Scott-Continuous Embeddings and Learning Families of Algebraic Structures from Text

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Conference on Mathematical Logic
Gjuletchiza, 2025

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[1] This study is financed by the European Union-NextGenerationEU, through the National Recovery and Resilience Plan of the Republic of Bulgaria, project BG-RRP-2.004-0008-C01

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Istanbul 2011

Introduction

A common theme in the study of countable mathematical structures is that isomorphism invariant properties have syntactic characterizations.
Examples:

- the existence of Scott sentences for countable structures (Scott [1963]),
- the Lopez-Escobar theorem, which says that every invariant Borel subset in the space of countable structures is definable in the infinitary logic $L_{\omega_1\omega}$ (Lopez-Escobar [1965]; Vaught [1974]),
- a relation on a structure that is Σ^0_α in every copy is definable by a computable Σ_α $L_{\omega_1\omega}$-formula (Ash, Knight, Manasse, Slaman [1989]; Chisholm [1990]).

Cantor topology and Scott topology on Mod(τ)

Fix a countable relational signature τ. Here we consider only τ-structures $\mathcal{A}$ with domain ω.
We fix an encoding of (atomic diagrams of) τ-structures. This allows us to identify τ-structures with elements of the Cantor space 2^ω. More formally, we talk about the space of τ-models $\mathrm{Mod}(\tau)$ which is homeomorphic to a closed subspace of 2^ω.

Consider a new topological space $\mathrm{Mod}_\mathrm{p}(\tau)$. Let τ contains $=$ and $\neq$. The elements of $\mathrm{Mod}_\mathrm{p}(\tau)$ are still τ-structures with domain ω, but the space is equipped with the Scott topology.

Cantor-continuous embeddings

The Cantor space, 2^ω is a collection of reals, equipped with the product topology of the discrete topology on $\{0, 1\}$. A basis for 2^ω is formed by the collection of $[\sigma] = \{\mathrm{f} \in 2^\omega : \sigma \subset \mathrm{f}\}$, for all finite binary strings σ.

Proposition (Folklore)

A function $\Psi : 2^\omega \to 2^\omega$ is Cantor-continuous if and only if there exists a Turing operator Φ_e and a set $\mathrm{A} \in 2^\omega$ such that $\Psi(\mathrm{X}) = \Phi_\mathrm{e}(\mathrm{A} \oplus \mathrm{X})$ for all $\mathrm{X} \in 2^\omega$.

Definition

A mapping Ψ is a Cantor-continuous embedding of the class of structures $\mathcal{K}_0$ into the class of structures $\mathcal{K}_1$, ($\Psi\colon \mathcal{K}_0 \leq_{\mathrm{Cantor}} \mathcal{K}_1$), if Ψ is Cantor-continuous and:

1. For any $\mathcal{A} \in \mathcal{K}_0$, $\Psi(\mathcal{D}(\mathcal{A}))$ is the characteristic function of the atomic diagram of a structure from $\mathcal{K}_1$.
2. For any $\mathcal{A}, \mathcal{B} \in \mathcal{K}_0$, we have $\mathcal{A} \cong \mathcal{B}$ if and only if $\Psi(\mathcal{A}) \cong \Psi(\mathcal{B})$.

The Lopez-Escobar theorem

In the classical setting the Lopez-Escobar theorem establishes a correspondence between subsets of $\mathrm{Mod}(\tau)$ defined by sentences in the infinitary logic $\mathrm{L}_{\omega_1\omega}$ and the Borel sets.

Theorem (Lopez-Escobar [1965], Vaught [1974])

Let $\mathcal{K}$ be a subclass of $\mathrm{Mod}(\tau)$ which is closed under isomorphisms. Let $\alpha > 0$ be a countable ordinal. Then $\mathcal{K}$ is $\mathbf{\Sigma}^0_\alpha$ (in the Borel hierarchy) if and only if $\mathcal{K}$ is axiomatizable by a Σ_α-sentence.

$L_{\omega_1\omega}$ formulas

Let I be a countable set.

- The Σ_0 formulas (Π_0 formulas) are quantifier free τ-formulas. For $\alpha \geq 1$:
- $\varphi(\bar{u})$ is Σ_α formula if it has the form

$$\varphi(\bar{u}) = \bigvee_{i \in I} \exists \bar{x}_i(\phi_i(\bar{u}, \bar{x}_i) \wedge \psi_i(\bar{u}, \bar{x}_i)),$$

  where $\phi_i(\bar{u}, \bar{x}_i)$ is a Σ_{β_i} and $\psi_i(\bar{u}, \bar{x}_i)$ is Π_{β_i}, for some $\beta_i < \alpha$.
- $\varphi(\bar{u})$ is Π_α formula if it has the form

$$\varphi(\bar{u}) = \bigwedge_{i \in I} \forall \bar{x}_i(\phi_i(\bar{u}, \bar{x}_i) \vee \psi_i(\bar{u}, \bar{x}_i)),$$

  where $\phi_i(\bar{u}, \bar{x}_i)$ is a Σ_{β_i} and $\psi_i(\bar{u}, \bar{x}_i)$ is a Π_{β_i}, for some $\beta_i < \alpha$.

Pullback theorem

When the embedding $\mathcal{K}_0 \leq_{\text{Cantor}} \mathcal{K}_1$ is given by a Turing operator Φ_e, then we say that $\mathcal{K}_0$ is Turing computable embeddable into $\mathcal{K}_1$, and we denote this by $\Phi_e \colon \mathcal{K}_0 \leq_{\text{tc}} \mathcal{K}_1$.

Theorem (Pullback theorem Knight, Miller, Vanden Boom [2007])

If $\Phi_e \colon \mathcal{K} \leq_{\text{tc}} \mathcal{K}'$ then for any computable infinitary sentence φ' in the signature of $\mathcal{K}'$, we can effectively find a computable infinitary sentence φ in the signature of $\mathcal{K}$ such that for all $\mathcal{A} \in \mathcal{K}$,

$$\mathcal{A} \models \varphi \text{ if and only if } \Phi_e(\mathcal{A}) \models \varphi'.$$

Moreover, for a nonzero $\alpha < \omega_1^{CK}$, if φ' is Σ^c_α (or Π^c_α), then so is φ.

Corollary (Relativization)

If $\Psi \colon \mathcal{K} \leq_{\text{Cantor}} \mathcal{K}'$ then for any infinitary sentence φ' in the signature of $\mathcal{K}'$, there exists an infinitary sentence φ in the signature of $\mathcal{K}$ such that for all $\mathcal{A} \in \mathcal{K}$, $\mathcal{A} \models \varphi$ if and only if $\Psi(\mathcal{A}) \models \varphi'$.

Scott-Continuous Embeddings

The Scott topology, denoted by $\mathcal{P}_{\mathrm{w}}$, can be characterized as the product topology of the Sierpiński space on $\{0,1\}$. The Sierpiński space on $\{0,1\}$ is the topological space with open sets $\{\emptyset, \{1\}, \{0,1\}\}$. A basis for $\mathcal{P}_{\mathrm{w}}$ is formed by the collection $[\mathrm{D}] = \{\mathrm{A} \subseteq \omega : \mathrm{D} \subseteq \mathrm{A}\}$, for all finite sets D.

Definition (Case,71)

A set $\mathrm{A} \in \mathcal{P}_{\mathrm{w}}$ defines a generalized enumeration operator $\Gamma_{\mathrm{A}} : \mathcal{P}_{\mathrm{w}} \to \mathcal{P}_{\mathrm{w}}$ if and only if for each set $\mathrm{B} \in \mathcal{P}_{\mathrm{w}}$,

$$\Gamma_{\mathrm{A}}(\mathrm{B}) = \{\mathrm{x} : \exists \mathrm{v}(\langle \mathrm{x}, \mathrm{v} \rangle \in \mathrm{A} \ \&\ \mathrm{D}_{\mathrm{v}} \subseteq \mathrm{B})\}.$$

When $\mathrm{A} = \mathrm{W}_{\mathrm{e}}$ for some c.e. set W_{e}, Γ_{e} is the usual enumeration operator.

Proposition (Folklore)

A mapping $\Gamma : \mathcal{P}_{\mathrm{w}} \to \mathcal{P}_{\mathrm{w}}$ is Scott-continuous if and only if Γ is a generalized enumeration operator.

Scott-Continuous Embeddings

Corollary

A mapping $\Psi : \mathcal{P}_{\mathrm{w}} \to \mathcal{P}_{\mathrm{w}}$ is Scott-continuous if and only if Ψ is

- (a) monotone, i.e., $A \subseteq B$ implies $\Psi(A) \subseteq \Psi(B)$, and
- (b) compact, i.e., $x \in \Psi(A)$ if and only if $x \in \Psi(D)$ for some finite $D \subseteq A$.

Definition

A mapping Γ is a Scott-continuous embedding of $\mathcal{K}_0$ into $\mathcal{K}_1$, denoted by $\Gamma\colon \mathcal{K}_0 \leq_{\mathrm{Scott}} \mathcal{K}_1$, if Γ is Scott-continuous and satisfies the following:

1. For any $\mathcal{A} \in \mathcal{K}_0$, $\Gamma(\mathcal{D}_+(\mathcal{A}))$ is the positive atomic diagram of a structure from $\mathcal{K}_1$.
2. For any $\mathcal{A}, \mathcal{B} \in \mathcal{K}_0$, we have $\mathcal{A} \cong \mathcal{B}$ if and only if $\Gamma(\mathcal{A}) \cong \Gamma(\mathcal{B})$.

If Γ_e is an enumeration operator, we say that Γ_e is a positive computable embedding of $\mathcal{K}_0$ into $\mathcal{K}_1$, $\Gamma_e\colon \mathcal{K}_0 \leq_{\mathrm{pc}} \mathcal{K}_1$.

A hierarchy of positive infinitary formulas

Let I be a countable set.

- Let $\alpha = 0$. Then:
  - the Σ^p_0 formulas are the finite conjunctions of atomic τ-formulas.
  - the Π^p_0 formulas are the finite disjunctions of negations of atomic τ-formulas.
- Let $\alpha = 1$. Then:
  - $\varphi(\bar{u})$ is a Σ^p_1 formula if it has the form

    $$\varphi(\bar{u}) = \bigvee_{i \in I} \exists \bar{x}_i \psi_i(\bar{u}, \bar{x}_i),$$

    where for each $i \in I$, $\psi_i(\bar{u}, \bar{x}_i)$ is a Σ^p_0 formula.
  - $\varphi(\bar{u})$ is a Π^p_1 formula if it has the form

    $$\varphi(\bar{u}) = \bigwedge_{i \in I} \forall \bar{x}_i \psi_i(\bar{u}, \bar{x}_i),$$

    where for each $i \in I$, $\psi_i(\bar{u}, \bar{x}_i)$ is a Π^p_0 formula.

A hierarchy of positive infinitary formulas

- Let $\alpha \geq 2$. Then:
  - $\varphi(\bar{u})$ is Σ^p_α formula if it has the form

$$\varphi(\bar{u}) = \bigvee_{i \in I} \exists \bar{x}_i(\phi_i(\bar{u}, \bar{x}_i) \wedge \psi_i(\bar{u}, \bar{x}_i)),$$

    where $\phi_i(\bar{u}, \bar{x}_i)$ is a $\Sigma^p_{\beta_i}$ and $\psi_i(\bar{u}, \bar{x}_i)$ is $\Pi^p_{\beta_i}$, for some $\beta_i < \alpha$.
  - $\varphi(\bar{u})$ is Π^p_α formula if it has the form

$$\varphi(\bar{u}) = \bigwedge_{i \in I} \forall \bar{x}_i(\phi_i(\bar{u}, \bar{x}_i) \vee \psi_i(\bar{u}, \bar{x}_i)),$$

    where $\phi_i(\bar{u}, \bar{x}_i)$ is a $\Sigma^p_{\beta_i}$ and $\psi_i(\bar{u}, \bar{x}_i)$ is a $\Pi^p_{\beta_i}$, for some $\beta_i < \alpha$.

Effective version: computable Σ^p_α (Π^p_α) formulas.

The Lopez-Escobar theorem for Scott-contionious domains

Reminder: In the classical setting the Lopez-Escobar theorem establishes a correspondence between subsets of $\mathrm{Mod}(\tau)$ defined by sentences in the infinitary logic $\mathrm{L}_{\omega_1\omega}$ and the Borel sets.

Theorem (Bazhenov, Fokina, Rossegger, S., and Vatev [2024])

Let $\mathcal{K}$ be a subclass of $\mathrm{Mod}_{\mathrm{p}}(\tau)$ which is closed under isomorphisms. Let $\alpha > 0$ be a countable ordinal. Then $\mathcal{K}$ is $\mathbf{\Sigma}^0_\alpha$ in the space $\mathrm{Mod}_{\mathrm{p}}(\tau)$ if and only if $\mathcal{K}$ is axiomatizable by a $\Sigma^{\mathrm{p}}_\alpha$-sentence.

We use a forcing relation used by Soskov in 2004 in order to characterize the relatively intrinsic relations for the enumeration reducibility, but our presentation is closer to Montalbán 2021.

A pullback theorem for positive computable embeddings

Theorem (Pullback Theorem: Bazhenov, Fokina, Rossegger, S., and Vatev [2024])

Let $\mathcal{K} \subseteq \mathrm{Mod_p}(\tau)$ and $\mathcal{K}' \subseteq \mathrm{Mod_p}(\tau')$ be closed under isomorphism. Let $\Gamma_e : \mathcal{K} \leq_{pc} \mathcal{K}'$, then for any computable Σ^p_α (or Π^p_α) τ'-sentence φ we can effectively find a computable Σ^p_α (or Π^p_α) τ-sentence φ^* such that for all $\mathcal{A} \in \mathcal{K}$,

$$\mathcal{A} \models \varphi^* \text{ iff } \Gamma_e(\mathcal{A}) \models \varphi.$$

Theorem (Pullback Theorem relativized: Bazhenov, Fokina, Rossegger, S., and Vatev [2024])

Let $\mathcal{K} \subseteq \mathrm{Mod_p}(\tau)$ and $\mathcal{K}' \subseteq \mathrm{Mod_p}(\tau')$ be closed under isomorphism and $\alpha > 0$ be a countable ordinal. Let Γ be a Scott-continuous embedding from $\mathcal{K}$ into $\mathcal{K}'$ ($\mathcal{K} \leq_{\mathrm{Scott}} \mathcal{K}'$). Then for any Σ^p_α (or Π^p_α) τ'-sentence φ we can find a Σ^p_α (or Π^p_α) τ-sentence φ^* such that for all $\mathcal{A} \in \mathcal{K}$,

$$\mathcal{A} \models \varphi^* \text{ iff } \Gamma(\mathcal{A}) \models \varphi.$$

Algorithmic learning theory

- The classical algorithmic learning theory goes back to the works of Putnam [1965], Gold [1967], Glymour [1991] mainly focused on learning for formal languages and for recursive functions.
- Within the framework of computable structure theory, the work of Stephan and Ventsov [2001] initiated investigations of learnability for classes of substructures of a given computable structure.
- Fokina, Kötzing, and San Mauro [2019] considered various classes $\mathcal{K}$ of computable equivalence relations.
- Bazhenov, Fokina, and San Mauro [2020] extended the notion of learnability from Informant to arbitrary countable families of computable structures.

Learning for families of algebraic structures

- Fix a computable signature τ. Let $\mathcal{K}$ be a countable family of countable τ-structures.
- Step-by-step, we obtain larger and larger finite pieces of a τ-structure $\mathcal{A}$. In addition, we assume that this $\mathcal{A}$ is isomorphic to some structure from the class $\mathcal{K}$.

Problem: Is it possible to identify (in the limit) the isomorphism type of $\mathcal{A}$.

The problem combines the approaches of algorithmic learning theory and computable structure theory:
We want to learn the family $\mathcal{K}$ up to isomorphism.

Learning structures from informant

Consider a family of τ-structures $\mathcal{K} = \{\mathcal{A}_i\}_{i\in\omega}$. We assume that the structures $\mathcal{A}_i$ are pairwise not isomorphic.

- The learning domain:

$$\mathrm{LD}(\mathcal{K}) = \{\mathcal{B} \mid \mathcal{B} \cong \mathcal{A}_i \text{ for some } i \in \omega; \text{ and } \mathrm{dom}(\mathcal{B}) = \omega\}$$

  The learning domain can be treated as a set of reals (i.e., $\mathrm{LD}(\mathcal{K}) \subseteq 2^\omega$).

- A learner M sees (stage by stage) finite pieces of data about a given structure from $\mathrm{LD}(\mathcal{K})$, and M outputs conjectures. More formally,

$$\text{M is a function from } 2^{<\omega} \text{ to } \omega.$$

  If $\mathrm{M}(\sigma) = i$, then this means: "the finite string σ looks like an isomorphic copy of $\mathcal{A}_i$".

Learning structures from informant

- For a τ-structure $\mathcal{A}$, an informant $\mathbb{I}$ for $\mathcal{A}$ is an arbitrary sequence $(\psi_0, \psi_1, \psi_2, \dots)$ containing elements from $\mathrm{D}(\mathcal{A})$.
- The learning is successful if for every $\mathcal{B} \in \mathrm{LD}(\mathcal{K})$ and any informant $\mathbb{I}_{\mathcal{B}}$ for $\mathcal{B}$, the learner eventually stabilizes to a correct conjecture about the isomorphism type of $\mathcal{B}$. More formally, there exists a limit

$$\lim_{n \to \omega} M(\mathbb{I}_{\mathcal{B}} \upharpoonright n) = i$$

belonging to ω, and $\mathcal{A}_i$ is isomorphic to $\mathcal{B}$.

Definition

The family $\mathcal{K}$ is Inf-learnable (up to isomorphism) if there exists a learner M that successfully learns the family $\mathcal{K}$.

Remark. More formally, the family $\mathcal{K}$ is $\mathrm{InfEx}_{\cong}$-learnable:

- Inf means learning from informant;
- Ex means "explanatory".

A syntactic characterization of Inf-learnability

Theorem (Bazhenov, Fokina, and San Mauro [2020])

The following conditions are equivalent:

- The family $\mathcal{K}$ is Inf-learnable.
- There are Σ_2-sentences ψ_i, $\mathrm{i} \in \omega$ such that

$$\mathcal{A}_\mathrm{i} \models \psi_\mathrm{j} \text{ if and only if } \mathrm{i} = \mathrm{j}.$$

If ω denotes the standard ordering of natural numbers, and ω^* denotes the standard ordering of negative integers then:

Example

The pair of linear orders $\{\omega, \omega^*\}$ is learnable from informant.
Key observation:
These orders are "separated" by Σ_2-sentences: (ω has a least element) vs. (ω^* has a greatest element).

A descriptive set-theoretic characterization of Inf-learning

One of the benchmark Borel equivalence relations on the Cantor space 2^ω is the relation E_0 (almost equality).

$$\alpha \mathrm{E}_0 \beta \leftrightarrow (\exists \mathrm{n})(\forall \mathrm{m} \geq \mathrm{n})(\alpha(\mathrm{m}) = \beta(\mathrm{m})).$$

Theorem (Bazhenov, Cipriani, and San Mauro [2023])

The following conditions are equivalent:

- The family $\mathcal{K}$ is Inf-learnable.
- There is a continuous function $\Gamma : 2^\omega \to 2^\omega$ such that for all $\mathcal{A}, \mathcal{B} \in \mathrm{LD}(\mathcal{K})$ we have:

$$\mathcal{A} \cong \mathcal{B} \leftrightarrow \Gamma(\mathcal{A}) \mathrm{E}_0 \Gamma(\mathcal{B}).$$

Learning structures from text

We assume that the relational signature τ contains both $=$ (equality) and $\neq$ (inequality).

[Notice that then the empty set is definable in $\mathcal{A}$ by an atomic formula $(\mathrm{x} \neq \mathrm{x})$.]

We want to learn a countable family $\mathcal{K} = \{\mathcal{A}_\mathrm{i}\}_{\mathrm{i}\in\omega}$. Now we learn from text, i.e., from the positive information about an input structure $\mathcal{A}$.

Definition

Let $\mathcal{A}$ be a τ-structure with domain ω. A text t for the structure $\mathcal{A}$ is an arbitrary sequence $\{\mathrm{t}(\mathrm{i})\}_{\mathrm{i}\in\omega}$ such that:

- for each $\mathrm{i} \in \omega$, $\mathrm{t}(\mathrm{i})$ is an atomic formula, i.e., a formula of the form $\mathrm{R}(\mathrm{a}_1, \ldots, \mathrm{a}_\mathrm{n})$, where $\mathrm{R} \in \tau$ and $\mathrm{a}_1, \ldots, \mathrm{a}_\mathrm{n} \in \omega$;
- the set $\{\mathrm{t}(\mathrm{i}) \mid \mathrm{i} \in \omega\}$ contains precisely all atomic formulas which are true in the structure $\mathcal{A}$.

Using an encoding, texts are elements of the Baire space ω^ω.

Learning structures from text

A learner M is a function from $\omega^{<\omega}$ to ω.

Definition

Txt-learning is successful if every $\mathcal{A}$ with $\mathrm{dom}(\mathcal{A}) = \omega$ satisfies the following: if $\mathcal{A}$ is an isomorphic copy of $\mathcal{A}_i$, then for any text t for the structure $\mathcal{A}$, we have

$$\lim_{k\to\infty} M(t \upharpoonright k) = i.$$

Definition

The family $\mathcal{K}$ is Txt-learnable (up to isomorphism) if there exists a Txt-learner M that successfully learns the family $\mathcal{K}$.

Simple observations

Proposition

If $\mathcal{K}$ is Txt-learnable, then $\mathcal{K}$ is Inf-learnable.

Example

Consider the following pair of equivalence structures:

- The structure $\mathcal{A}$ has one infinite class and nothing else.
- The structure $\mathcal{B}$ has two infinite classes and nothing else.

Then the family $\mathcal{K} = \{\mathcal{A}; \mathcal{B}\}$ is Inf-learnable, but not Txt-learnable.

A syntactic characterization of Txt-learnability

Theorem (Bazhenov, Fokina, Rossegger, S., and Vatev [2024])

The following conditions are equivalent:

- The family $\mathcal{K}$ is Txt-learnable.
- $\mathcal{K} \leq_{\mathrm{Scott}} \mathcal{K}_{\mathrm{univ}}$.
- There are Σ_2^p-sentences ψ_i, $i \in \omega$ such that

$$\mathcal{A}_i \models \psi_j \text{ if and only if } i = j.$$

Idea of the proof of the theorem

Given a learner M, one can build a Scott-continuous embedding from our class $\mathcal{K}$ into some "universal learnable" class $\mathcal{K}_{\mathrm{univ}} = \{\mathcal{B}_i \mid i \in \omega\}$, where $\mathcal{B}_i = (\omega; E)$ is an equivalence structure which has infinitely many infinite classes, infinitely many classes of size $i + 1$, and nothing else.
Note that each $\mathcal{B}_i$ has its own distinguishing Σ_2^p-sentence:

$$\psi_i = \exists x_0 \dots \exists x_i [\bigwedge_{j \neq k \ j,k \leq i} (x_j \neq x_k \ \& \ x_j E x_k) \\ \& \ \forall y (\neg (y E x_0) \vee \bigvee_{l \leq i} \neg (y \neq x_l)].$$

So, $\mathcal{K} \leq_{\mathrm{Scott}} \mathcal{K}_{\mathrm{univ}}$.
Applying the Pullback theorem, we get the desired sequence of distinguishing Σ_2^p-sentences for our family $\mathcal{K}$.

Idea of the proof of the theorem

Suppose that there exist Σ_2^p sentences φ_i such that $\mathcal{A}_i \models \varphi_j$ if and only if $i = j$. Then $\mathcal{K}$ is TxtEx-learnable:
Without loss of generality, suppose that the Σ_2^p sentence φ_i has the form

$$\exists \bar{x}_i \left(\alpha_i(\bar{x}_i) \wedge \bigwedge_{j \in J_i} \forall \bar{y}_j \neg(\beta_{i,j}(\bar{x}_i, \bar{y}_j)) \right)$$

For any sequence σ of positive atomic formulas we must determine the value of $M(\sigma)$.
Find the least $\langle i, \bar{a} \rangle$ $\alpha_i(\bar{a})$ is in σ and no sentence of the form $\beta_{i,j}(\bar{a}, \bar{b}_j)$ is in σ. Then we let $M(\sigma) = i$.
Suppose $\mathcal{A} \cong \mathcal{A}_i$ and consider some text $\mathbb{T}_{\mathcal{A}}$ for $\mathcal{A}$. Since $\mathcal{A} \models \varphi_i$, find the least tuple $\bar{a}$ such that $\mathcal{A} \models \alpha_i(\bar{a})$ and $\mathcal{A} \models \bigwedge_{j \in J_i} \forall \bar{y}_j \neg(\beta_{i,j}(\bar{a}, \bar{y}_j))$. It follows that the code of $\alpha_i(\bar{a})$ will appear in some initial segment of $\mathbb{T}_{\mathcal{A}}$ and none of the positive atomic sentences $\beta_{i,j}(\bar{a}, \bar{b}_j)$ will appear in $\mathbb{T}_{\mathcal{A}}$ for any $\bar{b}_j$ and any $j \in J_i$. It follows that $\lim_{n\to\infty} M(\mathbb{T}_{\mathcal{A}} \upharpoonright n) = i$.

TxtEx-Complete Classes

Definition

A countably infinite class $\mathcal{K}_0$ is TxtEx-complete if

- $\mathcal{K}_0$ is TxtEx-learnable, and
- for any countable TxtEx-learnable class $\mathcal{K}$, $\mathcal{K} \leq_{\mathrm{Scott}} \mathcal{K}_0$.

Corollary

The class $\mathcal{K}_{\mathrm{univ}}$ is TxtEx-complete.

Consider a signature $L_{st} = \{<, =, \neq\} \cup \{P_i(x) : i \in \omega\}$, $\mathcal{A}_i$ – a structure with $P_j^{\mathcal{A}_i}$ – disjoint infinite sets. For $x \in P_j$ and $y \in P_k$, $j \neq k$, x and y are incomparable under $<$. Let η denote the order type of the rationals, and, $\mathcal{A}_{i,j}$ is the restriction of $\mathcal{A}_i$ to the elements in P_j.

$$\mathcal{A}_{i,j} \cong \begin{cases} \eta, & \text{if } i \neq j \\ 1 + \eta, & \text{if } i = j. \end{cases}$$

Let us denote $\mathcal{K}_{st} = \{\mathcal{A}_i : i \in \omega\}$.

Theorem (Bazhenov, Fokina, and San Mauro [2020])

For a class $\mathcal{K} = \{\mathcal{B}_i : i \in \omega\}$, the following are equivalent:

1. The class $\mathcal{K}$ is InfEx-learnable.
2. $\mathcal{K} \leq_{\text{Cantor}} \mathcal{K}_{st}$.
3. There is a sequence of $L_{\omega_1\omega}$ Σ_2 sentences $\{\psi_i : i \in \omega\}$ such that for all i and j, $\mathcal{B}_j \models \psi_i$ if and only if $i = j$.

TxtEx-Complete Classes

Let $\mathcal{E}$ be a class of all equivalence structures $\mathcal{A}_{\mathrm{n}}$ with one class of infinitely many elements and one class of n elements, $\mathrm{n} \geq 1$. We prove the following:

- $\mathcal{K}_{\mathrm{univ}} \leq_{\mathrm{pc}} \mathcal{K}_{\mathrm{st}}$.
- $\mathcal{K}_{\mathrm{st}} \leq_{\mathrm{pc}} \mathcal{E}$.
- $\mathcal{E} \leq_{\mathrm{pc}} \mathcal{K}_{\mathrm{univ}}$.

Proposition

The classes $\mathcal{E}$, $\mathcal{K}_{\mathrm{univ}}$, and $\mathcal{K}_{\mathrm{st}}$ are TxtEx-complete.

Descriptive set-theoretic characterization for Txt-learnability

Consider the space $\mathcal{P}_w$ of all subsets of ω, with the Scott topology. For $X \in \mathcal{P}_w$ and $m \in \omega$, denote the m-th column of X by:

$$X^{[m]} = \{y \mid \langle m, y \rangle \in X\}.$$

The equivalence relation E_{set} is defined as follows:

$$X \; E_{set} Y \leftrightarrow \{X^{[m]} \mid m \in \omega\} = \{Y^{[m]} \mid m \in \omega\}.$$

Theorem (Bazhenov, Fokina, Rossegger, S., and Vatev [2024])

Let $\mathcal{K} = \{\mathcal{A}_i \mid i \in \omega\}$ be a family of countable τ-structures. Equivalent:

- The family $\mathcal{K}$ is Txt-learnable.
- There is a continuous function $\Gamma : \mathrm{Mod}_p(\tau) \to \mathcal{P}_w$ such that for all $\mathcal{A}, \mathcal{B} \in \mathrm{LD}(\mathcal{K})$:
  - $\mathcal{A} \cong \mathcal{B} \leftrightarrow \Gamma(\mathcal{A}) E_{set} \Gamma(\mathcal{B})$;
  - for each $i \in \omega$ we have $\{\Gamma(\mathcal{A}_i)^{[m]} \mid m \in \omega\} = \{\omega, \{i\}\}$.

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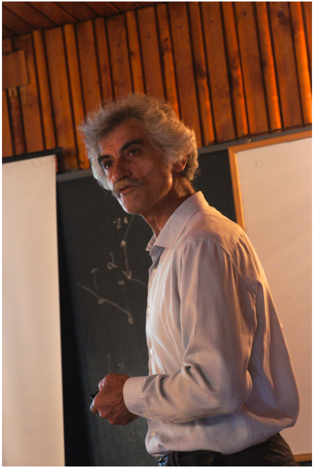
I wish you all the best!



Gjuletchica 2014

Health, joy and many more conferences!

Patra 2009