that*

$$(\exists \mathscr{D})[\mathscr{D} \approx_k \mathscr{C} \;\&\; \mathscr{D} \Vdash^X_\alpha F_e(z)] \;\leftrightarrow\; \mathcal{B}_{X(k)} \models \Phi^\alpha_{\mathscr{C},e,x}.$$

If $\mathcal{B}_0$ and $\mathcal{B}_1$ satisfy the same Σ^c_α sentences, then we can change the k-th bit in X and continue to force the same requirement $F_e(x)$.

Properties of the forcing relation

For a condition $\mathscr{C}$, we let $X_{\mathscr{C}} \in 2^{\omega}$ be such that $X_{\mathscr{C}}(i) = X(i)$ for $i < len(\mathscr{C})$ and $X_{\mathscr{C}}(i) = 0$ for $i \geq len(\mathscr{C})$.

Lemma

Let us fix a computable ordinal $\alpha \geq 1$. Let $\mathcal{B}_0$ and $\mathcal{B}_1$ be computable structures in the language $\mathscr{L}$ with equality and **both structures satisfy the same Σ^c_α sentences in $\mathscr{L}$.** *Then for every partial condition $\mathscr{C}$, $X \in 2^{\omega}$ and natural numbers e, z:*

1) $\mathscr{C} \Vdash^{X}_{\alpha} F_e(z) \ \leftrightarrow\ \mathscr{C} \Vdash^{X_{\mathscr{C}}}_{\alpha} F_e(z)$,
2) $\mathscr{C} \Vdash^{X}_{\alpha} \neg F_e(z) \ \leftrightarrow\ \mathscr{C} \Vdash^{X_{\mathscr{C}}}_{\alpha} \neg F_e(z)$.

For finite ordinals, this can also be done by the method of Marker's extensions (Soskov and A. Soskova).

Remark

We do not need α-friendliness here and hence $\mathcal{N}$ may **not** have a computable copy. From b) and c) we see that this construction does not work for limit ordinals. Soskov has an example of a structure without ω-jump invert for spectra.

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In the GHKMMS paper, the structures $\mathcal{B}_0$ and $\mathcal{B}_1$ are also **uniformly relatively Δ^0_α-categorical**, i.e. given an X-computable index for $\mathcal{C} \cong \mathcal{B}_i$, we can find a $\Delta^0_\alpha(X)$ computable index for an isomorphism from $\mathcal{B}_i$ onto $\mathcal{C}$. What is this property useful for ?

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We do not need α-friendliness here and hence $\mathcal{N}$ may **not** have a computable copy. From b) and c) we see that this construction does not work for limit ordinals. Soskov has an example of a structure without ω-jump invert for spectra.

I will talk about the limit case later.
In the GHKMMS paper, the structures $\mathcal{B}_0$ and $\mathcal{B}_1$ are also **uniformly relatively Δ^0_α-categorical**, i.e. given an X-computable index for $\mathcal{C} \cong \mathcal{B}_i$, we can find a $\Delta^0_\alpha(X)$ computable index for an isomorphism from $\mathcal{B}_i$ onto $\mathcal{C}$. What is this property useful for ?

- For GHKMMS, it is needed to show that there are Δ^0_α categorical structures, which are not relatively Δ^0_α categorical, α - succ. ordinal.
- Another application - in the study of categorocity spectrum of structures.

Definitions

Definition

The computable structure $\mathcal{A}$ is $\mathbf{d}$-*categorical* if for every computable copy $\mathcal{B}$ of $\mathcal{A}$, there exists an isomorphism $f : \mathcal{B} \cong \mathcal{A}$ such that $f \leq_T \mathbf{d}$.

Example

The structure $\mathcal{A} = (\mathbb{Q}, <)$ is computably categorical, whereas $\mathcal{B} = (\omega, <)$ is not computably categorical.

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The structure $\mathcal{A} = (\mathbb{Q}, <)$ is computably categorical, whereas $\mathcal{B} = (\omega, <)$ is not computably categorical.

Definition (Fokina, Kalimullin, Miller)

Let $\mathcal{A}$ be a computable structure. *The categoricity spectrum* of $\mathcal{A}$ is the set $CatSpec(\mathcal{A}) = \{\mathbf{d} \mid \mathcal{A} \text{ is } \mathbf{d}\text{-categorical}\}$. We say that $\mathbf{d}$ is *the degree of categoricity* of $\mathcal{A}$ if $\mathbf{d}$ is *the least* degree in $CatSpec(\mathcal{A})$.

There is also a relativised version.

Definition (F. K. M.)

Let $\mathbf{c}$ be the Turing degree of the structure $\mathcal{A}$. We define *the categoricity spectrum* of $\mathcal{A}$ relative to $\mathbf{c}$ to be the set $CatSpec_{\mathbf{c}}(\mathcal{A}) =$

$$\{\mathbf{d} \mid (\forall \mathcal{B} \cong \mathcal{A})[deg(\mathcal{B}) \leq_T \mathbf{c} \ \rightarrow \ (\exists f : \mathcal{B} \cong \mathcal{A})[f \leq_T \mathbf{d}]\}.$$

For computable $\mathcal{A}$, we have

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A question of type "jump inversion" is the following:

Question

Under what conditions for a $\mathbf{d}$-computable structure $\mathcal{A}$ can we claim the existence of a computable structure $\mathcal{B}$ such that

$$CatSpec(\mathcal{B}) = CatSpec_{\mathbf{d}}(\mathcal{A})?$$

This notion is relatively new and not well-studied. One interesting question is which degrees can be degrees of categoricity.

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For every $\kappa \leq \omega$, $\mathbf{0}^{(\kappa)}$ is the degree of categoricity.

In short, they build $\mathbf{0}^{(n)}$-computable graph $\mathcal{A}$, for which $CatSpec_{\mathbf{0}^{(n)}}(\mathcal{A})$ is the cone above $\mathbf{0}^{(n)}$. By applying the $(n+1)$-th Marker's extension of $\mathcal{A}$, they obtain the structure $\mathcal{M}$, for which

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This is a result of the type "jump inversion". Csima, Franklin and Shore generalise this result to any computable ordinal α. They use the $\mathbf{0}^{(\alpha)}$-computable graph $\mathcal{A}$ of F. K. M. and then they attach to some nodes of the graph certain "back-and-forth trees" of Hirschfeldt and White to obtain a computable sturcture $\mathcal{A}$ such that

$$CatSpec_{\mathbf{0}^{(\alpha)}}(\mathcal{A}) = CatSpec(\mathcal{M}).$$

An application of the strong coding construction

Lemma

Let α be a computable *successor* ordinal and $\mathcal{A}$ is $\mathbf{0}^{(\alpha)}$-computable structure, such that $CatSpec_{\mathbf{0}^{(\alpha)}}(\mathcal{A})$ is the cone above $\mathbf{0}^{(\alpha)}$. Then there exists a **computable** structure $\mathcal{N}$, obtained from $\mathcal{A}$ by the **strong coding** construction, for which

$$CatSpec(\mathcal{N}_\alpha) = CatSpec_{\mathbf{0}^{(\alpha)}}(\mathcal{A}).$$

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This lemma allows us to give a new proof to the following theorem.

Theorem (F. K. M. for $\alpha \leq \omega$, C. F. S. for $\alpha < \omega_1^{CK}$)

Let α is an arbitrary computable ordinal. There exists a computable structure $\mathcal{B}$ with degree of categoricity $\mathbf{0}^{(\alpha)}$.

About the structures $\mathcal{B}_0, \mathcal{B}_1$ in the lemma

The first four conditions are the old ones - those for building a strong jit structure.

- $\mathcal{B}_0$ and $\mathcal{B}_1$ are computable structures with domains in the same language $\mathscr{L}$;
- $\mathcal{B}_0, \mathcal{B}_1$ satisfy the same $\Sigma^{\text{inf}}_{\beta}$ sentences for every $\beta < \alpha$,
- each $\mathcal{B}_i$ satisfy some Σ^c_α sentence, which is not true in the other structuer $\mathcal{B}_{1-i}$.
- the pair $\{\mathcal{B}_0, \mathcal{B}_1\}$ is α-friendly;

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The limit case

Let us have a uniformly computable sequence of pairs of structures $\{(\mathcal{B}_0^n, \mathcal{B}_1^n)\}_n$. We say that the sequence $\{\mathcal{C}_n\}_n$ codes the set S if

$$\mathcal{C}_n \cong \begin{cases} \mathcal{B}_1^n, & \text{if } n \in S \\ \mathcal{B}_0^n, & \text{if } n \notin S. \end{cases}$$

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Let $\alpha = \lim \alpha_n$ be a computable limit ordinal and α_n are succ. ordinals. If we choose $\mathcal{B}_0^n$ and $\mathcal{B}_1^n$ to satisfy the conditions for α_n-weak jit, then we can build a sequence $\{\mathcal{C}_n\}_n$ such that

$$\Delta^0_\alpha(\bigoplus_n \mathcal{C}_n) \leq_T S.$$

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$$\Delta_\alpha^0(\bigoplus_n \mathcal{C}_n) \leq_T S.$$

When does this work for jump inversion of spectra of structures? We know that in the general case it does not for limit ordinals.

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- The spectrum of $\mathcal{A}$ has a **least degree** $\mathbf{d} \geq 0^{(\alpha)}$. Fix a copy $\mathcal{B}$ such that $\mathcal{D}(\mathcal{B})$ belongs to $\mathbf{d}$.
- If $\mathcal{N}$ is the weak jit structure built for $\mathcal{B}$, we obtain $Spec(\mathcal{A}) \subseteq Spec(\mathcal{N})$.

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- Let $\{(\Phi_0^n, \Phi_1^n)\}$ be the $\Sigma_{\alpha_n}^c$ sentences that help us distinguish between $\mathcal{B}_0^n$ and $\mathcal{B}_1^n$. Consider an element $\mathcal{C}$ of the sequence of structures $\mathscr{C}$. We need to be able to find n and i such that $\mathcal{C} \cong \mathcal{B}_i^n$, effectively relative to oracle Δ_α^0.
- To do that we require the pairs $(\mathcal{B}_0^n, \mathcal{B}_1^n)$ to be such that for $i = 0, 1$:

$$\mathcal{B}_i^n \models \Phi_i^n \ \& \ \bigwedge_{k \neq n} \neg \Phi_i^k \ \& \ \bigwedge_n \neg \Phi_{1-i}^n.$$

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- To do that we require the pairs $(\mathcal{B}_0^n, \mathcal{B}_1^n)$ to be such that for $i = 0, 1$:

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- We have this property. If $\alpha_n = 2\beta_n + 2$, $\mathcal{B}_0^n$ can be $\mathbb{Z}^{\beta_n} \cdot \omega$ and Φ_0^n says that there is a least $\mathbb{Z}^{\beta_n}$ block. Then B_1^n will be $\mathbb{Z}^{\beta_n} \cdot \omega^\star$ and Φ_1^n will say that there is a greatest $\mathbb{Z}^{\beta_n}$ block.

The limit case for weak jit

In this way we obtain a new proof of an old result.

Theorem

Let α be a computable limit ordinal and let $\mathcal{A}$ be a structure whose spectrum has a least degree $\mathbf{d} \geq \Delta^0_\alpha$. Then there exists a weak jit structure $\mathcal{N}$ such that

1) $Spec_\alpha(\mathcal{N}) = Spec(\mathcal{A})$, and
2) $(\forall X \subseteq A)[X \in \Sigma^c_\alpha(\mathcal{N}) \leftrightarrow X \in \Sigma^c_1(\mathcal{A})]$.

Yet another application

Definition

For a countable sequence of sets $\mathscr{R} = \{R_n\}_{n\in\omega}$ and a set B, $\mathscr{R} \leq_{c.e.} B$ if $R_n \leq_{c.e.} B^{(n)}$ uniformly in n;

Definition

For two sequences of sets $\mathscr{R}$ Ðÿ $\mathscr{U}$, we define:

$\mathscr{R} \leq_\omega \mathscr{U} \leftrightarrow (\forall X \subseteq \mathbb{N})[\mathscr{U} \leq_{c.e.} X \to \mathscr{R} \leq_{c.e.} X]$;

The equivalence classes under $\leq_\omega$ are called ω-enumeration degrees. Introduced by Soskov and studied by him and his students in Sofia in the past decade.

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The equivalence classes under $\leq_\omega$ are called ω-enumeration degrees. Introduced by Soskov and studied by him and his students in Sofia in the past decade.
This is a generalization of the enumeration reducibility.

Theorem (Selman)

$\mathcal{A} \leq_e B \leftrightarrow (\forall X \subseteq \mathbb{N})[B \leq_{c.e.} X \;\Rightarrow\; A \leq_{c.e.} X]$.

Embedding ω-degrees into Muchnick degrees

Theorem (Soskov 2013)

For every sequence $\mathscr{R}$, we can build a structures $\mathcal{N}_{\mathscr{R}}$ such that:

$$Spec(\mathcal{N}_{\mathscr{R}}) = \{d_T(B) \mid \mathscr{R} \leq_{c.e.} B\}.$$

Then we have the following characterization:

$$\mathscr{R} \leq_\omega \mathscr{U} \leftrightarrow \; Spec(\mathcal{N}_{\mathscr{U}}) \subseteq Spec(\mathcal{N}_{\mathscr{R}}).$$

Soskov uses the technique of Marker's extension in his proof. The structure $\mathcal{N}_{\mathscr{R}}$ is defined in an computable infinite language, because for every R_n he builds its n-th Marker's extensionĐšĐ" R'_n, which is a $(n+1)$-ary relation. The structure $\mathcal{N}_{\mathscr{R}}$ can also be built by coding the sequence $\mathscr{R}$ by pairs of structures. We apply the strong jit theorem for each R_n and take the join of the produced structures.

Concluding remarks

- It would be nice if we can choose $\mathcal{N}_{\mathscr{R}}$ to be something nice such us a linear ordering.
- The Marker's extension construction has nice model-theoretic properties, but from the point of view of computable structure theory, it seems that we can replace it by the pairs of structures construction.
- When is it true that $\mathcal{N}$ is Medvedev equivalent to $\mathcal{M}$, where $\mathcal{N}$ is the strong jit for $\mathcal{A}$, $\mathcal{M}$ is the Marker's extension for $\mathcal{A}$?

The end

Thank you for your attention!